The convergence of the solutions of a system of max-type difference equations of higher order

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Abstract. In this paper, we study the convergence of the solutions of the following system of max-type difference equations

$$x_n = \max\{\frac{A}{y_{n-m}}, \frac{1}{y_{n-r}^{\alpha}}\}, \quad y_n = \max\{\frac{A}{x_{n-m}}, \frac{1}{x_{n-r}^{\alpha}}\}, \quad n = 0, 1, \dots,$$

where $m, r \in \{1, 2, \dots\}$, $A \in (0, +\infty)$ and $\alpha \in (0, 1)$ and the initial values $x_{-d}, x_{-d+1}, \dots,$ $x_{-1}, y_{-d}, y_{-d+1}, \dots, y_{-1} \in (0, +\infty)$ with $d = \max\{m, r\}$. We show that: (1) If $0 < A \leq 1$ and $\{(x_n, y_n)\}_{n\geq -d}$ is a solution of the above system, then $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = 1$; (2) If $A > 1$ and $\{(x_n, y_n)\}_{n\geq -d}$ is a solution of the above system, then for any $0 \leq k \leq 2m - 1$, x_{2mn+k} and y_{2mn+k} are eventually monotone.

Keywords: System of max-type difference equations, positive solution, eventual monotonicity.

1. Introduction

Recently there has been a great interest in studying the properties of solutions of many max-type difference equations and systems, such as eventual periodicity, the boundedness character and eventual monotonicity (see [1-12]).

In 2009, Gelişken and Çinar [13] investigated the asymptotic behavior and the periodicity of the positive solutions of the following max-type difference equation

$$x_n = \max\{\frac{A}{x_{n-1}}, \frac{1}{x_{n-3}^{\alpha}}\}, \quad n \in \mathbf{N}_0 \equiv \{0, 1, 2, \dots\}, \tag{1.1}$$

where $A \in \mathbf{R}^+ \equiv (0, +\infty)$ and $\alpha \in (0, 1)$, and showed that every positive solution of (1.1) converges to 1 or is eventually periodic with period 2.

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In 2012, we [14] studied the convergence of the positive solutions of the following max-type difference equation

$$x_n = \max\{\frac{A}{x_{n-m}}, \frac{1}{x_{n-k}^{\alpha}}\}, \quad n \in \mathbf{N}_0, \tag{1.2}$$

where $A \in \mathbf{R}^+$, $m, k \in \mathbf{N} \equiv \{1, 2, \ldots\}$ and $\alpha \in (0, 1)$.

In 2008, Sun [15] studied the asymptotic behavior of the following max-type difference equation

$$x_n = \max\{\frac{A}{x_{n-1}^{\alpha}}, \frac{B}{x_{n-2}^{\beta}}\}, \quad n \in \mathbf{N}_0, \tag{1.3}$$

where $A, B \in \mathbf{R}^+$ and $\alpha, \beta \in (0, 1)$, and showed that every positive solution of (1.3) converges to $\max\{1/A^{\frac{1}{\alpha+1}}, 1/B^{\frac{1}{\beta+1}}\}$.

In 2009, Stević [16] showed that every positive solution of the following max-type difference equation

$$x_n = \max\{\frac{A_i}{x_{n-i}^{\alpha_i}} : 1 \le i \le k\}, \quad n \in \mathbf{N}_0 \tag{1.4}$$

converges to $\max\{1/A_i^{\frac{1}{\alpha_i+1}} : 1 \le i \le k\}$, where $A_i \in \mathbf{R}^+$ and $\alpha_i \in (0, 1)$ for every $1 \le i \le k$.

In 2011, we [17] showed that every positive solution of the following max-type difference equation

$$x_n = \max\{\frac{A_i}{x_{n-m_i}^{\alpha_i}} : 1 \le i \le k\}, \quad n \in \mathbf{N}_0 \tag{1.5}$$

converges to $\max\{1/A_i^{\frac{1}{\alpha_i+1}} : 1 \le i \le k\}$, where $m_i \in \mathbf{N}$, $A_i \in \mathbf{R}^+$ and $\alpha_i \in (0, 1)$ for every $1 \le i \le k$.

In this paper, we investigate the convergence of the following system of max-type difference equations

$$x_n = \max\{\frac{A}{y_{n-m}}, \frac{1}{y_{n-r}^{\alpha}}\}, \quad y_n = \max\{\frac{A}{x_{n-m}}, \frac{1}{x_{n-r}^{\alpha}}\}, \quad n \in \mathbf{N}_0, \tag{1.6}$$

where $m, r \in \mathbf{N}$, $A \in \mathbf{R}^+$ and $\alpha \in (0, 1)$ and the initial values $x_{-d}, x_{-d+1}, \ldots, x_{-1}$, $y_{-d}, y_{-d+1}, \ldots, y_{-1} \in \mathbf{R}^+$ with $d = \max\{m, r\}$. We show that:

(1) If $0 < A \le 1$ and $\{(x_n, y_n)\}_{n \ge -d}$ is a solution of (1.6), then $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = 1$;

(2) If $A > 1$ and $\{(x_n, y_n)\}_{n \ge -d}$ is a solution of (1.6), then for any $0 \le k \le 2m - 1$, x_{2mn+k} and y_{2mn+k} are eventually monotone.

2. The case $0 < A \leq 1$

In this section, we study the convergence of the solutions of (1.6) when $0<A\leq 1$.

