Predator-prey model of Holling-type II with harvesting and predator in disease

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Abstract. A modified predator-prey model is introduced with Holling -type II. Including a constant rate of harvesting in both infected predators by prey, and predators who are prone to disease. An existence of positive biological equilibrium and uniformly boundedness of the present system are obtained as well. Furthermore, the local stability conditions are defined based on Routh-Hurwitz criteria. Finally, an effective Lyapunov function was performed to check the global asymptotic stability of the interior equilibrium point.

Keywords: predator-prey, Holling-type II, harvesting, boundness, stability, Lyapunov function.

1. Introduction

The active research area on classical applications of mathematics to biology, is the study of interactions between populations of various species, by using autonomous differential equations modeling a predator-prey systems [1, 2, 11, 12]. The Lotka-Volterra model was the first to study interactions between predators and prey in 1927. Together with further developed and extended researches such as [3], who divided the prey populations into susceptible and infected. This dynamic relationship between predators and their prey will take into account some aspects that are considered essential to explain the dynamics. Lesile-Gower model has investigated several researchers such as [4]. They studied the boundedness of positive equilibrium points and global stability. Sufficient conditions for the existence and global stability of the model's positive periodic solutions were discussed in [13].

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The goal of this paper is to give a study of three-dimensional system incorporating a modified version of Lesile-Gower Holling-type II, $p(x) = \frac{x}{m+x}$, where m is the environment that provides production to prey. In our system there are preys and an infected predators which are harvested continuously. However, there are a susceptible predators living on prey and are not harvested. Harvesting infected mathematical dynamics because otherwise, it can lead population density to a dangerously low level of extinction. There are many researches on harvesting such as [5, 9, 10]. Important to note is that our system proposed predators are exposed to the risk of disease. In both mathematical and ecological terms, the effect of disease on the ecological system is an important issue. As a result, many researches [6, 7, 8] proposed and studied different predator-prey models in presence of disease.

This paper is organized as what follows. In section 2, we describe our system (1) then we reduce the number of parameters to get system (2). Next, in section 3, we studied the uniformly boundedness of all positive solutions. Existence of positive equilibrium points are discussed in section 4. After that in section 5, behavior of solutions at each equilibrium points are discussed. Finally, global stability of interior equilibrium point is studied in section 6.

2. The model

We will consider our model under the framework of the following nonlinear differential equations:

$$
\begin{aligned}
\frac{dx}{dt} &= rx(1 - \frac{x}{k}) - \frac{ayx}{m+x} - azx - h_1 x, \\
\frac{dy}{dt} &= bxy + \alpha yz + \frac{\gamma yx}{m+x} - h_2 y, \\
\frac{dz}{dt} &= bzx - \alpha yz - dz.
\end{aligned} \tag{1}
$$

Here, x, y and z are the prey, infected predator and susceptible predator, respectively and $r, k, a, b, \gamma, \alpha, h_1, h_2, d$ are assumed to be positive constants. From the biological point of view, we are only interested in the dynamics of system (1) in the closed octant $\mathbb{R}^3_+$. Thus, we consider the initial conditions are $x(0) = x_0 \geq 0, y(0) = y_0 \geq 0, z(0) = z_0 \geq 0$.

To reduce the number of parameters, we non-dimensionalize system (1) with the following scaling:

$$
X = \frac{x}{k},\ Y = \frac{ay}{rm},\ Z = \frac{az}{rm},\ T = rt.
$$

Table 1: **Definition of parameters in the model**

Parameter	**Definition**
r	The logistic growth rate of the prey in the absence of predators.
k	The enviromental carrying capacity.
a, b	The capture rates with $(a > b)$.
α	The interaction between y, z.
h_1, h_2	The rates of harvesting where $(h_1 > h_2)$.
γ	The interaction rate of infected predator species.
d	The natural death rate in the absence of prey.

