Properties of γ-P_S-R_0 and γ-P_S-R_1 spaces

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Abstract. This paper introduces some more γ-P_S- separation axioms called γ-P_S-R_0 and γ-P_S-R_1 by using γ-P_S-open sets and τ_γ-P_S-closure of a set. Some properties of these spaces are constructed.

Keywords: γ-P_S-open set, γ-P_S-R_0 space, γ-P_S-R_1 space.

1. Introduction

Kasahara [12] introduced the notion of an α operation approaches on a class τ of sets and studied the concept of α-continuous functions with α-closed graphs and α-compact spaces. After this, Jankovic [11] introduced the concept of α-closure of a set in X via α-operation and investigated further characterizations of

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function with α-closed graph. Later, Ogata [15] defined and studied the concept of γ-open sets, and applied it to investigate operation-functions and operation-separation axioms. Since that, γ operation on τ has attracted the attention of many researchers. Among them are γ-preopen sets [13] and γ-semiopen sets [14].

Recently, the notion of γ-P_S-open sets had been defined by Asaad, Ahmad and Omar [4]. This set is stronger than γ-preopen set. Besides that, they also introduced γ-locally indiscrete and γ-hyperconnected spaces [6]. In addition, Asaad, Ahmad and Omar ([3], [7]) introduced and studied the notion of γ-P_S-T_i spaces for $i = 0, \frac{1}{2}, 1, 2$. Later, in [1] they studied γ-P_S-regular and γ-P_S-normal spaces and investigated some of their characterizations.

The aim of this paper is to introduce some more γ-P_S- separation axioms called γ-P_S-R_0 and γ-P_S-R_1 by using γ-P_S-open sets and τ_γ-P_S-closure of a set. Some properties of these spaces are constructed. In this paper, the pairs (X, τ) and (Y, σ) (or simply X and Y) always mean topological spaces on which no separation axioms assumed unless otherwise mentioned.

2. Preliminaries and basic definitions

An operation γ on the topology τ on X is a mapping $\gamma\colon \tau \to P(X)$ such that $U \subseteq \gamma(U)$ for each $U \in \tau$, where $P(X)$ is the power set of X and $\gamma(U)$ denotes the value of γ at U [15]. A nonempty subset A of a topological space (X, τ) with an operation γ on τ is said to be γ-open [15] if for each $x \in A$, there exists an open set U containing x such that $\gamma(U) \subseteq A$. The complement of a γ-open set is called γ-closed. The class of all γ-open sets of X is denoted by τ_γ. The τ_γ-closure of a subset A of X with an operation γ on τ is defined as the intersection of all γ-closed sets containing A and it is denoted by τ_γ-$Cl(A)$ [15], and the τ_γ-interior of a subset A of X with an operation γ on τ is defined as the union of all γ-open sets containing A [14] and it is denoted by τ_γ-$Int(A)$ [14]. An operation γ on τ is said to be regular if for every open neighborhood U and V of each $x \in X$, there exists an open neighborhood W of x such that $\gamma(W) \subseteq \gamma(U) \cap \gamma(V)$ [15].

A subset A of a topological space (X, τ) with an operation γ on τ is said to be γ-preopen [13] (respectively, γ-semiopen [14]) if $A \subseteq \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(A))$ (respectively, $A \subseteq \tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(A))$). The complement of a γ-semiopen [14] set is called γ-semiclosed. A γ-preopen subset A of a topological space (X, τ) is called γ-P_S-open [4] if for each $x \in A$, there exists a γ-semiclosed set F such that $x \in F \subseteq A$. The complement of a γ-P_S-open set is called γ-P_S-closed. The τ_γ-P_S-closure [4] of a subset A of a space (X, τ) is defined as the intersection of all γ-P_S-closed sets of X containing A and it is denoted by τ_γ-$P_SCl(A)$. The τ_γ-P_S-interior [4] of a subset A of a space (X, τ) is defined as the union of all γ-P_S-open sets of X contained in A and it is denoted by τ_γ-$P_SInt(A)$. A subset A of X is γ-P_S-closed if and only if $\tau_\gamma\text{-}P_SCl(A) = A$ [4]. A point $x \in \tau_\gamma\text{-}P_SCl(A)$ if and only if $A \cap U \neq \emptyset$ for every γ-P_S-open set U of X containing x [4]. A subset A of a topological space (X, τ) with an operation γ on τ is said to be γ-P_S-generalized closed (shortly γ-P_S-g-closed) [3] if τ_γ-$P_SCl(A) \subseteq G$ whenever $A \subseteq G$ and G is γ-P_S-open set in X. For any topological space (X, τ) and any operation γ on τ, we denote the class of all γ-P_S-open (respectively, γ-P_S-closed) sets of X by τ_γ-$P_SO(X)$ or τ_γ-$P_SO(X, \tau)$ (respectively, τ_γ-$P_SC(X)$ or τ_γ-$P_SC(X, \tau)$).

A space (X, τ) with an operation γ on τ is said to be γ-P_S-T_0 [7] if for each pair of distinct points x, y in X, there exists a γ-P_S-open set G such that either $x \in G$ and $y \notin G$ or $y \in G$ and $x \notin G$. A space (X, τ) with an operation γ on τ is said to be γ-P_S-T_1 [7] (respectively, γ-semiT_1 [14]) if for each pair of distinct points x, y in X, there exist two γ-P_S-open (respectively, γ-semiopen) sets G and H such that $x \in G$ but $y \notin G$ and $y \in H$ but $x \notin H$. A space (X, τ) with an operation γ on τ is said to be γ-P_S-T_2 [7] if for each pair of distinct points x, y in X, there exist two γ-P_S-open sets G and H containing x and y respectively such that $G \cap H = \emptyset$. A space (X, τ) with an operation γ on τ is said to be γ-P_S-$T_{\frac{1}{2}}$ [3] if every γ-P_S-g-closed set in X is γ-P_S-closed.

3. Properties of γ-P_S-R_0 spaces

In this section, another type of γ-P_S- separation axioms which is γ-P_S-R_0 spaces will be presented. Some of its properties will be discussed.

The definition of γ-P_S-R_0 space in terms of γ-P_S-open and $\tau_\gamma P_S$-closure operator is defined as follows:

Definition 3.1. A topological space (X, τ) with an operation γ on τ is said to be γ-P_S-R_0 if every γ-P_S-open set contains the τ_γ-P_S-closure of each of its singletons.

In other words, a space (X, τ) is said to be γ-P_S-R_0 if G is a γ-P_S-open set and $x \in G$, then τ_γ-$P_SCl(\{x\}) \subseteq G$.

Some important properties of γ-P_S-R_0 spaces and relations between γ-P_S-R_0 space and other types of γ-P_S- separation axioms are established.