Let $\{(x_n, y_n)\}_{n\geq -d}$ be a solution of (1.6) with the initial values $x_{-d}, x_{-d+1}, \ldots, x_{-1}, y_{-d}, y_{-d+1}, \ldots, y_{-1} \in \mathbf{R}^+$. Then we obtain from (1.6) that for any $n \in \mathbf{N_0}$,

$$x_n y_{n-r}^{\alpha} \geq 1, \quad y_n x_{n-r}^{\alpha} \geq 1. \tag{2.1}$$

For all $n \geq d$, write

$$X_n = \max\{x_n, x_{n-1}, \ldots, x_{n-2d}, 1\}, \quad Y_n = \max\{y_n, y_{n-1}, \ldots, y_{n-2d}, 1\}. \tag{2.2}$$

Lemma 2.1.

(1) *$x_{n+1} \leq X_n$ and $y_{n+1} \leq Y_n$ for all $n \geq d$ and X_n, Y_n are decreasing $(n \geq d)$.*

(2) *$x_n \geq A/Y_d$ and $y_n \geq A/X_d$ for any $n \geq d+m+1$.*

Proof. By (1.6) and (2.1), we obtain that for any $n \geq d$,

$$\begin{aligned} x_{n+1} &= \max\{\tfrac{Ax_{n-m+1-r}^{\alpha}}{y_{n-m+1}x_{n-m+1-r}^{\alpha}}, \tfrac{x_{n-r+1-r}^{\alpha^2}}{y_{n-r+1}^{\alpha}x_{n-r+1-r}^{\alpha^2}}\} \\ &\leq \max\{x_{n-m+1-r}^{\alpha}, x_{n-r+1-r}^{\alpha^2}\} \\ &\leq \max\{x_{n-m+1-r}, x_{n-r+1-r}, 1\} \\ &\leq X_n. \end{aligned} \tag{2.3}$$

Thus

$$X_{n+1} = \max\{x_{n+1}, \ldots, x_{n+1-2d}, 1\} \leq \max\{x_{n+1}, X_n\} = X_n. \tag{2.4}$$

In same fashion, we also obtain that for any $n \geq d$,

$$y_{n+1} \leq Y_n, \quad Y_{n+1} \leq Y_n. \tag{2.5}$$

Hence it follows that for all $n \geq d+m+1$,

$$x_n \geq \frac{A}{y_{n-m}} \geq \frac{A}{Y_{n-m-1}} \geq \frac{A}{Y_d}, \quad y_n \geq \frac{A}{x_{n-m}} \geq \frac{A}{X_{n-m-1}} \geq \frac{A}{X_d}. \tag{2.6}$$

The proof is complete. □

By Lemma 2.1 we write

$$\liminf_{n\to\infty} y_n = y > 0, \quad \liminf_{n\to\infty} x_n = x > 0 \tag{2.7}$$

and

$$\lim_{n\to\infty} X_n = X, \quad \lim_{n\to\infty} Y_n = Y. \tag{2.8}$$

Then

$$X \geq 1, \quad Y \geq 1. \tag{2.9}$$

Remark 2.2. Note that from (2.1) we see that there exists a sequence $0 < k_1 < k_2 < \ldots$ such that $x_{k_n} \geq 1$ for any $n \in \mathbf{N}$ or $y_{k_n} \geq 1$ for any $n \in \mathbf{N}$.

Lemma 2.3.

(1) *If there exists a sequence* $0 < k_1 < k_2 < \ldots$ *such that* $x_{k_n} \geq 1$ *for any* $n \in \mathbf{N}$*, then* $\limsup_{n\to\infty} x_n = X$.

(2) *If there exists a sequence* $0 < k_1 < k_2 < \ldots$ *such that* $y_{k_n} \geq 1$ *for any* $n \in \mathbf{N}$*, then* $\limsup_{n\to\infty} y_n = Y$.

Proof. Assume that there exists a sequence $0 < k_1 < k_2 < \ldots$ such that $x_{k_n} \geq 1$ for any $n \in \mathbf{N}$. Then by Remark 2.2 we see that there exists a subsequence $x_{r_1}, x_{r_2}, \ldots, x_{r_n}, \ldots$ with $x_{r_n} = X_{k_n} \geq x_{k_n} \geq 1$ for any $n \in \mathbf{N}$, which implies

$$\limsup_{n\to\infty} x_n \geq X \geq 1. \tag{2.10}$$

On the other hand, by $x_{n+1} \leq X_n$ for all $n \geq d$ we obtain

$$\limsup_{n\to\infty} x_n \leq \limsup_{n\to\infty} X_n = X. \tag{2.11}$$

Thus $\limsup_{n\to\infty} x_n = X$. The other case is treated similarly, so we omit the detail. The proof is complete. □

Lemma 2.4. (1) *If there exists a sequence* $0 < k_1 < k_2 < \ldots$ *such that* $x_{k_n} \geq 1$ *for any* $n \in \mathbf{N}$*, then* $X = y = 1$.

(2) *If there exists a sequence* $0 < k_1 < k_2 < \ldots$ *such that* $y_{k_n} \geq 1$ *for any* $n \in \mathbf{N}$*, then* $Y = x = 1$.

