Studying the solutions of the delay Sturm Liouville problems

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Abstract. In this paper, the delay Sturm Liouville problems are introduced with the sufficient and necessary conditions, where the solutions of these problems are studied. Besides, definitions, remarks, examples, theorems, and corollaries are submitted to illustrate the delay Sturm Liouville problems properties. In addition, the inverse of the DSLP are shown in section two. Furthermore, the applications of the results in the second section are given in the third section.

Keywords: eventually positive, eventually negative, oscillatory

Introduction

Delay differential equations appear in a large number of fields in science, for instance in biology, mechanics, economic, mathematics, etc. Moreover, the solutions of the delay differential equations are studied by many authors like B. Mehmet, B. Azad, S. Erdogan [1] and K. Gopalsamy [2]. Besides, G. Ladas and I. P. Stavroulakis [3]. Note that, the concepts of eventually positive and eventually negative solutions are introduced by Peiguang W. [4], while the definition of oscillatory solution is given by P. Gimenes Luciene [5].

As well as, the Sturm Liouville Problems are also studied by numerous authors, for instance, they are studied by S.A. Buterin and two others in [6], besides B. Erdal, O. Ramazan [7]. Where M. Seyfollah [8], P. Milenko, V. Vladimir, M. Olivera [9], and C. Elmir, P. Milenko [10] are discussed the inverse SturmLiouville problems.

Studying the behavior of solutions of the delay Sturm Liouville Problems is interested in this work, where the delay Sturm Liouville Problems in this paper are denoted by DSLP. Add to that, the DSLP formulas are introduced with the sufficient and necessary conditions. Furthermore, definitions, remarks, examples, theorems, and corollaries are submitted to illustrate the delay Sturm Liouville problems properties. While the inverse of the DSLP are shown in section two. Also, the results in the second section are discussed on a form of applications in the third section, where they are shown as in examples form.

The Delay Sturm Liouville Problems (DSLP)

In this section, the DSLP as well as the sufficient and necessary conditions are introduced. Where definitions, remarks, examples, theorems, and corollaries are submitted to illustrate DSLP properties. Besides, the inverse of the DSLP is given.

Definition 1. *The DSLP are the following equation and the two inequalities with the boundary conditions, in addition to the sufficient and necessary conditions below*

$$[r(t)y'(t)]' + \alpha(t)y(t) + \lambda\rho(t)y(t-\tau) = 0, \tag{1}$$

$$[r(t)y'(t)]' + \alpha(t)y(t) + \lambda\rho(t)y(t-\tau) \leqslant 0, \tag{2}$$

$$[r(t)y'(t)]' + \alpha(t)y(t) + \lambda\rho(t)y(t-\tau) \geqslant 0. \tag{3}$$

Where $r'(t) > 0, \alpha(t) \geqslant 0, \rho(t) > 0$ are continuous functions on some interval $a \leqslant t \leqslant b$, λ is a positive parameter, and τ is a positive constant. The boundary conditions are

$$a_1 y(a) - a_2 y'(a) = 0, \quad b_1 y(b) - b_2 y'(b) = 0 \tag{4}$$

at the end points of the interval, and require that at least on coefficient in each equation be nonzero. Besides, the necessary and sufficient conditions are listed in theorem 1 below under which the following are satisfied:

- *Equation 1 has oscillatory solutions only;*
- *Inequality 2 has eventually negative solutions only;*
- *Inequality 3 has eventually positive solutions only.*

Remark 1. Notice that, the previous results are because of a lateness argument, which they do not exist when $\tau = 0$, as the following example shows

Example 1. Let the DSLP

$$y''(t) - (1+\lambda^2)y(t) \geqslant 0.$$

Then it has a negative solution $y(t) = -e^{-nt}, n = 1, 2, \dots$ such that $n < \lambda$

Theorem 1. *Suppose that the DSLP 2 exists with $r'(s) = 1$, moreover*

$$\lim_{\tau\to\infty} \inf \int_{t-\tau}^{t} \lambda\rho(s)ds > -\lim_{\tau\to\infty} \inf \int_{t-\tau}^{t} \alpha(s)ds \tag{5}$$

and

$$\lim_{\tau\to\infty} \inf \int_{t-\tau}^{t} \lambda\rho(s)ds > 0. \tag{6}$$

Then 2 has eventually negative solutions only.