Theorem 3.2. *For any topological space (X, τ) and any operation γ on τ. If τ_γ-$P_SCl(\{x\}) \subseteq F$ whenever F is γ-semiclosed set in X and $x \in F$. Then (X, τ) is γ-P_S-R_0 space.*

Proof. Let G be any γ-P_S-open set in X such that $x \in G$. Then there exists a γ-semiclosed set F in X such that $x \in F \subseteq G$. So by hypothesis, τ_γ-$P_SCl(\{x\}) \subseteq F$ and hence τ_γ-$P_SCl(\{x\}) \subseteq G$. This means that (X, τ) is γ-P_S-R_0. □

Remark 3.3. Let $A \subseteq (X, \tau)$, τ_γ-$Cl(\tau_\gamma$-$Int(A)) \subseteq \tau_\gamma$-$P_SCl(A)$.

Theorem 3.4. *If a topological space (X, τ) is γ-P_S-R_0, then τ_γ-$Cl(\tau_\gamma$-$Int(\{x\}))$ $\subseteq G$ whenever G is γ-P_S-open set in X and $x \in G$.*

Proof. Straightforward from Definition 3.1 and Remark 3.3. □

Theorem 3.5. *Let (X, τ) be a topological space and γ be an operation on τ. Then the following conditions are equivalent:*

(i) *X is γ-P_S-R_0.*

(ii) *If for each γ-P_S-closed set E in X such that $x \notin E$, then there exists a γ-P_S-open set G such that $E \subseteq G$ and $x \notin G$.*

(iii) *If for each γ-P_S-closed set E in X such that $x \notin E$, then τ_γ-$P_SCl(\{x\}) \cap E = \emptyset$.*

(iv) *If for each distinct points $x, y \in X$, then either τ_γ-$P_SCl(\{x\}) \cap \tau_\gamma$-$P_SCl(\{y\}) = \emptyset$ or τ_γ-$P_SCl(\{x\}) = \tau_\gamma$-$P_SCl(\{y\})$.*

Proof. (i) $\Rightarrow$ (ii) Let E be any γ-P_S-closed set in X and $x \notin E$. Since (X, τ) is γ-P_S-R_0, then τ_γ-$P_SCl(\{x\}) \subseteq X \backslash E$. Let $G = X \backslash \tau_\gamma$-$P_SCl(\{x\})$. Then G is γ-P_S-open set in X such that $E \subseteq G$ and $x \notin G$.

(ii) $\Rightarrow$ (iii) Let E be any γ-P_S-closed set in X does not containing x. Then by (ii), there exists $G \in \tau_\gamma$-$P_SO(X, \tau)$ such that $E \subseteq G$ and G does not containing x. This means that $\{x\} \cap G = \emptyset$ and hence τ_γ-$P_SCl(\{x\}) \cap G = \emptyset$ since G is γ-P_S-open set. That is, τ_γ-$P_SCl(\{x\}) \cap E = \emptyset$.

(iii) $\Rightarrow$ (iv) Let τ_γ-$P_SCl(\{x\}) \neq \tau_\gamma$-$P_SCl(\{y\})$ for distinct elements x and y in X. Then there exists an element $z \in \tau_\gamma$-$P_SCl(\{x\})$ such that $z \notin \tau_\gamma$-$P_SCl(\{y\})$ (or $z \in \tau_\gamma$-$P_SCl(\{y\})$ such that $z \notin \tau_\gamma$-$P_SCl(\{x\})$). Then there exists γ-P_S-open set G of X such that $z \in G$ and $z \notin G$. So $\{x\} \cap G \neq \emptyset$ and $\{y\} \cap G = \emptyset$. This means that $x \in G$ and $x \notin \tau_\gamma$-$P_SCl(\{y\})$. Therefore, by (iii), τ_γ-$P_SCl(\{x\}) \cap \tau_\gamma$-$P_SCl(\{y\}) = \emptyset$.

(iv) $\Rightarrow$ (i) Let G be any γ-P_S-open set in X such that $x \in G$. For each point $y \in G$ and $x \neq y$. Then $x \notin \tau_\gamma$-$P_SCl(\{y\})$ which implies that τ_γ-$P_SCl(\{x\}) \neq \tau_\gamma$-$P_SCl(\{y\})$. So by hypothesis, τ_γ-$P_SCl(\{x\}) \cap \tau_\gamma$-$P_SCl(\{y\}) = \emptyset$ for each $y \in X \backslash G$. Hence τ_γ-$P_SCl(\{x\}) \cap (\bigcup_{y \in X \backslash G} \tau_\gamma$-$P_SCl(\{y\})) = \emptyset$ for each $y \in X \backslash G$. On the other hand, since G is γ-P_S-open set in X and $y \in X \backslash G$ implies τ_γ-$P_SCl(\{y\}) \subseteq X \backslash G$. That is, $X \backslash G = \bigcup_{y \in X \backslash G} \tau_\gamma$-$P_SCl(\{y\})$. Therefore, τ_γ-$P_SCl(\{x\}) \cap X \backslash G = \emptyset$ and hence τ_γ-$P_SCl(\{x\}) \subseteq G$. Then by Definition 3.1, a space X is γ-P_S-R_0. □

Corollary 3.6. *A space (X, τ) is γ-P_S-R_0 if and only if τ_γ-$P_SCl(\{x\}) \neq \tau_\gamma$-$P_SCl(\{y\})$ implies τ_γ-$P_SCl(\{x\}) \cap \tau_\gamma$-$P_SCl(\{y\}) = \emptyset$ for every x and y in X.*

Proof. Directly follows from Theorem 3.5. □

Definition 3.7. Let A be any subset of a topological space (X, τ) and γ be an operation on τ. Then the γ-P_S-kernel of A is denoted by γ-$P_Sker(A)$ and is defined as follows:

$$\gamma\text{-}P_Sker(A) = \cap\{G : A \subseteq G \text{ and } G \in \tau_\gamma\text{-}P_SO(X, \tau)\}$$

In other words, γ-$P_S ker(A)$ is the intersection of all γ-P_S-open sets of (X,τ) containing A.

The following remark follows directly from Definition 3.7.

Remark 3.8. Let $A \subseteq (X,\tau)$ and γ be an operation on τ. Then the following are true:

(i) $A \subseteq \gamma$-$P_S ker(A)$.

(ii) If A is γ-P_S-open set in X, then γ-$P_S ker(A) = A$.

The following example shows that the converse of the above remark is not true in general.

Example 3.9. Let a space $X = \{a,b,c\}$ with the topology $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a,b\}, \{b,c\}\}$. Define an operation $\gamma\colon \tau \to P(X)$ as follows:
For every $A \in \tau$

$$\gamma(A) = \begin{cases} A, & \text{if } a \in A \\ Cl(A), & \text{if } a \notin A. \end{cases}$$

Obviously, $\tau_\gamma = \{\emptyset, X, \{a\}, \{a,b\}, \{b,c\}\}$. So τ_γ-$P_S O(X) = \{\emptyset, X, \{a\}, \{a,c\}, \{b,c\}\}$. Therefore, the γ-$P_S ker(\{b\}) = \{b,c\}$ and hence γ-$P_S ker(A) \nsubseteq A$. Again, γ-$P_S ker(\{c\}) = \{c\}$, but the set $\{c\}$ is not γ-P_S-open in X.