Proof. Assume that there exists a sequence $0 < k_1 < k_2 < \ldots$ such that $x_{k_n} \geq 1$ for any $n \in \mathbf{N}$. Then by Lemma 2.1 we may assume that there exist $0 < m_1 < m_2 < \ldots < m_n < \ldots$ and $0 < s_1 < s_2 < \ldots < s_n < \ldots$ such that

$$\begin{aligned} y_{s_n} &\to \liminf_{n\to\infty} y_n = y > 0, \\ x_{s_n - m} &\to A_1, \\ x_{s_n - r} &\to A_2, \\ x_{m_n} &\to \limsup_{n\to\infty} x_n = X \geq 1, \\ y_{m_n - m} &\to B_1, \\ y_{m_n - r} &\to B_2. \end{aligned} \tag{2.12}$$

By taking the limit in the following relationship

$$x_{m_n} = \max\{\frac{A}{y_{m_n - m}}, \frac{1}{y^{\alpha}_{m_n - r}}\} \tag{2.13}$$

as $n \to \infty$, it follows

$$X = \max\{\frac{A}{B_1}, \frac{1}{B_2^\alpha}\} \le \max\{\frac{A}{B_1}, \frac{1}{B_2}\} \le \frac{1}{y}. \tag{2.14}$$

Thus

$$Xy \le 1. \tag{2.15}$$

We claim $X = 1$. In fact, if $X > 1$, then by (2.15) and taking the limit in the following relationship

$$y_{s_n} = \max\{\frac{A}{x_{s_n - m}}, \frac{1}{x_{s_n - r}^\alpha}\} \tag{2.16}$$

as $n \to \infty$, we obtain

$$1 > \frac{1}{X} \ge y = \max\{\frac{A}{A_1}, \frac{1}{A_2^\alpha}\} \ge \frac{1}{A_2^\alpha} > \frac{1}{A_2} \ge \frac{1}{X}, \tag{2.17}$$

which is a contradiction. This implies $X = 1$. Again by (2.15) and taking the limit in (2.16) as $n \to \infty$, we obtain

$$1 \ge y = \max\{\frac{A}{A_1}, \frac{1}{A_2^\alpha}\} \ge \frac{1}{A_2^\alpha} \ge \frac{1}{A_2} \ge \frac{1}{X} \ge 1. \tag{2.18}$$

Thus $y = 1$. The other case is treated similarly, so we omit the detail. The proof is complete. □

Theorem 2.5. $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = 1$.

Proof. If there exists a sequence $0 < k_1 < k_2 < \ldots$ such that $x_{k_n} \ge 1$ for any $n \in \mathbf{N}$ and there exists a sequence $0 < s_1 < s_2 < \ldots$ such that $y_{s_n} \ge 1$ for any $n \in \mathbf{N}$, then by Lemma 2.4 we have $X = Y = x = y = 1$, which implies $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = 1$.

If there exists a sequence $0 < k_1 < k_2 < \ldots$ such that $x_{k_n} \ge 1$ for any $n \in \mathbf{N}$ and there exists a $N \in \mathbf{N}$ such that $y_n < 1$ for any $n \ge N$, then by Lemma 2.4 we have $X = y = 1$. Thus $1 = y \le \limsup_{n\to\infty} y_n \le 1$, which implies $\lim_{n\to\infty} y_n = 1$. By taking the limit in the following relationship

$$x_n = \max\{\frac{A}{y_{n-m}}, \frac{1}{y_{n-r}^\alpha}\} \tag{2.19}$$

as $n \to \infty$, it follows

$$\lim_{n\to\infty} x_n = \max\{\frac{A}{1}, \frac{1}{1}\} = 1. \tag{2.20}$$

In same fashion, we also show that if there exists a sequence $0 < k_1 < k_2 < \ldots$ such that $y_{k_n} \ge 1$ for any $n \in \mathbf{N}$ and there exists a $N \in \mathbf{N}$ such that $x_n < 1$ for any $n \ge N$, then $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = 1$. The proof is complete. □

2. The case $A > 1$

In this section, we study the convergence of the solutions of (1.6) when $A > 1$. Let $x_n = \sqrt{A}x'_n, y_n = \sqrt{A}y'_n$ and $A' = 1/A^{\frac{1+\alpha}{2}}$. Then (1.6) reduces to the system of the following difference equations

$$x'_n = \max\{\frac{1}{y'_{n-m}}, \frac{A'}{y'^{\alpha}_{n-r}}\}, \quad y'_n = \max\{\frac{1}{x'_{n-m}}, \frac{A'}{x'^{\alpha}_{n-r}}\}, \quad n \in \mathbf{N}_0.$$

In the following, we study the system of the following difference equations

$$x_n = \max\{\frac{1}{y_{n-m}}, \frac{A}{y^{\alpha}_{n-r}}\}, \quad y_n = \max\{\frac{1}{x_{n-m}}, \frac{A}{x^{\alpha}_{n-r}}\}, \quad n \in \mathbf{N}_0, \tag{3.1}$$

where $A \in (0,1)$. Let $\{(x_n, y_n)\}_{n \geq -d}$ be a solution of (3.1) with the initial values $x_{-d}, x_{-d+1}, \ldots, x_{-1}, y_{-d}, y_{-d+1}, \ldots, y_{-1} \in \mathbf{R}^+$. Then we have from (3.1) that for any $n \in \mathbf{N_0}$,

$$x_n y_{n-m} \geq 1, \quad y_n x_{n-m} \geq 1. \tag{3.2}$$

By (3.1) and (3.2) we obtain the following statements:

(S_1) For any $n \geq d$,

$$x_n \leq \max\{x_{n-2m}, Ax^{\alpha}_{n-r-m}\}, \quad y_n \leq \max\{y_{n-2m}, Ay^{\alpha}_{n-r-m}\}. \tag{3.3}$$

(S_2) If $x_n = 1/y_{n-m}$ (resp. $y_n = 1/x_{n-m}$) for some $n \geq m$, then

$$x_n = \frac{x_{n-2m}}{y_{n-m}x_{n-2m}} \leq x_{n-2m} \text{ (resp. } y_n \leq y_{n-2m}). \tag{3.4}$$