Then system (1) takes the form (after some simplification):

$$\begin{aligned}
\frac{dX}{dT} &= X(1-X) - \frac{YX}{1+\beta X} - mZX - \delta_1 X, \\
\frac{dY}{dT} &= cXY + eYZ + \frac{nYX}{1+\beta X} - \delta_2 Y, \\
\frac{dZ}{dT} &= cXZ - eYZ - wZ,
\end{aligned} \tag{2}$$

where

$$\frac{k}{m} = \beta,\ \frac{h_1}{r} = \delta_1,\ \frac{bk}{r} = c,\ \frac{d}{r} = w,\ \frac{h_2}{r} = \delta_2,\ \frac{\gamma k}{rm} = n,\ \frac{\alpha m}{a} = e.$$

3. Boundedness of all positive solutions

Theorem 1. *All solutions of system* (2) *that start in* $\mathbb{R}^3_+$ *are uniformly bounded and remain positive, in order to be meaningful from a biological viewpoint.*

Proof. Assuming that $(X(T), Y(T), Z(T))$ be any positive solution of system (2). Let $Q(T) = nX + Y + Z$. Hence

$$\frac{dQ}{dT} + \mu Q \leq -n\Big(X - \frac{D}{2}\Big)^2 + n\Big(\frac{D}{2}\Big)^2,$$

where $D = (1-\delta_1) + \mu$, $(\delta_1 < 1)$, therefore

$$\frac{dQ}{dT} + \mu Q \leq n\Big(\frac{D}{2}\Big)^2.$$

Solving the differential inequality, we obtain

$$Q(T) \leq \frac{nD^2}{4\mu} + ce^{-\mu T},$$

for $T \to \infty$, all solutions of system (2) enter into the region

$$B = \{(X, Y, Z) : 0 \leq Q \leq \frac{nD^2}{4\mu}\}.$$

□

4. Equilibrium points

System (2) has the following points of equilibrium:

(i) The trivial equilibrium $E_0(0,0,0)$.

(ii) The predators free equilibrium $E_1(1-\delta_1,0,0)$.

(iii) The infected predator free equilibrium $E_2(X_2,0,Z_2)$, where $X_2 = \frac{w}{c}$ and $Z_2 = \frac{c(1-\delta_1)-w}{m}$ if $\frac{c(1-\delta_1)}{w} > 1$.

(iv) The susceptible predator free equilibrium $E_3(X_3,Y_3,0)$, from system (2) we get $c\beta X^2 + ((c+n)-\delta_2\beta)X - \delta_2 = 0$, we have one positive real root given by $X_3 = \frac{-((c+n)-\delta_2\beta)+\sqrt{((c+n)-\delta_2\beta)^2+4c\beta\delta_2}}{2c\beta}$, therefore $Y_3 = ((1-\delta_1) - X_3)(1+\beta X_3)$ if $(1-\delta_1) > X_3$ hold.

(v) The interior equilibrium $E^*(X^*,Y^*,Z^*)$, given by:

$$\begin{aligned} 1 - X^* - \frac{Y^*}{1+\beta X^*} - mZ^* &= \delta_1, \\ cX^* + eZ^* + \frac{nX^*}{1+\beta X^*} &= \delta_2, \\ cX^* - eY^* &= w, \end{aligned} \tag{3}$$

from (3) we get $X^* = \frac{w+eY^*}{c}$, $Z^* = \frac{(1+\beta X^*)(\delta_2 - cX^*)-nX^*}{e(1+\beta X^*)}$, if $(X^* < \frac{\delta_2}{c})$. Therefore

$$D_1Y^2 + D_2Y - D_3 = 0, \tag{4}$$

where

$$\begin{aligned} D_1 &= \beta e^2(m - \frac{e}{c}), \\ D_2 &= me\big(c+n+\beta(2w-\delta_2) - \frac{c}{m}\big) - e^2(1-\beta(1-\frac{2w}{c}-\delta_1)), \\ D_3 &= m(c\delta_2 + w(\beta(\delta_2 - w) - (c+n))) \\ &\quad + ew(\beta((1-\delta_1) - \frac{w}{c}) - 1) - ec(1-\delta_1). \end{aligned} \tag{5}$$

We have one positive root for equation (4) given by:

$$Y^* = \frac{-D_2 + \sqrt{D_2^2 + 4D_1D_3}}{2D_1},$$

if the following conditions hold, $D_1 > 0 \iff mc > e$, and $D_3 > 0 \iff \beta\delta_2 > 1, \beta c(1-\delta_1) > 1$, and $ec(1-\delta_1) < 1$.

