On \((m,n)\)-fully stable Banach algebra modules

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Abstract. In this paper the concept of fully-\((m,n)\) stable Banach Algebra-module \(F - (m,n) - S - B - A\)-module, we study some properties of \(F - (m,n) - S - B - A\)-module and another characterization have been given.

Keywords: fully stable Banach \(A\)-module, fully \((m,n)\)-stable Banach \(A\)-module, multiplication \((m,n)\) \(A\)-module.

1. Introduction

A non-empty set \(A\) is an algebra if, \((A,+,\cdot)\) is a vector space over a field \(F\), \((A,+,\circ)\) is a ring and \((\alpha a) \circ b = \alpha (a \circ b) = a \circ (\alpha b)\) for every \(\alpha \in F\), for every \(a, b \in A\)” [1]. In [2]” a ring \(R\) is an algebra \((R,+,\cdot, -,0)\) where + and \(\cdot\) are two binary operations, – is unary and 0 is nullary element satisfying, \((R,+,-,0)\) is an abelian group, \((R,\cdot)\) is a semigroup and \(x.(y + z) = (x.y) + (x.z)\) and \((x + y).z = (x.z) + (y + z)\). ”Let \(A\) be an algebra, recall that a Banach space \(E\) is a Banach left \(A\)-module \((B - A\)-module\) if \(E\) is a left \(A\)-module, and \(\|a.x\| \leq \|a\| \|x\| (a \in A, x \in E)\)” [1]. Following [3] ”a map from a left \(B - A\)-module \(X\) into a left Banach \(A\)-module \(Y\) \((A\) is not necessarily commutative \) is said a multiplier (homomorphism) if it satisfies \(T(a.x) = a.Tx\) for all \(a \in A\), \(x \in X\). ” In [4], ”a submodule \(N\) of an \(R\)-module \(M\) is said to be stable, if \(f(N) \subseteq N\) for each \(R\)-homomorphism \(f : N \rightarrow M\). \(M\) is called a fully stable module, each submodule of \(M\) is stable”. ”A Banach algebra module \(M\) is called \(F - S - B - A\)-module if for every submodule \(N\) of \(M\) and for each multiplier \(\theta : N \rightarrow M\) such that \(\theta(N) \subseteq N\) [5]. We use the notation \(R^{m \times n}\) for the set of all \(m \times n\) matrices over \(R\). For \(A \in R^{m \times n}\), \(A^{T}\) will denote the transpose of \(A\). In general, for an \(R\)-module \(N\), we write \(N^{m \times n}\) for the set of
all formal \( m \times n \) matrices whose entries are elements of \( N \). Let \( M \) be a right Banach Algebra-module and \( N \) be a left \( R \)-module. For \( x \in M^{l \times m} \), \( s \in R^{m \times n} \) and \( y \in M^{n \times k} \), under the usual multiplication of matrices, \( xs \) (resp. \( sy \)) is a well defined element in \( M^{l \times m} \) (resp. \( N^{n \times k} \)). If \( X \subseteq M^{l \times m} \), \( S \subseteq R^{m \times n} \) and \( Y \subseteq N^{n \times k} \) define

\[
\ell_{M^{l \times m}}(S) = \left\{ u \in M^{l \times m} \mid us = 0; \forall s \in S \right\},
\]

\[
r_{N^{n \times k}}(S) = \left\{ v \in N^{n \times k} \mid sv = 0; \forall s \in S \right\},
\]

\[
\ell_{R^{m \times n}}(Y) = \left\{ s \in R^{m \times n} \mid sy = 0; \forall y \in Y \right\},
\]

\[
r_{R^{m \times n}}(X) = \left\{ s \in R^{m \times n} \mid xs = 0; \forall x \in X \right\}.
\]

We will write \( N^n = N^{1 \times n} \), \( N_n = N^{n \times 1} \) [6]. In this paper for two fixed positive integers \( n, m \) the concept of fully \((m, n)\)-stable Banach algebra modules has been introduced.

2. Fully \((m, n)\)-stable Banach algebra modules

"A left \( B - A \)-module \( X \) is \( n \)-generated for \( n \in N \) if there exists \( x_1, \ldots, x_n \in X \) such that each \( x \in X \) can represented as \( x = \sum_{k=1}^{n} a_k x_k \) for some \( a_1, \ldots, a_n \in A \). A module which is \( 1 \)-generated is called a cyclic module" [7].

