A new contraction and existence theorems on fuzzy metric space with a graph

Vishal Gupta*
Department of Mathematics
Maharishi Markandeshwar
Deemed to be University
Mullana, Haryana
India
vishal.gmn@gmail.com

Manu Verma
Department of Mathematics
Maharishi Markandeshwar
Deemed to be University
Mullana, Haryana
India
ammanu7@gmail.com

Jatinderdeep Kaur
School of Mathematics
Thapar Institute of Engineering and Technology
Patiala (Punjab)-147004
India
jkaur@thapar.edu

Abstract. In the present paper our aim is to develop coupled fixed point theorems in fuzzy metric space with graph. We introduce the concept of $\mathcal{J}$-γ-contraction mapping using the control function developed by Wardowski [16]. In current paper, we show the existence of coupled coincidence fixed point in fuzzy metric space with respect to graph. We also give the result having particular value of control function such that the $\mathcal{J}$-γ-contraction change to $\mathcal{J}$-fuzzy contraction.
Keywords: coupled fixed point, $\mathcal{J}$-γ-contraction mapping, fuzzy metric space($\mathcal{FM}$-space).

1. Introduction

Banach contraction principle remains as the backbone of fixed point theory to relate different areas like differential equations, integral equations, game theory etc. The concept of graph and fixed point theory were also combined to prove fixed point theorems in R-trees by Espinola and Kirk [5]. Jackymski [7] stepped in the fixed point theory with the language of graph theory and gave result with a directed graph on Banach contractions in a metric space. Metric space with

∗. Corresponding author

graph theory is the developing area in the field of research. Also, the coupled and common fixed point theorem were executed by Bhaskar and Lakshmikantham [1]. In 2014, Chifu and Petrusel [2] worked on coupled fixed point results in metric space endowed with directed graph. In the same year, Shukla [15] also defined G-fuzzy contraction on fuzzy metric space endowed with graph. Shukla's work in fuzzy metric space gave us a new direction to think with graphs. The justified extension of coupled fixed point result to fuzzy metric was done by Zhu et al. [23]. The control functions introduced by Wardowski [16] helped us to use the concept of coupled fixed point theorem in fuzzy metric space endowed with graph theory. Fuzzy metric space was introduced by Kramosil and Michalek [8] and modified by George and Veeramani [6] using the concept of fuzzy sets, introduced by Zadeh [22] on metric spaces. Different work has been done in fuzzy metric space by [10], [12]-[14], [17]-[21]. In present paper, we apply the concept of coupled fixed point theory on graph theory in fuzzy metric space to find common coupled fixed point.

Definition 1.1 ([11])**.** A binary operation $* : [0,1]^2 \to [0,1]$ is called continuous t-norm if the following properties are satisfied:

(i) $*$ is associative and commutative,

(ii) $u * 1 = u$ for all $u \in [0,1]$,

(iii) $u * v \le w * r$ whenever $u \le w$ and $v \le r$ for all $u, v, w, r \in [0,1]$,

(iv) $*$ is continuous.

George and Veeramani [6] introduced the following definition of fuzzy metric space. This definition of fuzzy metric space is utilized in our paper.

Definition 1.2 ([6])**.** The 3-tuple $(\mathcal{K}, M, *)$ is called a fuzzy metric space if $\mathcal{K}$ is an arbitrary non-empty set, $*$ is a continuous t-norm and M is a fuzzy set on $\mathcal{K}^2 \times (0,\infty)$ satisfying the following conditions for each $u, y, z \in \mathcal{K}$ and $t, s > 0$:

(FM1) $M(u, y, t) > 0$,

(FM2) $M(u, y, t) = 1$ if and only if $u = y$,

(FM3) $M(u, y, t) = M(y, u, t)$,

(FM4) $M(u, y, t) * M(y, z, s) \ge M(u, z, t + s)$, and

(FM5) $M(u, y, \cdot) : (0,\infty) \to [0,1]$ is continuous.

Definition 1.3 ([6])**.** Let $(\mathcal{K}, M, *)$ be a fuzzy metric space.

(i) A sequence $\{u_n\}$ in $\mathcal{K}$ is said to be convergent to a point $x \in \mathcal{K}$ if $\lim_{n\to\infty} M(u_n, x, t) = 1$ for all $t > 0$.

(ii) A sequence $\{u_n\}$ in $\mathcal{K}$ is called a Cauchy sequence if for each $0 < \epsilon < 1$ and $t > 0$, there exists a positive integer n_0 such that $M(u_n, u_m, t) > 1 - \epsilon$ for all $n, m \geq n_0$.

(iii) A fuzzy metric space in which every Cauchy sequence is convergent is said to be complete.

Definition 1.4 ([16])**.** Denote by W a family of mappings $\gamma : (0, 1] \to [0, \infty)$ satisfying the following two conditions:

(W1) γ transforms $(0, 1]$ onto $[0, \infty)$;

(W2) for all $s, t \in (0, 1]$, $s < t \Rightarrow \gamma(s) > \gamma(t)$ (i.e. γ is strictly decreasing).