5. Behaviour of solutions

First, we need to compute the Jacobian matrix of system (2) for general (X, Y, Z).

$$J(X,Y,Z) = \begin{bmatrix} 1-2X-\frac{Y}{(1+\beta X)^2}-mZ-\delta_1 & -\frac{X}{1+\beta X} & -mX \\ cY+\frac{nY}{(1+\beta X)^2} & cX+eZ+\frac{nX}{1+\beta X}-\delta_2 & eY \\ cZ & -eZ & cX-eY-w \end{bmatrix}.$$

We will evaluated $J(X,Y,Z)$ at each equilibrium points. Let J_0 denoted the Jacobian matrix at E_0,

$$J_0 = \begin{bmatrix} 1-\delta_1 & 0 & 0 \\ 0 & -\delta_2 & 0 \\ 0 & 0 & -w \end{bmatrix},$$

this immediately shows that E_0 is saddle point (unstable).

Let J_1 denoted the Jacobian matrix at E_1,

$$J_1 = \begin{bmatrix} -(1-\delta_1) & \frac{-(1-\delta_1)}{1+\beta(1-\delta_1)} & -m(1-\delta_1) \\ 0 & \frac{(1-\delta_1)[c(1+\beta(1-\delta_1))+n]}{1+\beta(1-\delta_1)}-\delta_2 & 0 \\ 0 & 0 & c(1-\delta_1)-w \end{bmatrix},$$

so the eigenvalues are:

$$\begin{aligned} &\lambda_1 = -(1-\delta_1) < 0, \\ &\lambda_2 = \frac{(1-\delta_1)[c(1+\beta(1-\delta_1))+n]}{1+\beta(1-\delta_1)} - \delta_2, \\ &\lambda_3 = c(1-\delta_1) - w > 0. \end{aligned} \tag{6}$$

thus, from (6)E_1 is saddle point (unstable).

Let J_2 denoted the Jacobian matrix at E_2,

$$J_2 = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ 0 & A_{22} & 0 \\ A_{31} & A_{32} & 0 \end{bmatrix},$$

where

$$A_{11} = 1-2X_2-mZ_2-\delta_1,\ A_{12} = \frac{-X_2}{1+\beta X_2},$$
$$A_{13} = -mX_2,\ A_{22} = cX_2+eZ_2+\frac{nX_2}{1+\beta X_2}-\delta_2,\ A_{31} = cZ_2,\ A_{32} = -eZ_2.$$

The characteristic equation of the Jacobian matrix J_2 is given by:

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0,$$

where

$$\begin{aligned} a_1 &= -(A_{11} + A_{22}), \\ a_2 &= A_{11}A_{22} - A_{13}A_{31}, \\ a_3 &= A_{31}A_{13}A_{22}. \end{aligned} \tag{7}$$

Hence,

$$a_1a_2 - a_3 = -A_{11}A_{22}(A_{11} + A_{22}) + A_{11}A_{13}A_{31}.$$

If $A_{11} < 0,\ A_{22} < 0$ then $a_1 > 0,\ a_3 > 0$, and $a_1a_2 - a_3 > 0$, then by using Routh-Hurwitz criterion E_2 is asymptotically stable.

Let J_3 denoted the Jacobian matrix at E_3,

$$J_3 = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ 0 & 0 & B_{33} \end{bmatrix},$$

where

$$\begin{aligned} &B_{11} = 1 - 2X_3 - \frac{Y_3}{(1+\beta X_3)^2} - \delta_1,\ B_{12} = \frac{-X_3}{1+\beta X_3,},\ B_{13} = -mX_3, \\ &B_{21} = cY_3 + \frac{nY_3}{(1+\beta X_3)^2},\ B_{22} = cX_3 + \frac{nX_3}{1+\beta X_3} - \delta_2, \\ &B_{23} = eY_3,\ B_{33} = cX_3 - eY_3 - w. \end{aligned}$$

The characteristic equation of the Jacobian matrix J_3 is given by:

$$\lambda^3 + b_1\lambda^2 + b_2\lambda + b_3 = 0,$$

where

$$\begin{aligned} b_1 &= -(B_{11} + B_{22} + B_{33}), \\ b_2 &= B_{11}B_{22} + B_{11}B_{33} + B_{22}B_{33} - B_{12}B_{21}, \\ b_3 &= B_{12}B_{21}B_{33} - B_{11}B_{22}B_{33}. \end{aligned} \tag{8}$$