**Definition 2.1.** Let \( K \) be \( B - A \)-module, \( K \) is called \((m, n)\)-fully stable \( B \)-A-module, if for every \( n \)-generated submodule \( L \) of \( K^m \) and for each multiplier \( \theta : L \to K^m \) satisfy \( \theta(L) \subseteq L \), for two fixed positive integers \( n, m \).

In [5] "for a nonempty subset \( M \) in a left \( B - A \) -module \( X \), the annihilator \( ann_A(M) \) of \( M \) is \( ann_A(M) = \{ a \in A \mid a \cdot x = 0 \forall x \in M \} \)."

**Notation.** Let \( X \) be a \( B - A \)-module

1. \( L_{x_1, x_2, \ldots, x_n} = \{ \oplus l_{x_i} \mid n \in N, x_i \in X, i = 1, 2, \ldots, n \} \),
   \( K_{y_1, y_2, \ldots, y_n} = \{ \oplus k_{y_i} \mid k \in K, y_i \in X, i = 1, 2, \ldots, n \} \),
2. \( \ell_{A^{m \times n}}L_{x_1, x_2, \ldots, x_n} = \{ a \in A^{m \times n}, a, (\oplus l_{x_i}) = 0, \forall \oplus l_{x_i} \in L_{x_1, x_2, \ldots, x_n} \} \),
   \( \ell_{A^{m \times n}}K_{y_1, y_2, \ldots, y_n} = \{ a \in A^{m \times n}, a, (\oplus k_{y_i}) = 0, \forall k_{y_i} \in K_{y_1, y_2, \ldots, y_n} \} \).

**Proposition 2.2.** A \( B - A \)-module \( M \) is fully-\((m, n)\) stable, if and only if any two \( m \)-element subsets \( \{ L_{x_1, L_{x_1,x_2}, \ldots, L_{x_1,x_2,\ldots,x_m} \} \) and \( \{ K_{y_1, K_{y_1,y_2}, \ldots, K_{y_1,y_2,\ldots,y_m} \} \) of \( M_n \), if \( \beta_j \notin \sum_{i=1}^{n} A \alpha_i \), for each \( j = 1, \ldots, m \) implies \( \ell_{A^n}(\{ L_{x_1, L_{x_1,x_2}, \ldots, L_{x_1,x_2,\ldots,x_m} \}) \subseteq \ell_{A^n}(\{ K_{y_1, K_{y_1,y_2}, \ldots, K_{y_1,y_2,\ldots,y_m} \}).

**Proof.** Assume that \( K \) is \( F - (m, n) - S - B - A \)-module and there exist two \( m \)-element subsets \( \{ L_{x_1, L_{x_1,x_2}, \ldots, L_{x_1,x_2,\ldots,x_m} \} \) and \( \{ K_{y_1, K_{y_1,y_2}, \ldots, K_{y_1,y_2,\ldots,y_m} \} \) of \( M_n \) such that if \( K_{y_1} \notin \sum_{i=1}^{n} A \alpha_i \), for each \( j = 1, \ldots, m \) and