Theorem 3.10. *Let A be a subset of a topological space (X,τ) and γ be an operation on τ. Then γ-$P_S ker(A) = \{x \in X\colon \tau_\gamma\text{-}P_S Cl(\{x\}) \cap A \neq \emptyset\}$.*

Proof. Let $x \in \gamma$-$P_S ker(A)$. Suppose that τ_γ-$P_S Cl(\{x\}) \cap A = \emptyset$. Then $x \notin X \backslash \tau_\gamma$-$P_S Cl(\{x\})$ and $X \backslash \tau_\gamma$-$P_S Cl(\{x\})$ is γ-P_S-open set in X such that $A \subseteq X \backslash \tau_\gamma$-$P_S Cl(\{x\})$. Hence $x \notin \gamma$-$P_S ker(A)$. This is contradiction of the hypothesis. So τ_γ-$P_S Cl(\{x\}) \cap A \neq \emptyset$.

Now, for the other part, let $x \in X$ such that τ_γ-$P_S Cl(\{x\}) \cap A \neq \emptyset$. Suppose that $x \notin \gamma$-$P_S ker(A)$. Then there exists a γ-P_S-open set G containing A and $x \notin G$ and hence $x \notin A$. So, let $y \in \tau_\gamma$-$P_S Cl(\{x\}) \cap A$. Then G is a γ-P_S-open set containing y but does not contain x. This means that τ_γ-$P_S Cl(\{x\}) \cap A = \emptyset$ which is contradiction of the assumption. Therefore, $x \in \gamma$-$P_S ker(A)$. □

Theorem 3.11. *Let (X,τ) be a topological space with an operation γ on τ and let $x, y \in X$. Then $y \in \gamma$-$P_S ker(\{x\})$ if and only if $x \in \tau_\gamma$-$P_S Cl(\{y\})$.*

Proof. Suppose that $y \notin \gamma$-$P_S ker(\{x\})$. Then there exists a γ-P_S-open set G containing x such that $y \notin G$. That is, $x \notin \tau_\gamma$-$P_S Cl(\{y\})$.

Conversely, the proof is similar to the above case. □

Theorem 3.12. *Let $A \subseteq (X,\tau)$ and γ be an operation on τ. Then A is γ-P_S-g-closed if and only if τ_γ-$P_S Cl(A) \subseteq \gamma$-$P_S ker(A)$.*

Proof. Suppose that A is γ-P_S-g-closed. Then τ_γ-$P_SCl(A) \subseteq G$, whenever $A \subseteq G$ and G is γ-P_S-open. Let $x \in \tau_\gamma$-$P_SCl(A)$. Suppose that $x \notin \gamma$-$P_Sker(A)$. Then there exists a γ-P_S-open set G containing A and $x \notin G$. This implies that $x \notin \tau_\gamma$-$P_SCl(A)$. Which is contradiction to the hypothesis. Hence $x \in \gamma$-$P_Sker(A)$ and so τ_γ-$P_SCl(A) \subseteq \gamma$-$P_Sker(A)$.

Conversely, let τ_γ-$P_SCl(A) \subseteq \gamma$-$P_Sker(A)$. If G is any γ-P_S-open set containing A. Then γ-$P_Sker(A) \subseteq G$. That is τ_γ-$P_SCl(A) \subseteq \gamma$-$P_Sker(A) \subseteq G$. Therefore, A is γ-P_S-g-closed set in X. □

Definition 3.13 ([7])**.** A topological space (X, τ) with an operation γ on τ, is said to be γ-P_S-symmetric if for each x, y in X, then $x \in \tau_\gamma$-$P_SCl(\{y\})$ implies that $y \in \tau_\gamma$-$P_SCl(\{x\})$.

Theorem 3.14 ([7])**.** *Let (X, τ) be a topological space and γ be an operation on τ. Then X is γ-P_S-symmetric if and only if the singleton set $\{x\}$ is γ-P_S-g-closed, for each $x \in X$.*

Corollary 3.15. *Let (X, τ) be a topological space and γ be an operation on τ. Then X is γ-P_S-symmetric if and only if τ_γ-$P_SCl(\{x\}) \subseteq \gamma$-$P_Sker(\{x\})$ for every $x \in X$.*

Proof. The proof is immediate consequence of Theorem 3.12 and Theorem 3.14. □

Recall that a topological space (X, τ) with an operation γ on τ is γ-locally indiscrete if every γ-open subset of X is γ-closed, or every γ-closed subset of X is γ-open [6].

Lemma 3.16 ([7])**.** *If (X, τ) is γ-locally indiscrete space, then (X, τ) is γ-P_S-symmetric.*

Recall that a topological space (X, τ) with an operation γ on τ is γ-hyperconnected if τ_γ-$Cl(G) = X$ for every γ-open set G of X [6].

Lemma 3.17 ([7])**.** *If (X, τ) is γ-hyperconnected space and γ is a regular operation on τ, then (X, τ) is γ-P_S-symmetric.*

Corollary 3.18. *If (X, τ) is either γ-locally indiscrete or γ-hyperconnected space, then τ_γ-$P_SCl(\{x\}) \subseteq \gamma$-$P_Sker(\{x\})$ for every $x \in X$.*

Proof. Straightforward from Lemma 3.16, Lemma 3.17 and Corollary 3.15. □

Theorem 3.19. *For any points x and y in a topological space (X, τ) with an operation γ on τ. Then τ_γ-$P_SCl(\{x\}) \neq \tau_\gamma$-$P_SCl(\{y\})$ if and only if γ-$P_Sker(\{x\}) \neq \gamma$-$P_Sker(\{y\})$.*

Proof. Let τ_γ-$P_SCl(\{x\}) \neq \tau_\gamma$-$P_SCl(\{y\})$. This means that there is a point p in X such that $p \in \tau_\gamma$-$P_SCl(\{x\})$ and $p \notin \tau_\gamma$-$P_SCl(\{y\})$. Then there exists a γ-P_S-open set containing p and x, but not containing y. Hence by Definition 3.7, $y \notin \gamma$-$P_Sker(\{x\})$ and so γ-$P_Sker(\{x\}) \neq \gamma$-$P_Sker(\{y\})$.