(S_3) If $x_n = A/y^{\alpha}_{n-r} > 1/y_{n-m}$ (resp. $y_n = A/x^{\alpha}_{n-r} > 1/x_{n-m}$) for some $n \geq d$, then

$$\begin{aligned} x_{n-2m} &< x_{n-2m}x_n y_{n-m} \\ &= \max\{x_n, \tfrac{x_n x_{n-2m} y^{\alpha}_{n-r} A}{x^{\alpha}_{n-r-m} y^{\alpha}_{n-r}}\} \\ &\leq \max\{x_n, x_{n-2m}A^2\} \\ &= x_n \\ \text{(resp. } y_{n-2m} &\leq y_n). \end{aligned} \tag{3.5}$$

Lemma 3.1. *If there exists $M \in \mathbf{N}$ such that $\{y_{M+2mn}\}_{n \geq 0}$ (respectively $\{x_{M+2mn}\}_{n \geq 0}$) is monotone, then $\{x_{M+2mn+r}\}_{n \geq 0}$ (respectively $\{y_{M+2mn+r}\}_{n \geq 0}$) is eventually monotone.*

Proof. If there exists $K \in \mathbf{N}$ such that $x_{M+2mn+r} = 1/y_{M+2mn-m+r}$ for all $n \geq K$, then by (3.4) we know that $x_{M+2mn+r} \leq x_{M+2m(n-1)+r}$ for all $n \geq K$. Thus $\{x_{M+2mn+r}\}_{n \geq K}$ is decreasing.

If there exists $K \in \mathbf{N}$ such that $x_{M+2mn+r} > 1/y_{M+2mn-m+r}$ for all $n \geq K$, then by (3.5) we know that $x_{M+2mn+r} > x_{M+2m(n-1)+r}$ for all $n \geq K$. Thus $\{x_{M+2mn+r}\}_{n \geq K}$ is increasing.

In the following, we assume that there exists a sequence $1 < s_1 < t_1 < s_2 < t_2 < \ldots < s_n < t_n < \ldots$ such that

$$x_{M+2mk+r} = \frac{A}{y^{\alpha}_{M+2mk}} > \frac{1}{y_{M+2mk+r-m}}, \quad \text{for every} \quad s_n \leq k < t_n \tag{3.6}$$

and

$$x_{M+2mk+r} = \frac{1}{y_{M+2mk+r-m}}, \quad \text{for every} \quad t_n \leq k < s_{n+1}. \tag{3.7}$$

By (3.5) we see that for any $n \in \mathbf{N}$,

$$x_{M+2ms_{n+1}+r} = \frac{A}{y^{\alpha}_{M+2ms_{n+1}}} > x_{M+2m(s_{n+1}-1)+r} \geq \frac{A}{y^{\alpha}_{M+2m(s_{n+1}-1)}}. \tag{3.8}$$

Then $y_{M+2ms_{n+1}} < y_{M+2m(s_{n+1}-1)}$ and $\{y_{M+2mn}\}_{n\geq 0}$ is decreasing.

For every $s_n \leq k < t_n$, by (3.5) we have $x_{M+2m(k-1)+r} < x_{M+2mk+r}$. For every $t_n \leq k < s_{n+1}$, it follows from $A^2 < 1 \leq y^{\alpha}_{M+2mk}x^{\alpha}_{M+2mk-m}$ and (3.4) that

$$\begin{aligned} \frac{A}{y^{\alpha}_{M+2mk}} &\geq \frac{A}{y^{\alpha}_{M+2m(t_n-1)}} = x_{M+2m(t_n-1)+r} \\ &\geq x_{M+2mk+r} = \frac{1}{y_{M+2mk+r-m}} \\ &= \min\{x_{M+2m(k-1)+r}, \frac{x^{\alpha}_{M+2mk-m}}{A}\} \\ &= x_{M+2m(k-1)+r} \geq x_{M+2mk+r}, \end{aligned} \tag{3.9}$$

which implies $x_{M+2m(k-1)+r} = x_{M+2mk+r}$. Thus $\{x_{M+2mn+r}\}_{n\geq 0}$ is eventually increasing. The other case is treated similarly, so we omit the detail. The proof is complete.

For all $n \geq d$, write

$$X_n = \max\{x_n, x_{n-1}, \ldots, x_{n-2d}, 1\}, \quad Y_n = \max\{y_n, y_{n-1}, \ldots, y_{n-2d}, 1\}. \tag{3.10}$$

Lemma 3.2. *$x_{n+1} \leq X_n$ and $y_{n+1} \leq Y_n$ for all $n \geq d$ and X_n, Y_n are decreasing $(n \geq d)$.*

Proof. By (3.3) we obtain that for any $n \geq d$,

$$\begin{aligned} x_{n+1} &\leq \max\{x_{n-2m+1}, x_{n-r+1-m}, 1\} \leq X_n, \\ y_{n+1} &\leq \max\{y_{n-2m+1}, y_{n-r+1-m}, 1\} \leq Y_n. \end{aligned} \tag{3.11}$$

Thus

$$\begin{aligned} X_{n+1} &= \max\{x_{n+1}, \ldots, x_{n+1-2d}, 1\} \leq \max\{x_{n+1}, X_n\} = X_n, \\ Y_{n+1} &= \max\{y_{n+1}, \ldots, y_{n+1-2d}, 1\} \leq \max\{y_{n+1}, Y_n\} = Y_n. \end{aligned} \tag{3.12}$$

The proof is complete.