Proof. Let $y(t)$ be a solution to 2. To show that $y(t)$ is eventually positive, which leads to a contradiction. So,

$$y(t) > 0, \quad t > t_0 \quad \Rightarrow \quad y(t-\tau) > 0, \qquad\qquad t > t_0 + \tau.$$

Also from 2,

$$y''(t) < 0, \quad t > t_0 + \tau \quad \Rightarrow \quad y(t) < y(t-\tau), \quad t > t_0 + 2\tau.$$

Now, dividing both sides of 2 by $y(t)$ and get the following

$$\frac{[r(t)y'(t)]'}{y(t)} + \alpha(t) + \lambda\rho(t)\frac{y(t-\tau)}{y(t)} \leqslant 0, t > t_0 + 2\tau,$$

$$\int_{t-\tau}^{t} \frac{[r(s)y'(s)]'}{y(s)} ds + \int_{t-\tau}^{t} \alpha(s) ds + \frac{y(t-\tau)}{y(t)} \int_{t-\tau}^{t} \lambda\rho(s) ds \leqslant 0, t > t_0 + 3\tau. \tag{7}$$

But $r'(s) = 1$ by assumption, besides to solve the first integral from the left of 7 consider

$$y(s) = \tan\theta, \quad y'(s) = \sec^2\theta, \quad y''(s) = 2\sec\theta\sec\theta\tan\theta.$$

The above compensation is using as follows

$$\int_{t-\tau}^{t} \frac{[r(s)y'(s)]'}{y(s)} ds = \int_{t-\tau}^{t} \frac{2\sec\theta\sec\theta\tan\theta}{\tan\theta} d\theta = 2y(t) - 2y(t-\tau).$$

Now, substitute it in 7

$$2y(t) - 2y(t-\tau) + \int_{t-\tau}^{t} \alpha(s) ds + \frac{y(t-\tau)}{y(t)} \int_{t-\tau}^{t} \lambda\rho(s) ds \leqslant 0, \qquad\qquad t > t_0 + 3\tau.$$

Dividing the previous inequality by $y(t-\tau)$, then $\forall t > t_0 + 3\tau$

$$2\frac{y(t)}{y(t-\tau)} - 2 + \frac{1}{y(t-\tau)} \int_{t-\tau}^{t} \alpha(s) ds + \frac{1}{y(t)} \int_{t-\tau}^{t} \lambda\rho(s) ds \leqslant 0.$$

So,

$$\frac{1}{y(t-\tau)} \int_{t-\tau)}^{t} \alpha(s) ds + \frac{1}{y(t)} \int_{t-\tau}^{t} \lambda\rho(s) ds \leqslant 2 - 2\frac{y(t)}{y(t-\tau)}.$$

Furthermore, since $\frac{y(t)}{y(t-\tau)} < 1$, then

$$\frac{1}{y(t-\tau)} \int_{t-\tau)}^{t} \alpha(s) ds + \frac{1}{y(t)} \int_{t-\tau}^{t} \lambda\rho(s) ds < 0.$$

Hence,

$$\int_{t-\tau}^{t} \lambda\rho(s) ds < -\frac{y(t)}{y(t-\tau)} \int_{t-\tau}^{t} \alpha(s) ds,$$

and again since $\frac{y(t)}{y(t-\tau)} < 1$, then

$$\int_{t-\tau}^{t} \lambda\rho(s)ds < -\int_{t-\tau}^{t} \alpha(s)ds.$$

Here, take the limit inferiors on both sides of the preceding inequalities, that is caused to get the following result

$$\lim_{t\to\infty} \inf \int_{t-\tau}^{t} \lambda\rho(s)ds < -\lim_{t\to\infty} \inf \int_{t-\tau}^{t} \alpha(s)ds,$$

which contradicts the hypothesis 5. Presently you can contradict 6 as follows

Take the integrals in 7 from $t - \frac{\tau}{2}$ to t for all $t > t_0 + \frac{\tau}{2}$ and follow the same way of steps in the current proof to get

$$\lim_{t\to\infty} \inf \int_{t-\frac{\tau}{2}}^{t} \lambda\rho(s)ds < -\lim_{t\to\infty} \inf \int_{t-\frac{\tau}{2}}^{t} \alpha(s)ds.$$