Conversely, suppose that γ-$P_Sker(\{x\}) \neq \gamma$-$P_Sker(\{y\})$. That is, there is a point p in X such that $p \in \gamma$-$P_Sker(\{x\})$ and $p \notin \gamma$-$P_Sker(\{y\})$. When the point $p \in \gamma$-$P_Sker(\{x\})$, then by Theorem 3.10, τ_γ-$P_SCl(\{p\}) \cap \{x\} \neq \emptyset$. This means that $x \in \tau_\gamma$-$P_SCl(\{p\})$. Now when the point $p \notin \gamma$-$P_Sker(\{y\})$. Then by Theorem 3.10, τ_γ-$P_SCl(\{p\}) \cap \{y\} = \emptyset$. Since $x \in \tau_\gamma$-$P_SCl(\{p\})$ which implies that τ_γ-$P_SCl(\{x\}) \subseteq \tau_\gamma$-$P_SCl(\{p\})$. So τ_γ-$P_SCl(\{x\}) \cap \{y\} = \emptyset$. Therefore, τ_γ-$P_SCl(\{x\}) \neq \tau_\gamma$-$P_SCl(\{y\})$. This completes the proof. □

Theorem 3.20. *A space (X, τ) is γ-P_S-R_0 if and only if γ-$P_Sker(\{x\}) \neq \gamma$-$P_Sker(\{y\})$ implies γ-$P_Sker(\{x\}) \cap \gamma$-$P_Sker(\{y\}) = \emptyset$ for every x and y in X.*

Proof. Assume that (X, τ) is γ-P_S-R_0 space. By Theorem 3.19, if γ-$P_Sker(\{x\}) \neq \gamma$-$P_Sker(\{y\})$, then τ_γ-$P_SCl(\{x\}) \neq \tau_\gamma$-$P_SCl(\{y\})$ for every points x and y in X. Hence by Corollary 3.6, τ_γ-$P_SCl(\{x\}) \cap \tau_\gamma$-$P_SCl(\{y\}) = \emptyset$ for every points x and y in X. To prove γ-$P_Sker(\{x\}) \cap \gamma$-$P_Sker(\{y\}) = \emptyset$. Suppose there is a point $p \in X$ such that $p \in \gamma$-$P_Sker(\{x\}) \cap \gamma$-$P_Sker(\{y\})$. Then $p \in \gamma$-$P_Sker(\{x\})$ and $p \in \gamma$-$P_Sker(\{y\})$. When $p \in \gamma$-$P_Sker(\{x\})$, then by Theorem 3.11, $x \in \tau_\gamma$-$P_SCl(\{p\})$. Since $x \in \tau_\gamma$-$P_SCl(\{x\})$. This means that τ_γ-$P_SCl(\{x\}) \cap \tau_\gamma$-$P_SCl(\{p\}) \neq \emptyset$. Thus by Theorem 3.5, τ_γ-$P_SCl(\{x\}) = \tau_\gamma$-$P_SCl(\{p\})$. When $p \in \gamma$-$P_Sker(\{y\})$, then by the same way, we can obtain τ_γ-$P_SCl(\{y\}) = \tau_\gamma$-$P_SCl(\{p\})$. This means that τ_γ-$P_SCl(\{x\}) = \tau_\gamma$-$P_SCl(\{y\})$ which is contradiction. Therefore, γ-$P_Sker(\{x\}) \cap \gamma$-$P_Sker(\{y\}) = \emptyset$ for every x and y in X.

Conversely, suppose that γ-$P_Sker(\{x\}) \neq \gamma$-$P_Sker(\{y\})$ implies γ-$P_Sker(\{x\}) \cap \gamma$-$P_Sker(\{y\}) = \emptyset$ for every x and y in a space (X, τ). If τ_γ-$P_SCl(\{x\}) \neq \tau_\gamma$-$P_SCl(\{y\})$, then by Theorem 3.19, γ-$P_Sker(\{x\}) \neq \gamma$-$P_Sker(\{y\})$. So by hypothesis, γ-$P_Sker(\{x\}) \cap \gamma$-$P_Sker(\{y\}) = \emptyset$. This implies that τ_γ-$P_SCl(\{x\}) \cap \tau_\gamma$-$P_SCl(\{y\}) = \emptyset$ since $p \in \tau_\gamma$-$P_SCl(\{x\}) \cap \tau_\gamma$-$P_SCl(\{y\})$ which implies that $p \in \tau_\gamma$-$P_SCl(\{x\})$ and hence $x \in \gamma$-$P_Sker(\{p\})$., So γ-$P_Sker(\{x\}) \cap \gamma$-$P_Sker(\{p\}) \neq \emptyset$. Then by hypothesis, γ-$P_Sker(\{x\}) = \gamma$-$P_Sker(\{p\})$. Similarly, we have γ-$P_Sker(\{y\}) \cap \gamma$-$P_Sker(\{p\}) \neq \emptyset$. Then by hypothesis, γ-$P_Sker(\{x\}) = \gamma$-$P_Sker(\{p\}) = \gamma$-$P_Sker(\{y\})$. This is a contradiction. Therefore, τ_γ-$P_SCl(\{x\}) \cap \tau_\gamma$-$P_SCl(\{y\}) = \emptyset$. Consequently, by Corollary 3.6, (X, τ) is γ-P_S-R_0 space. □

Theorem 3.21. *The following properties of a topological space (X, τ) with an operation γ on τ are equivalent:*

(*i*) *X is γ-P_S-R_0.*

(*ii*) *If for every nonempty subset A of X and every γ-P_S-open set U in X such that $A \cap U \neq \emptyset$, then there exists a γ-P_S-closed set E such that $A \cap E \neq \emptyset$ and $E \subseteq U$.*

(*iii*) $U = \cup\{E : E \subseteq U$ *and* $E \in \tau_\gamma$-$P_SC(X, \tau)\}$ *for every* $U \in \tau_\gamma$-$P_SO(X, \tau)$.

(*iv*) $E = \cap\{U : E \subseteq U$ *and* $U \in \tau_\gamma$-$P_SO(X, \tau)\}$ *for every* $E \in \tau_\gamma$-$P_SC(X, \tau)$.

(*v*) τ_γ-$P_SCl(\{x\}) \subseteq \gamma$-$P_Sker(\{x\})$ *for every element x in X.*

Proof. (i) $\Rightarrow$ (ii) Let A be a nonempty subset of X and U is γ-P_S-open set in X such that $A \cap U \neq \emptyset$. Then there exists a point $x \in A \cap U$ and hence $x \in U$. Since (X, τ) is γ-P_S-R_0, then τ_γ-$P_SCl(\{x\}) \subseteq U$. Let $E = \tau_\gamma$-$P_SCl(\{x\})$. Then E is γ-P_S-closed set in X such that $A \cap E \neq \emptyset$ and $E \subseteq U$.

(ii) $\Rightarrow$ (iii) Let $U \in \tau_\gamma$-$P_SO(X, \tau)$. Then $\cup\{E : E \subseteq U$ and $E \in \tau_\gamma$-$P_SC(X, \tau)\} \subseteq U$. Let x be any point in U. Then there exists $E \in \tau_\gamma$-$P_SC(X, \tau)\}$ such that $x \in E$ and $E \subseteq U$. So $x \in E \subseteq \cup\{E : E \subseteq U$ and $E \in \tau_\gamma$-$P_SC(X, \tau)\}$. Therefore, $U = \cup\{E : E \subseteq U$ and $E \in \tau_\gamma$-$P_SC(X, \tau)\}$.

(iii) $\Rightarrow$ (iv) It is clear.