\[
\ell_{A^n}(\{ L_{x_1, L_{x_1,x_2}, \ldots, L_{x_1,x_2,\ldots,x_m} \}) \subseteq \ell_{A^n}(\{ K_{y_1, K_{y_1,y_2}, \ldots, K_{y_1,y_2,\ldots,y_m} \}).
\]
Let $A$ be a ring and $M = \sum_{i=1}^{n} a_i L_{x_i}$ be a module over $A$. Let $L_{x_i} = (k_{i1}, k_{i2}, \ldots, k_{in})$. If $\sum_{i=1}^{n} a_i k_{ij} = 0$, then $\sum_{i=1}^{n} a_i k_{ij} = 0$, $j = 1, 2, \ldots, m$, implies that $r L_{x_j} = 0$ where $r = (r_1, \ldots, r_n)$ and hence $r \in \ell_A \{L_{x_1}, L_{x_2}, \ldots, L_{x_1}, x_{2}, \ldots, x_m\}$. By assumption $rK_{y_1} = 0$, $j = 1, \ldots, m$ so $\sum_{i=1}^{n} r_i K_{y_i} = 0$. This shows that $f$ is well defined. It is an easy matter to see that $f$ is a multiplier. Fully $(m, n)$ stability of $M$ implies that there exists $t = (t_1, \ldots, t_n) \in A^n$ such that $f(\sum_{i=1}^{n} t_i L_{x_i}) = \sum_{i=1}^{n} t_i K_{y_i}$ for each $\sum_{i=1}^{n} t_i L_{x_i} \in \sum_{i=1}^{n} AL_{x_i}$. Let $r_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in A^n$ where 1 is in the $i$-th position and 0 otherwise. $K_{y_i} = f(L_{x_i}) = \sum_{k=1}^{n} t_k L_{x_i} \in \sum_{i=1}^{n} AL_{x_i}$, which is contradiction. Conversely assume that there exists $n$-generated $B - A$-submodule of $M^m$ and multiplier $\mu : \sum_{i=1}^{n} AL_{x_i} \rightarrow M^m$ such that $\mu(\sum_{i=1}^{n} AL_{x_i}) \notin \sum_{i=1}^{n} AL_{x_i}$. Then there exists an element $\beta(= \sum_{i=1}^{n} r_i L_{x_i}) \in \sum_{i=1}^{n} AL_{x_i}$ such that $\mu(\beta) \notin \sum_{i=1}^{n} AL_{x_i}$. Take $K_{y_i} = K_{y_j}$, $j = 1, \ldots, m$, then we have $m$-element subset $\{\mu(K_{y_1}), \ldots, \mu(K_{y_m})\}$, such that $\mu(\mu(K_{y_j}) \notin \sum_{i=1}^{n} AL_{x_i}$, $j = 1, \ldots, m$. Let $\eta = (t_1, \ldots, t_n) \in \ell_A \{L_{x_1}, L_{x_2}, \ldots, L_{x_1}, x_{2}, \ldots, x_m\}$, then $\eta \alpha_j = 0$, i.e $\sum_{i=1}^{n} t_i \alpha_{ij} = 0$, for each $j = 1, \ldots, m$. $L_{x_i} = (a_{i1}, a_{i2}, \ldots, a_{in})$ and $\{\mu(K_{y_1}), \ldots, \mu(K_{y_m})\} \eta = \sum_{i=1}^{n} t_i \mu(K_{y_i}) = \sum_{i=1}^{n} t_i \mu(\sum_{k=1}^{n} t_k L_{x_i}) = \sum_{i=1}^{n} \mu(\sum_{k=1}^{n} t_k L_{x_i}) = 0$ hence $\ell_A \{L_{x_1}, L_{x_2}, \ldots, L_{x_1}, x_{2}, \ldots, x_m\} \subseteq \ell_A \{\mu(K_{y_1}), \ldots, \mu(K_{y_m})\}$, thus $\ell_A \{L_{x_1}, L_{x_2}, \ldots, L_{x_1}, x_{2}, \ldots, x_m\} \subseteq \ell_A \{\mu(K_{y_1}), \ldots, \mu(K_{y_1}, y_2, \ldots, y_m)\}$ which is a contradiction. Thus $M$ is $F - (m, n) - S - B - A$-module.

**Corollary 2.3.** Let $M$ be an $F - (m, n) - S - B - A$-module, then for any two $m$-element subsets $\{L_{x_1}, L_{x_2}, \ldots, L_{x_1}, x_{2}, \ldots, x_m\}$ and $\{K_{y_1}, K_{y_2}, \ldots, K_{y_1}, y_2, \ldots, y_m\}$ of $M_n$, $\ell_A \{L_{x_1}, L_{x_2}, \ldots, L_{x_1}, x_{2}, \ldots, x_m\} \subseteq \ell_A \{K_{y_1}, K_{y_2}, \ldots, K_{y_1}, y_2, \ldots, y_m\}$ implies that $AL_{x_1} + AL_{x_2} + \ldots + AL_{x_1}, x_{2}, \ldots, x_m = AK_{y_1} + AK_{y_2} + AK_{y_1}, y_2, \ldots, y_m$.

In [9], "$AB - A$-module $X$ is said to satisfy Baer criterion if each submodule of $X$ satisfies Baer criterion, that is for every submodule $N$ of $X$ and $A$-multiplier $\theta : N \rightarrow X$, there exists an element $a$ in $A$ such that $\theta(n) = an$ for all $n \in N$".

**Definition 2.4.** A $B - A$-module $X$ is said to satisfy Baer $(m, n)$-criterion if each submodule of $X$ satisfies Baer $(m, n)$-criterion, that is for every $n$-generated submodule $L$ of $X$ and $A$-multiplier $\theta : L \rightarrow X^m$, there exists an element $a$ in $A$ such that $\theta(l) = al$ for all $l \in L$.