By Lemma 3.2 we write

$$\lim_{n\to\infty} X_n = X, \quad \lim_{n\to\infty} Y_n = Y. \tag{3.13}$$

Then

$$X \geq 1, \quad Y \geq 1. \tag{3.14}$$

Remark 3.3. Note that from (3.2) we see that there exists a sequence $0 < k_1 < k_2 < \dots$ such that $x_{k_n} \geq 1$ for any $n \in \mathbf{N}$ or $y_{k_n} \geq 1$ for any $n \in \mathbf{N}$.

Lemma 3.4. (1) *If there exists a sequence* $0 < k_1 < k_2 < \dots$ *such that* $x_{k_n} \geq 1$ *for any* $n \in \mathbf{N}$*, then:*

(i) $\limsup_{n\to\infty} x_n = X$.

(ii) *There exists a* $M \in \mathbf{N}$ *such that* x_{M+2nm} *is decreasing with* $\lim_{n\to\infty} x_{M+2nm} = X$ *and* $x_{M+2nm} = 1/y_{M+2nm-m}$.

(2) *If there exists a sequence* $0 < k_1 < k_2 < \dots$ *such that* $y_{k_n} \geq 1$ *for any* $n \in \mathbf{N}$*, then*

(i) $\limsup_{n\to\infty} y_n = Y$.

(ii) *There exists a* $M \in \mathbf{N}$ *such that* y_{M+2nm} *is decreasing with* $\lim_{n\to\infty} y_{M+2nm} = Y$ *and* $y_{M+2nm} = 1/x_{M+2nm-m}$.

Proof. If there exists a sequence $0 < k_1 < k_2 < \dots$ such that $x_{k_n} \geq 1$ for any $n \in \mathbf{N}$, then there exists a subsequence $x_{r_1}, x_{r_2}, \dots, x_{r_n}, \dots$ with $x_{r_n} = X_{k_n} \geq x_{k_n} \geq 1$ for any $n \in \mathbf{N}$. Thus we see that $x_{r_n} = X_{k_n} \geq X$ and

$$\limsup_{n\to\infty} x_n \geq X \geq 1. \tag{3.15}$$

On the other hand, by $x_{n+1} \leq X_n$ for all $n \geq d$ we obtain

$$\limsup_{n\to\infty} x_n \leq \limsup_{n\to\infty} X_n = X. \tag{3.16}$$

Thus $\limsup_{n\to\infty} x_n = X$.

By $\limsup_{n\to\infty} x_n = X$ we see that there exists a $N \in \mathbf{N}$ such that $x_n < X/A < (X/A)^{\frac{1}{\alpha}}$ for any $n \geq N$. Then $Ax_n^{\alpha} < X$ for any $n \geq N$. We can assume that there exist a sequence $N + 2d \leq r_1 < r_2 < \dots < r_n < \dots$ and some $0 \leq c < 2m$ such that $x_{2r_n m+c} \geq X$ and

$$\lim_{n\to\infty} x_{2k_n m+c} = X. \tag{3.17}$$

From (3.3) we have

$$\begin{aligned} X &\le x_{2mr_n+c} \le x_{2mr_n-2m+c} \le \dots \\ &\le x_{2mr_{n-1}+c} \le x_{2mr_{n-1}-2m+c} \le \dots \\ &\dots \\ &\le x_{2mr_1+c}. \end{aligned} \tag{3.18}$$

Let $M = 2mr_1 + c$. Then x_{M+2nm} is decreasing and $\lim_{n\to\infty} x_{M+2nm} = X$.

By (3.2) we obtain

$$\frac{A}{y^{\alpha}_{M+2mn-r}} \le Ax^{\alpha}_{M+2mn-m-r} < X \le x_{M+2nm}. \tag{3.19}$$

Then $x_{M+2nm} = 1/y_{M+2nm-m}$. The other case is treated similarly, so we omit the detail. The proof is complete.

If m and r are odd and (1) of Lemma 3.4 holds, then for all $n \ge d$, write

$$\begin{aligned} X'_n &= \max\{x_{M+2n-1}, x_{M+2n-3}, \dots, x_{M+2n-2d-1}, 1\}, \\ Y'_n &= \max\{y_{M+2n}, y_{M+2n-2}, \dots, y_{M+2n-2d}, 1\}. \end{aligned} \tag{3.20}$$

If m and r are odd and (2) of Lemma 3.4 holds, then for all $n \ge d$, write

$$\begin{aligned} X'_n &= \max\{x_{M+2n}, x_{M+2n-2}, \dots, x_{M+2n-2d}, 1\}, \\ Y'_n &= \max\{y_{M+2n-1}, y_{M+2n-3}, \dots, y_{M+2n-2d-1}, 1\}. \end{aligned} \tag{3.21}$$

where M is as in Lemma 3.4.

Lemma 3.5. *If* (1) *of Lemma* 3.4 *holds, then* $x_{M+2n+1} \le X'_n$ *and* $y_{M+2n+2} \le Y'_n$ *for all* $n \ge d$ *and* X'_n, Y'_n *are decreasing* ($n \ge d$). *If* (2) *of Lemma* 3.4 *holds, then* $x_{M+2n+2} \le X'_n$ *and* $y_{M+2n+1} \le Y'_n$ *for all* $n \ge d$ *and* X'_n, Y'_n *are decreasing* ($n \ge d$).