(iv) $\Rightarrow$ (v) Let x be any element of X and let $y \in X$ such that $y \notin \gamma$-$P_Sker(\{x\})$. This means that there exists a γ-P_S-open set V in X containing x such that $y \notin V$ and hence τ_γ-$P_SCl(\{y\}) \cap V = \emptyset$. Then by (iv), $\cap\{U : \tau_\gamma$-$P_SCl(\{y\}) \subseteq U$ and $U \in \tau_\gamma$-$P_SO(X, \tau)\} \cap V = \emptyset$.This means that there exists γ-P_S-open set U in X such that $x \notin U$ and τ_γ-$P_SCl(\{y\}) \subseteq U$. Then $\{x\} \cap U = \emptyset$ and hence τ_γ-$P_SCl(\{x\}) \cap U = \emptyset$ and $y \notin \tau_\gamma$-$P_SCl(\{x\})$. Therefore, $y \notin \tau_\gamma$-$P_SCl(\{x\})$. Hence τ_γ-$P_SCl(\{x\}) \subseteq \gamma$-$P_Sker(\{x\})$.

(v) $\Rightarrow$ (i) Let G be any γ-P_S-open set in X containing x. Let $y \in \gamma$-$P_Sker(\{x\})$, then by Theorem 3.11, $x \in \tau_\gamma$-$P_SCl(\{y\})$ and hence $y \in U$. This follows that γ-$P_Sker(\{x\}) \subseteq U$. Thus $x \in \tau_\gamma$-$P_SCl(\{x\}) \subseteq \gamma$-$P_Sker(\{x\}) \subseteq U$. Then a space (X, τ) is γ-P_S-R_0. □

Theorem 3.22. *A topological space (X, τ) is γ-P_S-R_0 if and only if τ_γ-$P_SCl(\{x\})$ $= \gamma$-$P_Sker(\{x\})$ for all x in X.*

Proof. Let (X, τ) be a γ-P_S-R_0 space. Then by Theorem 3.21 (v), τ_γ-$P_SCl(\{x\})$ $\subseteq \gamma$-$P_Sker(\{x\})$ for every element x in X. Let $y \in \gamma$-$P_Sker(\{x\})$ and by Theorem 3.11, $x \in \tau_\gamma$-$P_SCl(\{y\})$. So by Corollary 3.6, τ_γ-$P_SCl(\{x\}) = \tau_\gamma$-$P_SCl(\{y\})$. Thus $y \in \tau_\gamma$-$P_SCl(\{x\})$. This means that γ-$P_Sker(\{x\}) \subseteq \tau_\gamma$-$P_SCl(\{x\})$. It follows that τ_γ-$P_SCl(\{x\}) = \gamma$-$P_Sker(\{x\})$ for all x in X.

Conversely, the proof is follows directly from Theorem 3.21 and hence it is omitted. □

Theorem 3.23. *The following properties are equivalent for a topological space (X, τ) with an operation γ on τ:*

(*i*) *X is γ-P_S-R_0.*

(ii) $x \in \tau_\gamma$-$P_SCl(\{y\})$ *if and only if* $y \in \tau_\gamma$-$P_SCl(\{x\})$.

Proof. (i) $\Rightarrow$ (ii) Let X be a γ-P_S-R_0 space and let $x \in \tau_\gamma$-$P_SCl(\{y\})$. Then τ_γ-$P_SCl(\{x\}) \cap \tau_\gamma$-$P_SCl(\{y\}) \neq \emptyset$. Hence by Corollary 3.6, τ_γ-$P_SCl(\{x\}) = \tau_\gamma$-$P_SCl(\{y\})$. So $y \in \tau_\gamma$-$P_SCl(\{x\})$. Similarly, $y \in \tau_\gamma$-$P_SCl(\{x\})$ implies $x \in \tau_\gamma$-$P_SCl(\{y\})$.

(ii) $\Rightarrow$ (i) Let G be any γ-P_S-open set in X containing x. Let $y \notin G$, then $x \notin \tau_\gamma$-$P_SCl(\{y\})$ and hence $y \notin \tau_\gamma$-$P_SCl(\{x\})$ (by hypothesis). So τ_γ-$P_SCl(\{x\}) \subseteq G$. This shows that X is γ-P_S-R_0 space. □

Corollary 3.24. *Let* (X, τ) *be a topological space and* γ *be an operation on* τ. *Then* (X, τ) *is* γ-P_S-R_0 *if and only if* (X, τ) *is* γ-P_S*-symmetric.*

Proof. This is an immediate consequence of Theorem 3.23 and Definition 3.13. □

Corollary 3.25 can be constructed by applying Corollary 3.15, Corollary 3.6 and Corollary 3.24.

Corollary 3.25. *Let* (X, τ) *be a topological space and* γ *be an operation on* τ. *Then* X *is* γ-P_S*-symmetric if and only if* τ_γ-$P_SCl(\{x\}) = \gamma$-$P_Sker(\{x\})$ *for every* $x \in X$.

Proof. Straightforward. □

Consequently, some results on γ-P_S-R_0 space are obtained by using Corollary 3.24.

Corollary 3.26. *Let* (X, τ) *be a topological space and* γ *be an operation on* τ. *Then* X *is* γ-P_S-R_0 *if and only if every singleton sets in* X *is* γ-P_S*-g-closed.*

Proof. Directly follows from Corollary 3.24 and Theorem 3.14. □

Lemma 3.27 ([7])**.** *For any topological space* (X, τ) *with an operation* γ *on* τ. *If* τ_γ-$P_SO(X) = \tau_\gamma$-$P_SC(X)$, *then* (X, τ) *is* γ-P_S*-symmetric.*

From Corollary 3.24 and Lemma 3.27, we have the following corollary.

Corollary 3.28. *If* τ_γ-$P_SO(X, \tau) = \tau_\gamma$-$P_SC(X, \tau)$, *then the topological space* (X, τ) *is* γ-P_S-R_0.

Proof. Obvious. □

The proof of the next corollary follows directly from Corollary 3.24, Lemma 3.16 and Lemma 3.17.

Corollary 3.29. *If* (X, τ) *is either* γ*-locally indiscrete or* γ*-hyperconnected space, then the topological space* (X, τ) *is* γ-P_S-R_0.

Proof. It is clear. □

Theorem 3.30 ([7]). *If (X,τ) is γ-P_S-T_1 space, then it is γ-P_S-symmetric.*

Theorem 3.31 ([7]). *A space (X,τ) is γ-P_S-T_1 if and only if (X,τ) is γ-P_S-T_0 and γ-P_S-symmetric.*

The relation between the γ-P_S-R_0 and γ-P_S-T_1 spaces are shown in the following theorem.

Theorem 3.32. *If (X,τ) is γ-P_S-T_1 space, then it is γ-P_S-R_0.*

Proof. The proof is an immediate consequence of Corollary 3.24 and Theorem 3.30. □

The converse of Theorem 3.32 does not true in general as stated in the following Example 3.33.

Example 3.33. Let a space $X = \{a, b, c\}$ with the topology $\tau = \{\emptyset, \{a, b\}, \{c\}, X\}$. Then $P_SO(X) = \tau$. Define an operation γ on τ by:

For every $A \in \tau$

$$\gamma(A) = \begin{cases} A, & \text{if } A = \{c\} \\ X, & \text{if } A \neq \{c\}. \end{cases}$$

Clearly, $\tau_\gamma = \{\emptyset, X, \{c\}\}$ and τ_γ-$P_SO(X) = \{\emptyset, X\}$. Then it is easy to show that the space (X,τ) is γ-P_S-R_0, but it is not γ-P_S-T_1.