But

$$-\lim_{t\to\infty} \inf \int_{t-\frac{\tau}{2}}^{t} \alpha(s)ds < 0.$$

That leads to

$$\lim_{t\to\infty} \inf \int_{t-\frac{\tau}{2}}^{t} \lambda\rho(s)ds < 0.$$

As you see, it contradicts 6. □

Theorem 2. *Suppose that the DSLP 3 exists and*

$$\lim_{t\to\infty} \inf \int_{t-\tau}^{t} \lambda\rho(s)ds > 0.$$

Then, 3 has eventually positive solutions only.

Proof. Firstly assume that $y(t)$ is a solution to 3. To prove that $-y(t)$ is eventually negative, which leads to a contradiction. Then

$$-y(t) < 0, \qquad t > t_0.$$

By multiplying both sides by -1, so the results will be

$$y(t) > 0, \qquad t > t_0 \quad \Rightarrow \quad y(t-\tau) > 0, \qquad t > t_0 + \tau$$

3 gives

$$y''(t) > 0, \quad t > t_0 + \tau \quad \Rightarrow \quad y(t) < y(t-\tau), \quad t > t_0 + 2\tau.$$

And continue the proof by the similar steps as in a proof of theorem 1 to get a contradiction. □

Corollary 1. *Consider that the DSLP 1 exists and*

$$\liminf_{t\to\infty} \int_{t-\tau}^{t} \lambda\rho(s)ds > 0.$$

Then 1 has oscillatory solutions only.

Proof. Remember that a solution is oscillatory, if it is neither eventually nor negative. Hence the proof is similar to the steps of a proof of the theorem 1 and theorem 2 by assuming the converse to get a contradiction. □

Definition 2. *Let* $r'(t) > 0, \alpha(t) \geqslant 0, \rho(t) > 0$ *be constants on some interval* $a \leqslant t \leqslant b$, λ *is a positive parameter, and* τ *is a positive constant.Then the DSLP becomes*

$$[ry'(t)]' + \alpha y(t) + \lambda\rho y(t-\tau) = 0, \tag{8}$$

$$[ry'(t)]' + \alpha y(t) + \lambda\rho y(t-\tau) \leqslant 0, \tag{9}$$

$$[ry'(t)]' + \alpha y(t) + \lambda\rho y(t-\tau) \geqslant 0. \tag{10}$$

The boundary conditions are

$$a_1 y(a) - a_2 y'(a) = 0, \quad b_1 y(b) - b_2 y'(b) = 0.$$

Where the conditions 5 and 6 are reduced to

$$\lambda\rho\tau > \alpha, \qquad \alpha \geqslant 0. \tag{11}$$

Corollary 2. *The necessary and sufficient condition is* $\lambda\rho\tau > \alpha, \alpha \geqslant 0$, *for which the statements below hold:*

- *Equation 8 has oscillatory solutions only;*
- *Inequality 9 has eventually negative solutions only;*
- *Inequality 10 has eventually positive solutions only.*

Theorem 3. *Suppose that the inverse of the DSLP as follows*

$$[r(t)y'(t)]' - \alpha(t)y(t) - \lambda^4\rho(t)y(t-2n\tau) = 0,$$

where $r(t) = 1, \alpha(t) \geqslant 0, \rho(t) = 1, n = 1, 2, \ldots$ *are continuous functions on some interval* $a \leqslant t \leqslant b$. *Where* λ *is a positive parameter, and* τ *is a positive constant with the boundary condition 4. Then each bounded solutions of the inverse of the DSLP are oscillatory.*