Proof. If (1) of Lemma 3.4 holds, then by (3.3) we obtain that for any $n \ge d$,

$$\begin{aligned} x_{M+2n+1} &\le \max\{x_{M+2n-2m+1}, x_{M+2n-r+1-m}, 1\} \le X'_n, \\ y_{M+2n+2} &\le \max\{y_{M+2n-2m+2}, y_{M+2n-r+2-m}, 1\} \le Y'_n. \end{aligned} \tag{3.22}$$

Thus

$$\begin{aligned} X'_{n+1} &= \max\{x_{M+2n+1}, \dots, x_{M+2n+1-2d}, 1\} \le X'_n, \\ Y'_{n+1} &= \max\{y_{M+2n+2}, \dots, y_{M+2n+2-2d}, 1\} \le Y'_n. \end{aligned} \tag{3.23}$$

The other case is treated similarly, so we omit the detail. The proof is complete.

Let

$$\lim_{n\to\infty} X'_n = X', \quad \lim_{n\to\infty} Y'_n = Y'. \tag{3.24}$$

Remark 3.6. Note that from (3.2) we see that there exists a sequence $0 < k_1 < k_2 < \dots$ such that $x_{M+2k_n-1} \ge 1$ for any $n \in \mathbf{N}$ or $y_{M+2k_n} \ge 1$ for any $n \in \mathbf{N}$ or $x_{M+2k_n} \ge 1$ for any $n \in \mathbf{N}$ or $y_{M+2k_n-1} \ge 1$ for any $n \in \mathbf{N}$.

Using arguments similar to ones developed in the proof of Lemma 2.4, we can show the following Lemma 3.7.

Lemma 3.7. (1) *Assume that* (1) *of Lemma* 3.4 *holds. If there exists a sequence* $0 < k_1 < k_2 < \dots$ *such that* $x_{M+2k_n-1} \geq 1$ *for any* $n \in \mathbf{N}$*, then*

(i) $\limsup_{n\to\infty} x_{M+2n-1} = X'$.

(ii) *There exists a* $N \in \mathbf{N}$ *such that* $x_{M+2N+2nm-1}$ *is decreasing with* $\lim_{n\to\infty} x_{M+2N+2nm-1} = X'$ *and* $x_{M+2N+2nm-1} = 1/y_{M+2N+2nm-1-m}$.

If there exists a sequence $0 < k_1 < k_2 < \dots$ *such that* $y_{M+2k_n} \geq 1$ *for any* $n \in \mathbf{N}$*, then*

(iii) $\limsup_{n\to\infty} y_{M+2n} = Y'$.

(iv) *There exists a* $N \in \mathbf{N}$ *such that* $y_{M+2N+2nm}$ *is decreasing with* $\lim_{n\to\infty} y_{M+2N+2nm} = Y'$ *and* $y_{M+2N+2nm} = 1/x_{M+2N+2nm-m}$.

(2) *Assume that (2) of Lemma 3.4 holds. If there exists a sequence* $0 < k_1 < k_2 < \dots$ *such that* $x_{M+2k_n} \geq 1$ *for any* $n \in \mathbf{N}$*, then*

(i) $\limsup_{n\to\infty} x_{M+2n} = X'$.

(ii) *There exists a* $N \in \mathbf{N}$ *such that* $x_{M+2N+2nm}$ *is decreasing with* $\lim_{n\to\infty} x_{M+2N+2nm} = X'$ *and* $x_{M+2N+2nm} = 1/y_{M+2N+2nm-m}$.

If there exists a sequence $0 < k_1 < k_2 < \dots$ *such that* $y_{M+2k_n-1} \geq 1$ *for any* $n \in \mathbf{N}$*, then*

(iii) $\limsup_{n\to\infty} y_{M+2n-1} = Y'$.

(iv) *There exists a* $N \in \mathbf{N}$ *such that* $y_{M+2N+2nm-1}$ *is decreasing with* $\lim_{n\to\infty} y_{M+2N+2nm-1} = Y'$ *and* $y_{M+2N+2nm-1} = 1/x_{M+2N+2nm-1-m}$.

Theorem 3.8. *For any* $0 \leq k \leq 2m-1$*,* $\{x_{2nm+k}\}_{n\geq 0}$ *and* $\{y_{2nm+k}\}_{n\geq 0}$ *are eventually monotone.*

Proof. First we suppose that $\gcd(m,r) = 1$.

If m is even and r is odd, or m is odd and r is even, then by Lemma 3.4 there are two cases to consider.

Case 1. There exists a $M \in \mathbf{N}$ such that x_{M+2nm} is decreasing and $x_{M+2nm} = 1/y_{M+2nm-m}$. Using Lemma 3.1 repeatedly, it follows that for every $0 \leq k \leq m-1$, $\{x_{M+2nm+2kr}\}_{n\geq 0}$, $\{x_{M+2nm+(2k+1)r-m}\}_{n\geq 0}$, $\{y_{M+2nm+(2k+1)r}\}_{n\geq 0}$ and $\{y_{M+2nm+2kr-m}\}_{n\geq 0}$ are eventually monotone.

Case 2. There exists a $M \in \mathbf{N}$ such that y_{M+2nm} is decreasing and $y_{M+2nm} = 1/x_{M+2nm-m}$. Using Lemma 3.1 repeatedly, it follows that for every $0 \leq k \leq$

$m-1$, $\{y_{M+2nm+2kr}\}_{n\ge 0}$, $\{y_{M+2nm+(2k+1)r-m}\}_{n\ge 0}$, $\{x_{M+2nm+(2k+1)r}\}_{n\ge 0}$ and $\{x_{M+2nm+2kr-m}\}_{n\ge 0}$ are eventually monotone.