However, the converse of Theorem 3.32 is true when a space (X,τ) is γ-P_S-T_0 as shown in the following Corollary 3.34.

Corollary 3.34. *Let (X,τ) be a topological space and γ be an operation on τ. Then (X,τ) is γ-P_S-T_1 if and only if (X,τ) is γ-P_S-R_0 and γ-P_S-T_0.*

Proof. This is an immediate consequence of Corollary 3.24 and Theorem 3.31. □

Theorem 3.35. *The following conditions are equivalent for a topological space (X,τ) with an operation γ on τ:*

(i) *X is γ-P_S-R_0.*

(ii) *If F is γ-P_S-closed set in X, then $F = \gamma$-$P_Sker(F)$.*

(iii) *If F is γ-P_S-closed set in X containing x, then γ-$P_Sker(\{x\}) \subseteq F$.*

(iv) *γ-$P_Sker(\{x\}) \subseteq \tau_\gamma$-$P_SCl(\{x\})$ for every element $x \in X$.*

Proof. (i) ⇒ (ii) Let F be a γ-P_S-closed set in a γ-P_S-R_0 space (X,τ) and $x \notin F$. Then $X \backslash F$ is γ-P_S-open set such that $x \in X \backslash F$. By hypothesis, τ_γ-$P_SCl(\{x\}) \subseteq X \backslash F$. This implies that τ_γ-$P_SCl(\{x\}) \cap F = \emptyset$. Hence by Theorem 3.10, $x \notin \gamma$-$P_Sker(F)$. So γ-$P_Sker(F) \subseteq F$. But in general, $F \subseteq \gamma$-$P_Sker(F)$ (by Remark 3.8 (i)). This follows that $F = \gamma$-$P_Sker(F)$.

(ii) ⇒ (iii) For any γ-P_S-closed set F in X containing x, $\{x\} \subseteq F$ which implies that γ-$P_S ker(\{x\}) \subseteq \gamma$-$P_S ker(F) = F$.

(iii) ⇒ (iv) Since τ_γ-$P_S Cl(\{x\})$ is γ-P_S-closed containing x. Then by (3), γ-$P_S ker(\{x\}) \subseteq \tau_\gamma$-$P_S Cl(\{x\})$.

(iv) ⇒ (i) Suppose $x \in \tau_\gamma$-$P_S Cl(\{y\})$, then by Theorem 3.11, $y \in \gamma$-$P_S ker(\{x\})$. By using (iv), we get $y \in \tau_\gamma$-$P_S Cl(\{x\})$. So $x \in \tau_\gamma$-$P_S Cl(\{y\})$ implies $y \in \tau_\gamma$-$P_S Cl(\{x\})$. Similarly, we can show that $y \in \tau_\gamma$-$P_S Cl(\{x\})$ implies that $x \in \tau_\gamma$-$P_S Cl(\{y\})$. Therefore, by Theorem 3.23, a space (X, τ) is γ-P_S-R_0. □

Definition 3.36 ([5])**.** Let (X, τ) and (Y, σ) be two topological spaces. A function $f: (X, \tau) \to (Y, \sigma)$ is called (γ, β)-P_S-irresolute at a point $x \in X$ if for each β-P_S-open set V of Y containing $f(x)$, there exists a γ-P_S-open set U of X containing x such that $f(U) \subseteq V$. If f is (γ, β)-P_S-irresolute at every point x in X, then f is said to be (γ, β)-P_S-irresolute.

Theorem 3.37 ([5])**.** *The following properties are equivalent for any function $f: (X, \tau) \to (Y, \sigma)$, where γ and β are operations on τ and σ respectively.*

(*i*) *f is (γ, β)-P_S-irresolute.*

(*ii*) *The inverse image of every β-P_S-open set of Y is γ-P_S-open set in X.*

(*iii*) *The inverse image of every β-P_S-closed set of Y is γ-P_S-closed set in X.*

Definition 3.38 ([5])**.** A function $f: (X, \tau) \to (Y, \sigma)$ is called (γ, β)-P_S-closed if for every γ-P_S-closed set V of X, then $f(V)$ is β-P_S-closed set in Y.

Theorem 3.39. *Let γ and β be operations on τ and σ respectively. If $f: (X, \tau) \to (Y, \sigma)$ is a surjective (γ, β)-P_S-irresolute and (γ, β)-P_S-closed function and (X, τ) is γ-P_S-R_0, then (Y, σ) is also β-P_S-R_0.*

Proof. Let U be any β-P_S-open set in (Y, σ) and y be any point in U. Since f is (γ, β)-P_S-irresolute, then by Theorem 3.37 (ii), $f^{-1}(U)$ is γ-P_S-open set in (X, τ). Since (X, τ) is γ-P_S-R_0 space, for a point $x \in f^{-1}(y)$, τ_γ-$P_S Cl(\{x\}) \subseteq f^{-1}(U)$. Since f is (γ, β)-P_S-closed, then σ_β-$P_S Cl(\{y\}) = \sigma_\beta$-$P_S Cl(\{f(x)\}) \subseteq f(\tau_\gamma$-$P_S Cl(\{x\})) \subseteq U$. Therefore, (Y, σ) is β-P_S-R_0 space. □

Recall that a function $f: (X, \tau) \to (Y, \sigma)$ with an operation γ on τ is γ-continuous [9] if $f^{-1}(V)$ is γ-open set in X, for every open set V of Y. Recall that a function $f: (X, \tau) \to (Y, \sigma)$ with an operation β on σ is β-open [2] if the image of every open set in X is open set in Y.

Lemma 3.40 ([5])**.** *If a function $f: (X, \tau) \to (Y, \sigma)$ is both γ-continuous and β-open, then f is (γ, β)-P_S-irresolute.*

Corollary 3.41. *Let γ and β be operations on τ and σ respectively. If $f: (X, \tau) \to (Y, \sigma)$ is a surjective (γ, β)-P_S-closed, γ-continuous and β-open function and (X, τ) is γ-P_S-R_0, then (Y, σ) is also β-P_S-R_0.*

Proof. Follows directly from Theorem 3.39 and Lemma 3.40. □

Definition 3.42. A topological space (X,τ) with an operation γ on τ is said to be weakly γ-P_S-R_0 if $\bigcap_{x\in X} \tau_\gamma$-$P_SCl(\{x\}) = \emptyset$.

Theorem 3.43. *Let (X,τ) be a topological space and γ be an operation on τ. Then (X,τ) is weakly γ-P_S-R_0 if and only if γ-$P_Sker(\{x\}) \neq X$ for every $x \in X$.*

Proof. Let (X,τ) be a weakly γ-P_S-R_0 space. Suppose that γ-$P_Sker(\{y\}) = X$ for every $y \in X$. Then $\{y\} \neq G$ where G is any proper γ-P_S-open subset of X. Then $y \in \tau_\gamma$-$P_SCl(\{x\})$ for every $x \in X$ and hence $y \in \bigcap_{x\in X} \tau_\gamma$-$P_SCl(\{x\})$. This is a contradiction.