Proof. Suppose the converse, that there exists a bounded solution $y(t)$ such that

$$y(t) > 0, \qquad t > t_0.$$

Hence

$$y(t-n\tau) > 0, \qquad t > t_0 + n\tau, \quad n = 1, 2, \dots$$

$$y''(t) > 0, \qquad t > t_0 + n\tau, \quad n = 1, 2, \dots$$

Because of $y(t)$ is bounded, it follows that

$$y'(t) < 0, \qquad t > t_0 + n\tau, \quad n = 1, 2, \dots$$

Put the following equation for sufficiently large t is negative, where $n = 1, 2, \dots$

$$x(t) = r(t)y'(t) - \lambda^2 \rho(t) y(t-n\tau). \tag{12}$$

At the present time, derivation of both sides of 12, with considering that $r(t) = 1, \rho(t) = 1$ by assumption, so 12 becomes as follows

$$x(t) = y'(t) - \lambda^2 y(t-n\tau),$$

$$x'(t) = y''(t) - \lambda^2 y'(t-n\tau),$$

$$x'(t) + \lambda^2 x(t-n\tau) = y''(t) - \lambda^2 y'(t-n\tau) + \lambda^2 y'(t-n\tau) - \lambda^4 y(t-2n\tau)$$

and substitute instead of $y''(t)$, then it follows

$$x'(t) + \lambda^2 x(t-n\tau) = \alpha(t)y(t) \geqslant 0.$$

Hence

$$x'(t) + \lambda^2 x(t-n\tau) \geqslant 0. \tag{13}$$

Now, notice that $\lambda^2 \rho\tau > \alpha, \alpha \geqslant 0$ is satisfied. So 13 has eventually positive solutions only. This results leads to a contradiction with the assumption 12. □

Applications

This section consists of some applications on the results in the second section, which is represented by the examples that are listed in the current section. Let the DSLP mentioned as below.

Example 2. Let the DSLP mentioned as below

$$y''(t) + 2y(t) + \lambda y(t-\pi) = 0,$$

where $y(0) = 0, y(\pi) = 0, \lambda = 1$. Then DSLP has the following oscillatory solutions, when $\lambda = 1$.

1. y(t)= sint

2. y(t)= -sint
3. y(t)= cost
4. y(t)= -cost

Also it satisfies condition 11, because $r(t) = 1, \alpha(t) = 2, \rho(t) = 1, \lambda\rho\tau = \pi > \alpha = 2$.

Example 3. Put the DSLP

$$y''(t) + \lambda y(t - \pi) \leqslant 0,$$

where $y(0) = 0, y(\pi) = 0, \lambda = k^2, k = 1, 2, \ldots$. Then DSLP when $\lambda = 1$ has an eventually negative solution $y(t) = -cost$, where $t = (2n + 1)\pi, \quad n = 0, \mp1, \mp2, \mp3, \ldots$ Since, when we substitute $y(t) = -cost$ and $y''(t) = cost$ in $y''(t) + \lambda y(t - \pi)$, then we get it equal to $2cost$. And it satisfies condition 11.

Example 4. Suppose that the DSLP

$$y''(t) + \lambda y(t - \pi) \geqslant 0,$$

where

$$y(0) = 0, y(\pi) = 0, \lambda = k^2, k = 1, 2, \ldots$$

Then the DSLP when $\lambda = 1$ has an eventually positive solution

$$y(t) = cost, \quad when \quad t = (2n + 1)\pi, \quad n = 0, \mp1, \mp2, \mp3, \ldots$$

Since, when we substitute $y(t) = cost$ and $y''(t) = -cost$ in $y''(t) + \lambda y(t - \pi)$, then we get it equal to $-2cost$. furthermore, it satisfies condition 11.

Example 5. Consider

$$y''(t) + e^{-t}y(t) + \lambda y(t - 1) \geqslant 0, \qquad \lambda = e^{-2t}.$$

Then the DSLP does not have eventually positive solution, since the condition 11 does not hold That is, $\lambda\rho\tau = e^{-2t}$ is not greater than $\alpha = e^{-t}$

Conclusion

The aim of this paper is studying the behaviour of solutions of the DSLP, and how they would be oscillatory, eventually positive, and eventually negative. Where Theorems, corollaries, examples, and remarks are given to explain each case. Moreover, the sufficient and necessary conditions are introduced with the DSLP forms. The inverse of the DSLP are also introduced in section two. Besides, the applications of the results in the second section are given, which they are represented by examples in the third section.

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