Since $\gcd(m,r)=1$, it follows that

$$\{2kr : 0\le k\le m-1\} = \{0,2,\dots,2m-2\} \pmod{2m}, \tag{3.25}$$

which implies

$$\{2kr+r-m : 0\le k\le m-1\} = \{1,3,\dots,2m-1\} \pmod{2m} \tag{3.26}$$

and

$$\begin{aligned}&\{2kr+r : 0\le k\le m-1\}\cup\{2kr-m : 0\le k\le m-1\}\\ &= \{0,1,2,\dots,2m-1\} \pmod{2m}.\end{aligned} \tag{3.27}$$

Thus $\{x_{M+2nm+k}\}_{n\ge 0}$ and $\{y_{M+2nm+k}\}_{n\ge 0}$ are eventually monotone for every $k\in\{0,1,2,\dots,2m-1\}$. In the following, we assume that m and r are odd. Then by Lemma 3.7 there are four cases to consider.

Case 1. There exist $M,N\in\mathbf{N}$ such that x_{M+2nm} and $x_{M+2N+2nm-1}$ are decreasing. Using Lemma 3.1 repeatedly, it follows that for every $0\le k\le m-1$, $\{x_{M+2nm+2kr}\}_{n\ge 0}$, $\{x_{M+2N+2nm+2kr-1}\}_{n\ge 0}$, $\{y_{M+2nm+(2k+1)r}\}_{n\ge 0}$ and $\{y_{M+2N+2nm+(2k+1)r-1}\}_{n\ge 0}$ are eventually monotone.

Case 2. There exist $M,N\in\mathbf{N}$ such that x_{M+2nm} and $y_{M+2N+2nm}$ are decreasing. Using Lemma 3.1 repeatedly, it follows that for every $0\le k\le m-1$, $\{x_{M+2nm+2kr}\}_{n\ge 0}$, $\{x_{M+2N+2nm+(2k+1)r}\}_{n\ge 0}$, $\{y_{M+2nm+(2k+1)r}\}_{n\ge 0}$ and $\{y_{M+2N+2nm+2kr}\}_{n\ge 0}$ are eventually monotone.

Case 3. There exist $M,N\in\mathbf{N}$ such that y_{M+2nm} and $y_{M+2N+2nm-1}$ are decreasing. Using Lemma 3.1 repeatedly, it follows that for every $0\le k\le m-1$, $\{y_{M+2nm+2kr}\}_{n\ge 0}$, $\{y_{M+2N+2nm+2kr-1}\}_{n\ge 0}$, $\{x_{M+2nm+(2k+1)r}\}_{n\ge 0}$ and $\{x_{M+2N+2nm+(2k+1)r-1}\}_{n\ge 0}$ are eventually monotone.

Case 4. There exist $M,N\in\mathbf{N}$ such that y_{M+2nm} and $x_{M+2N+2nm}$ are decreasing. Using Lemma 3.1 repeatedly, it follows that for every $0\le k\le m-1$, $\{y_{M+2nm+2kr}\}_{n\ge 0}$, $\{y_{M+2N+2nm+(2k+1)r}\}_{n\ge 0}$, $\{x_{M+2nm+(2k+1)r}\}_{n\ge 0}$ and $\{x_{M+2N+2nm+2kr}\}_{n\ge 0}$ are eventually monotone.

Since $\gcd(m,r)=1$, it follows that

$$\begin{aligned}&\{2kr : 0\le k\le m-1\}\cup\{2N+2kr-1 : 0\le k\le m-1\}\\ &= \{(2k+1)r : 0\le k\le m-1\}\cup\{2N+(2k+1)r-1 : 0\le k\le m-1\}\\ &= \{2kr : 0\le k\le m-1\}\cup\{2N+(2k+1)r : 0\le k\le m-1\}\\ &= \{(2k+1)r : 0\le k\le m-1\}\cup\{2N+2kr : 0\le k\le m-1\}\\ &= \{0,1,2,\dots,2m-1\} \bmod 2m.\end{aligned} \tag{3.28}$$

Thus $\{x_{M+2nm+k}\}_{n\ge 0}$ and $\{y_{M+2nm+k}\}_{n\ge 0}$ are eventually monotone for every $k\in\{0,1,2,\dots,2m-1\}$.

If $\gcd(m,r)=d>1$, then we write $m=dm_1$ and $r=dr_1$ with $\gcd(m_1,r_1)=1$. Consider the system of difference equations

$$x_n=\max\{\frac{1}{y_{n-dm_1}},\frac{A}{y^{\alpha}_{n-dr_1}}\},\quad y_n=\max\{\frac{1}{x_{n-dm_1}},\frac{A}{x^{\alpha}_{n-dr_1}}\},\quad n\in\mathbf{N}_0. \tag{3.29}$$

Write $x_{n,i}=x_{nd+i}$ and $y_{n,i}=y_{nd+i}$ for every $0\le i\le d-1$ and $n\in\mathbf{N}_0$. Then (3.29) reduces to the equations

$$x_{n,i}=\max\{\frac{1}{y_{n-m_1,i}},\frac{A}{y^{\alpha}_{n-r_1,i}}\},\quad y_{n,i}=\max\{\frac{1}{x_{n-m_1,i}},\frac{A}{x^{\alpha}_{n-r_1,i}}\},$$
$$0\le i\le d-1,\ n\in\mathbf{N}_0. \tag{3.29,i}$$

By an analogous way as in the above, we obtain that if $\{x_{n,i},y_{n,i}\}_{n\ge0}$ is a positive solution of $(3.29,i)$ for every $0\le i\le d-1$, then $\{x_{2m_1n+k,i}\}_{n\ge0}$ and $\{y_{2m_1n+k,i}\}_{n\ge0}$ are eventually monotone for every $0\le k\le 2m_1-1$. Thus for every $0\le k\le 2m-1$, $\{x_{2mn+k}\}_{n\ge0}$ and $\{y_{2mn+k}\}_{n\ge0}$ are eventually monotone. The proof is complete.