Conversely, let γ-$P_Sker(\{x\}) \neq X$ for every $x \in X$. If there exists a point y in X such that $y \in \bigcap_{x\in X} \tau_\gamma$-$P_SCl(\{x\})$. Then every γ-P_S-open set containing y must contain every point of x. Hence X is the unique γ-P_S-open set containing y. So γ-$P_Sker(\{y\}) = X$. This is a contradiction. Therefore, (X,τ) is weakly γ-P_S-R_0 space. □

The following Remark 3.44 follows directly from Definitions 3.1 and 3.42.

Remark 3.44. Every γ-P_S-R_0 space is weakly γ-P_S-R_0.

But in general the converse of the above Remark 3.44 does not hold as shown by the following Example 3.45.

Example 3.45. In Example 3.9, the space (X,τ) is weakly γ-P_S-R_0, but it is not γ-P_S-R_0 since $\{a,c\}$ is γ-P_S-open set in X containing c such that τ_γ-$P_SCl(\{c\}) = \{b,c\} \nsubseteq \{a,c\}$.

Remark 3.46. Let γ and β be operations on τ and σ respectively. If $f\colon (X,\tau) \to (Y,\sigma)$ is an injective (γ,β)-P_S-closed function and (X,τ) is weakly γ-P_S-R_0, then (Y,σ) is also weakly β-P_S-R_0.

4. Properties of γ-P_S-R_1 spaces

Now, γ-P_S-R_1 space in terms of γ-P_S-open and $\tau_\gamma P_S$-closure operator is defined as follows:

Definition 4.1. A topological space (X,τ) with an operation γ on τ is said to be γ-P_S-R_1 if for x and y in X with τ_γ-$P_SCl(\{x\}) \neq \tau_\gamma$-$P_SCl(\{y\})$, there exist disjoint γ-P_S-open sets G and H such that τ_γ-$P_SCl(\{x\}) \subseteq G$ and τ_γ-$P_SCl(\{y\}) \subseteq H$.

The relation between the γ-P_S-R_0 and γ-P_S-R_1 spaces are shown in the following theorem.

Theorem 4.2. *If (X,τ) is γ-P_S-R_1 space, then it is γ-P_S-R_0.*

Proof. Let G be any γ-P_S-open set in γ-P_S-R_1 space (X, τ) containing x. Let $y \notin G$. If $y \notin G$, and since $x \notin \tau_\gamma$-$P_SCl(\{y\})$. Then τ_γ-$P_SCl(\{x\}) \neq \tau_\gamma$-$P_SCl(\{y\})$. Thus, there exists a γ-P_S-open set H such that τ_γ-$P_SCl(\{y\}) \subseteq H$ and $x \notin H$. This implies that $y \notin \tau_\gamma$-$P_SCl(\{x\})$ and hence τ_γ-$P_SCl(\{x\}) \subseteq G$. Therefore, the space (X, τ) is γ-P_S-R_0. □

But the converse of the above Theorem 4.2 does not true in general as seen from the following example.

Example 4.3. Let a space $X = \{x, y, z\}$ with the discrete topology τ on X. Define an operation γ on τ by: for every $A \in \tau$

$$\gamma(A) = \begin{cases} A & \text{if } A = \{x, y\} \text{ or } \{x, z\} \text{ or } \{y, z\} \\ X & \text{if } otherwise \end{cases}$$

Clearly, $\tau_\gamma = \{\emptyset, \{x, y\}, \{x, z\}, \{y, z\}, X\} = \tau_\gamma$-$P_SO(X)$. Then the space (X, τ) is γ-P_S-R_0, but it is not γ-P_S-R_1 since for the points $x, y \in X$ such that τ_γ-$P_SCl(\{x\}) = \{x\} \neq \{y\} = \tau_\gamma$-$P_SCl(\{y\})$, but there is no disjoint γ-P_S-open sets G and H such that τ_γ-$P_SCl(\{x\}) \subseteq G$ and τ_γ-$P_SCl(\{y\}) \subseteq H$.

Theorem 4.4 ([7]). *Let (X, τ) be a topological space and γ be an operation on τ. Then X is γ-P_S-T_1 if and only if every singleton set in X is γ-P_S-closed.*

Lemma 4.5 ([7]). *Let (X, τ) be a topological space and γ be an operation on τ. Then the following statements are holds:*

(*i*) *If X is γ-P_S-T_2, then it is γ-P_S-T_1.*

(*ii*) *If X is γ-P_S-T_1, then it is γ-P_S-$T_{\frac{1}{2}}$.*

(*iii*) *If X is γ-P_S-$T_{\frac{1}{2}}$, then it is γ-P_S-T_0.*

Theorem 4.6. *If (X, τ) is γ-P_S-T_2 space, then it is γ-P_S-R_1.*

Proof. Let x and y be any elements in γ-P_S-T_2 space (X, τ) such that τ_γ-$P_SCl(\{x\}) \neq \tau_\gamma$-$P_SCl(\{y\})$. Then by Lemma 4.5 (i), (X, τ) is γ-P_S-T_1 and hence by Theorem 4.4, τ_γ-$P_SCl(\{x\}) = \{x\}$ and τ_γ-$P_SCl(\{y\}) = \{y\}$ for every $x, y \in X$. Then $x \neq y$ and hence by hypothesis, there exist disjoint γ-P_S-open sets G and H containing x and y respectively. This follows that τ_γ-$P_SCl(\{x\}) \subseteq G$ and τ_γ-$P_SCl(\{y\}) \subseteq H$. Consequently, (X, τ) is γ-P_S-R_1 space. □

So, the converse of the Theorem 4.6 is not true in general as described in the following Example 4.7.

Example 4.7. Consider the space $X = \{a, b, c, d\}$ with the topology $\tau = \{\emptyset, X, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}\}$. Define an operation $\gamma\colon \tau \to P(X)$ by $\gamma(A) = A$ for all $A \in \tau$. Then $\tau_\gamma = \tau$. Hence τ_γ-$P_SO(X) = \{\emptyset, X, \{a\}, \{b, c, d\}\}$. Hence the space (X, τ) is γ-P_S-R_1, but it is not γ-P_S-T_2.

The converse of the Theorem 4.6 is only true when a space (X,τ) is γ-P_S-T_0 as shown in the following Theorem 4.8.