**Proposition 2.5.** If $X$ satisfies Baer $(m, 1)$-criterion and $\ell_A (L \cap M) = \ell_A (L) + \ell_A (M)$ for each $n$-generated submodules of $X^m$, then $X$ satisfies Baer $(m, n)$-criterion.

**Proof.** Let $P = Ax_1 + Ax_2 + \ldots + Ax_n$ be an $n$-generated submodule of $X^m$ and $f : P \rightarrow X^m$ a multiplier. We use induction on $n$. It is clear that $M$ satisfies Baer $(m, n)$-criterion, if $n = 1$. Suppose that $M$ satisfies Baer $(m, n)$-criterion for all $k$-generated submodule of $X^m$, for $k \leq n - 1$. Write $L = Ax_1, M = Ax_2 + \ldots + Ax_n$, then for each $w_1 \in L$ and $w_2 \in M$, $f(w_1) = y_1 w_1$, $y_2 w_2$.
Let $w$ be a $B - A$ - module. Then $X$ satisfies Baer $(m, n)$-criterion if and only if $r_{X_n}(AL_{x_1} + AL_{x_1,x_2} + \ldots + AL_{x_1,x_2,\ldots,x_n}) = AL_{x_1} + AL_{x_1,x_2} + \ldots + AL_{x_1,x_2,\ldots,x_n}$ for $n$-element subset $\{L_{x_1}, L_{x_1,x_2}, \ldots, L_{x_1,x_2,\ldots,x_n}\}$ of $X_n$.

**Proof.** Suppose that Baer $(m, n)$-criterion holds for $n$-generated submodule of $X^m$ let $L_{x_i} = (k_{i1}; k_{i2}, \ldots, k_{in})$, for each $i = 1, \ldots, n$ and $K_y = \{K_{y1}, K_{y1,y2}, \ldots, K_{y1,y2,\ldots,y_n}\} \in r_{X_n}(AL_{x_1} + AL_{x_1,x_2} + \ldots + AL_{x_1,x_2,\ldots,x_n})$, $K_y = (a_{11}, a_{21}, \ldots, a_{n1})$.

Define $f : AL_{x_1} + AL_{x_1,x_2} + \ldots + AL_{x_1,x_2,\ldots,x_n} \to X_m$ by $\mu(\sum_{i=1}^{n} a_iL_{x_i}) = \sum_{i=1}^{n} a_iK_{y_i}$. If $\sum_{i=1}^{n} a_iL_{x_i}$, then $\sum_{i=1}^{n} a_iK_{y_i} = 0$. $j = 1, \ldots, m$, this implies that $r_{L_{x_i}} = 0$ where $r = (r_1, \ldots, r_m)$ and hence $r \in \ell_A(\ell_{x_1} + \ell_{x_1,x_2} + \ldots + \ell_{x_1,x_2,\ldots,x_n})$. By assumption $r_{L_{x_i}} = 0$, $i = 1, \ldots, n$ so $\sum_{i=1}^{n} a_iK_{y_i} = 0$. This show that $f$ is well defined. It is an easy matter to see that $\mu$ is a multiplier. By assumption there exists $t \in A$ such that $\mu(\sum_{i=1}^{n} a_iL_{x_i}) = t(\sum_{i=1}^{n} a_iK_{y_i}) = \sum_{i=1}^{n} (ta_i)K_{y_i}$ for each $\sum_{i=1}^{n} a_iL_{x_i} \in \sum_{i=1}^{n} AL_{x_i}$. Let $r_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in A^n$ where $1$ in the $i$-th position and $0$ otherwise. $K_{y_i} = \mu(\sum_{i=1}^{n} L_{x_i}) = \sum_{i=1}^{n} t_iL_{x_i} \in \sum_{i=1}^{n} AL_{x_i}$ which is contradiction. This implies that $r_{X_n}(AL_{x_1} + AL_{x_1,x_2} + \ldots + AL_{x_1,x_2,\ldots,x_n}) \subseteq AL_{x_1} + AL_{x_1,x_2} + \ldots + AL_{x_1,x_2,\ldots,x_n}$, the other inclusion is trivial.

Conversely, assume that $r_{X_n}(AL_{x_1} + AL_{x_1,x_2} + \ldots + AL_{x_1,x_2,\ldots,x_n}) = AL_{x_1} + AL_{x_1,x_2} + \ldots + AL_{x_1,x_2,\ldots,x_n}$, for each $\{L_{x_1}, L_{x_1,x_2}, \ldots, L_{x_1,x_2,\ldots,x_n}\}$ in $X_n$.