Hence,

$$\begin{aligned} b_1b_2 - b_3 = [&-(B_{11} + B_{22} + B_{33})(B_{11}B_{22} + B_{11}B_{33} + B_{22}B_{33}) + B_{12}B_{21}B_{11} \\ &+ B_{12}B_{21}B_{22}] + B_{11}B_{22}B_{33}. \end{aligned} \tag{9}$$

Let $M_1 = B_{11}B_{22}B_{33}$. If $B_{11} < 0,\ B_{22} < 0$, and $B_{33} < 0$, then $b_1 > 0,\ b_3 > 0, M_1 < 0$, and the first bracket in (9) is positive.

Thus, if $M_1 < [-(B_{11}+B_{22}+B_{33})(B_{11}B_{22}+B_{11}B_{33}+B_{22}B_{33})+B_{12}B_{21}B_{11}+B_{12}B_{21}B_{22}]$, then by using Routh-Hurwitz criterion E_3 is asymptotically stable.

Finally, let J^* denoted the Jacobian matrix at E^*,

$$J^* = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix},$$

where

$$\begin{aligned}
&C_{11} = 1 - 2X^* - \frac{Y^*}{(1+\beta X^*)^2} - mZ^* - \delta_1,\ C_{12} = \frac{-X^*}{1+\beta X^*},\ C_{13} = -mX^*,\\
&C_{21} = cY^* + \frac{nY^*}{(1+\beta X^*)^2},\\
&C_{22} = cX^* + eZ^* + \frac{nX^*}{1+\beta X^*} - \delta_2,\ C_{23} = eY^*,\ C_{31} = cZ^*,\ C_{32} = -eZ^*,\\
&C_{33} = cX^* - eY^* - w.
\end{aligned}$$

The characteristic equation of the Jacobian matrix J^* is given by:

$$\lambda^3 + c_1\lambda^2 + c_2\lambda + c_3 = 0,$$

where

$$\begin{aligned}
c_1 &= -(C_{11} + C_{22} + C_{33}),\\
c_2 &= (C_{11}C_{22} + C_{11}C_{33} + C_{22}C_{33}) - (C_{12}C_{21} + C_{13}C_{31} + C_{23}C_{32}),\\
c_3 &= (C_{12}C_{21}C_{33} + C_{13}C_{31}C_{22} + C_{11}C_{23}C_{32})\\
&\quad - (C_{11}C_{22}C_{33} + C_{12}C_{23}C_{31} + C_{13}C_{21}C_{32}).
\end{aligned} \tag{10}$$

Hence,

$$\begin{aligned}
c_1c_2 - c_3 = {} & [-(C_{11} + C_{22} + C_{33})(C_{11}C_{22} + C_{11}C_{33} + C_{22}C_{33})\\
& + C_{11}C_{12}C_{21} + C_{11}C_{13}C_{31} + C_{22}C_{12}C_{21} + C_{22}C_{23}C_{32} + C_{33}C_{13}C_{31}\\
& + C_{33}C_{23}C_{32}] + (C_{11}C_{22}C_{33} + C_{12}C_{23}C_{31} + C_{13}C_{21}C_{32}).
\end{aligned} \tag{11}$$

If $C_{11} < 0,\ C_{22} < 0$, and $C_{33} < 0$, then $c_1 > 0$, also the first parenthesis of c_3 in (10) and the square bracket in (11) are positive.

Let $M_2 = [-(C_{11} + C_{22} + C_{33})(C_{11}C_{22} + C_{11}C_{33} + C_{22}C_{33}) + C_{11}C_{12}C_{21} + C_{11}C_{13}C_{31} + C_{22}C_{12}C_{21} + C_{22}C_{23}C_{32} + C_{33}C_{13}C_{31} + C_{33}C_{23}C_{32}]$, and $M_3 = C_{11}C_{22}C_{33} + C_{12}C_{23}C_{31} + C_{13}C_{21}C_{32}$,then M_2 maybe positive or negative or zero. We have the following theorem using the Routh-Hurwitz criterion.