Remark 3.9. It follows from Theorem 3.8 that if $A>1$ and $\{(x_n,y_n)\}_{n\ge-d}$ is a positive solution of (1.6), then for any $0\le k\le 2m-1$, $\{x_{2nm+k}\}_{n\ge0}$ and $\{y_{2nm+k}\}_{n\ge0}$ are eventually monotone.

Remark 3.10. In [18], we showed that if $\alpha,A\in(0,1)$ and $k\in\mathbf{N}$, then the following equation

$$x_n=\max\{\frac{1}{x_{n-1}},\frac{A}{x_{n-2k-1}}\} \tag{3.30}$$

has a positive solution $\{z_n\}_{n\ge-2k-1}$ satisfying the following conditions:

(1) $z_{2n+1}=Az^{\alpha}_{2n-2k-1}$ for any $n\in\mathbf{N}$.

(2) $z_{2n+2}=1/z_{2n+1}$ for any $n\in\mathbf{N}$.

(3) $z_{2n+2}<z_{2n}$ for any $n\in\mathbf{N}$.

Let $x_n=y_n=z_n$ for any $n\ge-2k-1$. Then $\{(x_n,y_n)\}_{n\ge-2k-1}$ is a solution of the following equation

$$x_n=\max\{\frac{1}{y_{n-1}},\frac{A}{y^{\alpha}_{n-2k-1}}\},\qquad y_n=\max\{\frac{1}{x_{n-1}},\frac{A}{x^{\alpha}_{n-2k-1}}\},\quad n\in\mathbf{N}_0. \tag{3.31}$$

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References

[1] E. M. Elsayed, B. D. Iričanin, and S. Stević, *On the max-type equation* $x_{n+1} = \max\{A_n/x_n, x_{n-1}\}$, Ars Combinatoria, 95 (2010), 187-192.

[2] T. Sauer, *Global convergence of max-type equations*, Journal of Difference Equations and Applications, 17 (2011), 1-8.

[3] M. Sharif and A. Jawad, *Interacting generalized dark energy and reconstruction of scalar field models*, Modern Physics Letters A, 28 (2013), 38, Article ID 1350180.

[4] S. Stević, *On a generalized max-type difference equation from automatic control theory,* Nonlinear Analysis. Theory, Methods and Applications. An International Multidisciplinary Journal, 72 (2010), 3-4, 1841-1849.

[5] S. Stević, *Periodicity of max difference equations*, Utilitas Mathematica, 83 (2010), 69-71.

[6] S. Stević, *Product-type system of difference equations of second-order solvable in closed form*, Electronic Journal of Qualitative Theory of Differential Equations, 1-16, 2015.

[7] S. Stević, M. A. Alghamdi, A. Alotaibi, and N. Shahzad, *Boundedness character of a max-type system of difference equations of second order*, Electronic Journal of Qualitative Theory of Differential Equations, 45 (2014), 1-12.

[8] G. Su, T. Sun, and B. Qin, *Eventually periodic solutions of a max-type system of difference equations of higher order*, Discrete Dynamics in Nature and Society, Article ID 8467682, 2018.

[9] T. Sun, J. Liu, Q. He, X. Liu, and C. Tao, *Eventually periodic solutions of a max-type difference equation*, The Scientific World Journal, Article ID 219437, 2014.

[10] T. Sun, B. Qin, H. Xi, and C. Han, *Global behavior of the max-type difference equation* $x_{n+1} = \max\{1/x_n, A_n/x_{n-1}\}$, Abstract and Applied Analysis, Article ID 152964, 2009.

[11] T. Sun, H. Xi, C. Han, and B. Qin, *Dynamics of the max- type difference equation* $x_{n+1} = \max\{1/x_{n-m}, A_n/x_{n-r}\}$, Journal of Applied Mathematics and Computing, 38 (2012), 173-180.

[12] Q. Xiao and Q. Shi, *Eventually periodic solutions of a max- type equation*, Mathematical and Computer Modelling, 57 (2013), 3-4, 992-996.

[13] A. Gelişken and C. Çinar, *On the global attractivity of a max-type difference equation*, Discrete Dynamics in Nature and Society, Article ID 812674, 2009.

[14] T. Sun, H. Xi, and B. Qin, *Global behavior of the max-type difference equation* $x_{n+1} = \max\{A/x_{n-m}, 1/x_{n-k}^{\alpha}\}$, Journal of Concrete and Applicable Mathematics, 10 (2012), 1-2, 32-39.

[15] F. Sun, *On the asymptotic behavior of a difference equation with maximum*, Discrete Dynamics in Nature and Society, Article ID 243291, 2008.

[16] S. Stević, *Global stability of a difference equation with maximum*, Applied Mathematics and Computation, 210 (2009), 525-529.

[17] B. Qin, T. Sun, and H. Xi, *Global behavior of the max-type difference equation* $x_n = \max\{A_1/x_{n-m_1}^{\alpha_1}, A_2/x_{n-m_2}^{\alpha_2}, \ldots, A_k/x_{n-m_k}^{\alpha_k}\}$, International Journal of Mathematical Analysis, 5 (2011), 1859-1865.

[18] T. Sun and G. Su, *Dynamics of a difference equation with maximum*, Journal of Computational Analysis and Applications, 23 (2017), 401-407.

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