Then for each multiplier $f : AL_{x_1} + AL_{x_1,x_2} + \ldots + AL_{x_1,x_2,\ldots,x_n} \to X_m$ and $s = (s_1, \ldots, s_n) \in \ell_A(AL_{x_1} + AL_{x_1,x_2} + \ldots + AL_{x_1,x_2,\ldots,x_n})$, $\sum_{k=1}^{n} s_k(\sum_{i=1}^{n} t_iL_{x_i}) = 0$, for each $\sum_{i=1}^{n} t_iL_{x_i} \in \sum_{i=1}^{n} AL_{x_i}$, hence

$$\sum_{k=1}^{n} s_k \cdot f(\sum_{i=1}^{n} t_iL_{x_i}) = \sum_{k=1}^{n} f(\sum_{i=1}^{n} s_k t_iL_{x_i}) = 0,$$

thus $f(\sum_{i=1}^{n} t_iL_{x_i}) \in r_{X_n}(AL_{x_1} + AL_{x_1,x_2} + \ldots + AL_{x_1,x_2,\ldots,x_n}) = AL_{x_1} + AL_{x_1,x_2} + \ldots + AL_{x_1,x_2,\ldots,x_n}$, for some $t \in A$. Then $X$ satisfies Baer $(m, n)$-criterion.

**Corollary 2.7.** Let $X$ be a $B - A$ - module. Then $X$ is $F - (m, n) - S - B - A$ - module if and only if $r_{X_n}(AL_{x_1} + AL_{x_1,x_2} + \ldots + AL_{x_1,x_2,\ldots,x_n}) = AL_{x_1} + AL_{x_1,x_2} + \ldots + AL_{x_1,x_2,\ldots,x_n}$ for $n$-element subset $\{L_{x_1}, L_{x_1,x_2}, \ldots, L_{x_1,x_2,\ldots,x_n}\}$ of $X_n$.

Following [8], let $A$ be a unital Banach algebra and let $\alpha > 1$. $A$-module $X$ is called quasi $\alpha$-injective if, $\varphi : N \to X$ is $A$-module homomorphisms such that
\[ \| \varphi \| \leq 1, \] there exists \( A \)-module homomorphism \( \theta : X \to X \), such that \( \theta \circ i = \varphi \) and \( \| \theta \| \leq \alpha \) where \( i \) is an isometry from submodule \( N \) of \( X \). We shall say that \( X \) is quasi injective if it is quasi \( \alpha \) - injective for some \( \alpha \). The concepts quasi \( (m, n) - \alpha \) - injective for some \( \alpha \) and multiplication \( (m, n) - B - A \) - module has been introduced.

**Definition 2.8.** Let \( A \) be a unital Banach algebra and let \( \alpha > 1 \). \( A \)-module \( X \) is called quasi \( (m, n) - \alpha \) - injective if, \( \| \varphi \| \leq 1 \), there exists \( A \)-module homomorphism \( \theta : X \to X \), such that \( \theta \circ i = \varphi \) and \( \| \theta \| \leq \alpha \) where \( i \) is an isometry from \( n \)-generated submodule \( N \) of \( X \). We shall say that \( X \) is quasi \( (m, n) \) - injective if it is quasi \( (m, n) - \alpha \) - injective for some \( \alpha \).

**Definition 2.9.** \( B - A \)-module \( X \) is called multiplication \( (m, n) - A \)-module if each \( n \)-generated submodule of \( X \) is of the form \( KX_n \) for some ideal \( K \) of \( A^{m \times n} \).

**Proposition 2.10.** Let \( X \) be multiplication \( (m, n) - B - A \)-module. If \( X \) is quasi \( (m, n) - B - A \)-module then \( X \) is \( F - (m, n) - S - B - A \)-module.

**Proof.** Let \( N \) be \( n \)-generated submodule of \( X \), let \( \alpha > 1 \) and \( f \) be any \( A \)-module homomorphism from \( N \) to \( X^m \) such that \( \| f \| \leq 1 \). Since \( X \) is multiplication \( (m, n) - B - A \)-module, then \( N = KX_n \), and since \( X \) is quasi \( (m, n) - B - A \)-module, then there exist \( A \)-module homomorphism \( g : X^m \to X^m \) such that \( f(N) = g(N) = g(KX_n) = Kg(X_n) \subseteq KX_n = N \).

**References**