Theorem 2. *Suppose that $E^*(X^*, Y^*, Z^*)$ exist. Let $C_{11} < 0,\ C_{22} < 0$, and $C_{33} < 0$. Then E^* is asymptotically stable in one of the following cases:*

(i) M_3=0, *or*

(ii) $0 < M_3 < C_{12}C_{21}C_{33} + C_{13}C_{31}C_{22} + C_{11}C_{23}C_{32}$, *or*

(iii) $-M_2 < M_3 < 0$

6. Global stability of interior equilibrium point

Theorem 3. *The interior equilibrium E^* is globally asymptotically stable if the following conditions hold.*

$$1+\beta X^* > \frac{n(4c\mu\epsilon + D^2(n+1))}{4\mu\epsilon(n(1+m)-c)}, \tag{12}$$

$$1+\beta X^* < \max\Big\{\frac{4ne\mu\epsilon - nD^2}{4\mu\epsilon(c+e)}, \frac{n(e-c)}{e-nm}\Big\}, \tag{13}$$

$$c < n + nm < e + n. \tag{14}$$

Proof. We will construct a Lyapunov function V which is continuous and defined on $\mathbb{R}^3_+$. The function V should be zero at E^* and positive for all other values X, Y and Z. Let us define the function V as follow:

$$V(X,Y,Z) = L_1(X - X^* - X^* \ln\frac{X}{X^*}) + L_2(Y - Y^* - Y^* \ln\frac{Y}{Y^*}) + (Z - Z^* - Z^* \ln\frac{Z}{Z^*}),$$

where $L_1 = 1+\beta X^*$ and $L_2 = \frac{1+\beta X^*}{n}$.

The time derivative of V along the solution of (2) is:

$$\begin{aligned}\frac{dV}{dt} =& L_1(X-X^*)\Big(1 - X - \frac{Y}{1+\beta X} - mZ - \delta_1\Big)\\ &+ L_2(Y-Y^*)\Big(cX + eZ + \frac{nX}{1+\beta X} - \delta_2\Big)\\ &+ (Z-Z^*)(cX - eY - w),\end{aligned} \tag{15}$$

and using system (2), we get:

$$\begin{aligned}\frac{dV}{dt} =& L_1(X-X^*)\Big[-(X-X^*) - m(Z-Z^*)\\ &+ \frac{Y^*(1+\beta X) - Y(1+\beta X^*)}{(1+\beta X)(1+\beta X^*)}\Big]\\ &+ L_2(Y-Y^*)\Big[c(X-X^*) + e(Z-Z^*)\\ &+ \frac{nX(1+\beta X^*) - nX^*(1+\beta X)}{(1+\beta X)(1+\beta X^*)}\Big]\\ &+ (Z-Z^*)\Big[c(X-X^*) - e(Y-Y^*)\Big].\end{aligned} \tag{16}$$

Using some manipulation with the result of theorem 1 and assuming $\epsilon = \frac{1}{\beta}$, equation (16) takes the form

$$\begin{aligned}\frac{dV}{dt} \le& \Big[(1+\beta X^*)\Big(-1+\frac{c}{n} - m\Big) + c + \frac{D^2(n+1)}{4\mu\epsilon}\Big](X-X^*)^2\\ &+ \Big[(1+\beta X^*)\Big(\frac{c+e}{n}\Big) - \Big(e - \frac{D^2}{4\mu\epsilon}\Big)\Big](Y-Y^*)^2\\ &+ \Big[(1+\beta X^*)\Big(\frac{e}{n} - m\Big) + c - e\Big](Z-Z^*)^2.\end{aligned} \tag{17}$$

From (12)-(14), $\frac{dV}{dt}$ is negative definite. Finally, E^* is globally asymptotically stable. □

7. Conclusion

In this paper, we deal with a modified Holling-type II predator-prey model of one prey and two predators whom are exposed to the risk of disease. As well as constants of harvesting ($h_1 > h_2$) in both prey and infected predator species. Positive biological equilibrium points(E_0, E_1) are direct show unstability, where the local stability of (E_2, E_3, E^*) are discussed by using the Routh-Hurwitz criterion. In addition, global stability of interior equilibrium point (E^*) has been investigated by using a suitable Lyapunov function. In our study it is important to realize that illegal harvesting on prey occures a risk even in the absence of predator.

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