Theorem 4.8. *Let (X,τ) be a topological space and γ be an operation on τ. Then (X,τ) is γ-P_S-T_2 if and only if (X,τ) is γ-P_S-R_1 and γ-P_S-T_0.*

Proof. Let x and y be any distinct points in X and (X,τ) be a γ-P_S-R_1 and γ-P_S-T_0 space. Then by Theorem 4.2, (X,τ) is γ-P_S-R_0 and hence by Corollary 3.34, (X,τ) is γ-P_S-T_1. Then by Theorem 4.4, τ_γ-$P_SCl(\{x\}) = \{x\}$ and τ_γ-$P_SCl(\{y\}) = \{y\}$ for every distinct points $x, y \in X$. This means that τ_γ-$P_SCl(\{x\}) \neq \tau_\gamma$-$P_SCl(\{y\})$ and hence by Definition 4.1, there exist disjoint γ-P_S-open sets G and H such that τ_γ-$P_SCl(\{x\}) \subseteq G$ and τ_γ-$P_SCl(\{y\}) \subseteq H$. It follows that $x \in G$ and $y \in H$. Therefore, (X,τ) is γ-P_S-T_2.

Conversely, if (X,τ) is γ-P_S-T_2 space, then by Lemma 4.5 and Theorem 4.6, (X,τ) is γ-P_S-T_0 and γ-P_S-R_1 respectively. This completes the proof. □

Therefore, from Theorem 4.6, Theorem 4.8 and Lemma 4.5, we have the following corollary.

Corollary 4.9. *Let (X,τ) be any topological space and γ be an operation on τ. Then the following statements are equivalent:*

(i) *(X,τ) is γ-P_S-R_1 and γ-P_S-T_0.*

(ii) *(X,τ) is γ-P_S-R_1 and γ-P_S-$T_{\frac{1}{2}}$.*

(iii) *(X,τ) is γ-P_S-R_1 and γ-P_S-T_1.*

(iv) *(X,τ) is γ-P_S-T_2.*

Proof. It is clear and hence it is omitted. □

Recall that a topological space (X,τ) with an operation γ on τ is γ-P_S-regular if for each γ-P_S-closed set F of X not containing x, there exist disjoint γ-P_S-open sets G and H such that $x \in G$ and $F \subseteq H$.

Lemma 4.10 ([1])**.** *If (X,τ) is γ-P_S-regular and γ-P_S-T_1 space. Then it is γ-P_S-T_2.*

Corollary 4.11. *If (X,τ) is γ-P_S-T_1 space. Then every γ-P_S-regular space (X,τ) is γ-P_S-R_1.*

Proof. This is an immediate consequence of Lemma 4.10 and Theorem 4.6. □

Recall that a topological space (X,τ) with an operation γ on τ is γ-P_S-normal [1] if for each pair of disjoint γ-P_S-closed sets E, F of X, there exist disjoint γ-P_S-open sets G and H such that $E \subseteq G$ and $F \subseteq H$.

Lemma 4.12 ([1])**.** *If (X,τ) is γ-P_S-T_1 space. Then every γ-P_S-normal space X is γ-P_S-regular.*

The proof of the following corollary is follows directly from Corollary 4.11 and Lemma 4.12.

Corollary 4.13. *If (X, τ) is γ-P_S-normal and γ-P_S-T_1 space. Then it is γ-P_S-R_1.*

Proof. Obvious. □

The following are some significant characterizations of γ-P_S-R_1 spaces.

Remark 4.14. A topological space (X, τ) is γ-P_S-R_1 if and only if for x and y in X with τ_γ-$P_SCl(\{x\}) \neq \tau_\gamma$-$P_SCl(\{y\})$, there exist disjoint γ-P_S-closed sets E and F such that $x \in E$, $y \notin E$, $y \in F$, $x \notin F$ and $E \cup F = X$.

Theorem 4.15. *A topological space (X, τ) is γ-P_S-R_1 if and only if for x and y in X with γ-$P_Sker(\{x\}) \neq \gamma$-$P_Sker(\{y\})$, there exist disjoint γ-P_S-open sets G and H such that τ_γ-$P_SCl(\{x\}) \subseteq G$ and τ_γ-$P_SCl(\{y\}) \subseteq H$.*

Proof. Follows directly from Theorem 3.19. □

Theorem 4.16. *A topological space (X, τ) is γ-P_S-R_1 if and only if for $x \in X \backslash \tau_\gamma$-$P_SCl(\{y\})$ implies that x and y have disjoint γ-P_S-open neighborhoods.*

Proof. Suppose that $x \in X \backslash \tau_\gamma$-$P_SCl(\{y\})$. Then τ_γ-$P_SCl(\{x\}) \neq \tau_\gamma$-$P_SCl(\{y\})$ and hence x and y have disjoint γ-P_S-open neighborhoods.

Conversely, first, to show that (X, τ) is γ-P_S-R_0. Let W be a γ-P_S-open set and $x \in W$. Suppose that $y \notin W$. Then τ_γ-$P_SCl(\{y\}) \cap W = \emptyset$ and $x \notin \tau_\gamma$-$P_SCl(\{y\})$. There exist γ-P_S-open sets U and V such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$. Hence, τ_γ-$P_SCl(\{x\}) \subseteq \tau_\gamma$-$P_SCl(U)$ and τ_γ-$P_SCl(\{x\})$ $\cap$ $V \subseteq \tau_\gamma$-$P_SCl(U) \cap V = \emptyset$. Therefore, $y \notin \tau_\gamma$-$P_SCl(\{x\})$. Consequently, τ_γ-$P_SCl(\{x\}) \subseteq G$ and (X, τ) is γ-P_S-R_0. Next, to show that (X, τ) is γ-P_S-R_1. Suppose that τ_γ-$P_SCl(\{x\}) \neq \tau_\gamma$-$P_SCl(\{y\})$. Then, we can assume that there exists $p \in \tau_\gamma$-$P_SCl(\{x\})$ such that $p \notin \tau_\gamma$-$P_SCl(\{y\})$. There exist γ-P_S-open sets G and H such that $p \in G$, $y \in H$ and $G \cap H = \emptyset$. Since $p \in \tau_\gamma$-$P_SCl(\{x\})$, $x \in G$. Since (X, τ) is γ-P_S-R_0, we obtain τ_γ-$P_SCl(\{x\}) \subseteq G$, τ_γ-$P_SCl(\{y\}) \subseteq H$ and $G \cap H = \emptyset$. This shows that (X, τ) is γ-P_S-R_1 space. □

Corollary 4.17. *A topological space (X, τ) is γ-P_S-R_1 if and only if τ_γ-$P_SCl(\{x\})$ $\neq \tau_\gamma$-$P_SCl(\{y\})$ implies that x and y have disjoint γ-P_S-open neighborhoods.*

Proof. Assume that $x \in X \backslash \tau_\gamma$-$P_SCl(\{y\})$. Then τ_γ-$P_SCl(\{x\}) \neq \tau_\gamma$-$P_SCl(\{y\})$ and hence by hypothesis, x and y have disjoint γ-P_S-open neighborhoods. Thus by Theorem 4.16, (X, τ) is γ-P_S-R_1 space. The proof of the converse part is obvious. □

5. Conclusion

This study introduces some types of γ-P_S- separation axioms such as γ-P_S-R_0 and γ-P_S-R_1 by using γ-P_S-open sets and τ_γ-P_S-closure of a set. Some properties of these spaces have been constructed.

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