Note that (W1) and (W2) imply $\gamma(1) = 0$ and $\gamma(\alpha_n) \to 0$ whenever $\alpha_n \to 1$ as $n \to \infty$.

Example of γ-function is $\gamma(t) = \frac{1}{t} - 1$, $t \in (0, 1]$.

Lemma 1.1 ([3])**.** *Let $(\mathcal{K}, M, *)$ be a fuzzy metric space and let $\gamma \in H$. The sequences $\{x_n\}$ and $\{y_n\}$ in $\mathcal{K}$ are convergent to the points $x \in \mathcal{K}$ and $y \in \mathcal{K}$ if $\lim_{n\to\infty} \gamma(M(x_n, x, t) * M(y_n, y, t)) = 0$ for all $t > 0$.*

Lemma 1.2 ([3])**.** *The sequence $\{x_n\}$ and $\{y_n\}$ in $\mathcal{K}$ are Cauchy sequences if for each $0 < \epsilon < 1$ and $t > 0$, there exists a positive integer n_0 such that $\gamma(M(x_n, x_m, t) * M(y_n, y_m, t)) \leq \epsilon$ for all $n, m \geq n_0$.*

The concepts of graphs are similar to those in [7]. Let $(\mathcal{K}, M, *)$ be a fuzzy metric space. Let a directed graph $\mathcal{J}$ such that the set $V(\mathcal{J})$ of its vertices, consider as elements of $\mathcal{K}$, the set $\mathcal{E}(\mathcal{J})$ of its edges contains all loops, i.e. $\mathcal{E}(\mathcal{J}) \supseteq \Delta$, where Δ denote the diagonals of Cartesian product $\mathcal{K} \times \mathcal{K}$. We identify $\mathcal{J}$ with the pair $(V(\mathcal{J}), \mathcal{E}(\mathcal{J}))$ having no parallel edges. We also treat $\mathcal{J}$ as weighted graph by assigning to each edge the fuzzy distance between its vertices.

Also, $\mathcal{J}^{-1}$ be the graph obtained from $\mathcal{J}$ by reversing the direction of edges, i.e.

$$\mathcal{E}(\mathcal{J}^{-1}) = \{(x, y) \in \mathcal{K} \times \mathcal{K} : (y, x) \in \mathcal{E}(\mathcal{J})\}.$$

Definition 1.5 ([9])**.** An element $(x, y) \in \mathcal{K} \times \mathcal{K}$ is called a coupled coincidence point of the functions $H : \mathcal{K} \times \mathcal{K} \to \mathcal{K}$ and $g : \mathcal{K} \to \mathcal{K}$ if

$$H(x, y) = gx \quad \text{and} \quad H(y, x) = gy.$$

Let us denote the set of all coupled coincidence points of H and g by $C(Hg)$.

Definition 1.6 ([9])**.** An element $(x, y) \in \mathcal{K} \times \mathcal{K}$ is called a coupled common fixed point of the functions $H : \mathcal{K} \times \mathcal{K} \to \mathcal{K}$ and $g : \mathcal{K} \to \mathcal{K}$ if

$$H(x, y) = g(x) = x \quad \text{and} \quad H(y, x) = g(y) = y.$$

Definition 1.7 ([9]). Let $\mathcal{K}$ be a non-empty set. Then the function $H : \mathcal{K}\times\mathcal{K} \to \mathcal{K}$ and $g : \mathcal{K} \to \mathcal{K}$ are said to be commutative if

$$g(H(x,y)) = H(gx, gy) \quad \text{for all } x, y \in \mathcal{K}.$$

Definition 1.8 ([7]). A function $\mathcal{J} : \mathcal{K} \to \mathcal{K}$ is $\mathcal{J}$-continuous if

(i) for all $x, x^* \in \mathcal{K}$ and any sequence $\{n_i\}_{i\in N}$ of positive integers, $\{x_{n_i}\} \to x^*$ and $(x_{n_i}, x_{n_{i+1}}) \in \mathcal{E}(\mathcal{J})$ for $n \in N$ implies $\{g(x_{n_i})\} \to gx^*$.

(ii) for all $y, y^* \in \mathcal{K}$ and any sequence $\{n_i\}_{i\in N}$ of positive integers, $\{y_{n_i}\} \to y^*$ and $(y_{n_i}, y_{n_{i+1}}) \in \mathcal{E}(\mathcal{J}^{-1})$, for $n \in N$, implies $\{g(y_{n_i})\} \to gy^*$.

Definition 1.9 ([2]). A function $H : \mathcal{K} \times \mathcal{K} \to \mathcal{K}$ in $\mathcal{J}$-continuous if for all $(x, y), (x^*, y^*) \in \mathcal{K} \times \mathcal{K}$ and any sequence $\{n_i\}_{i\in N}$ of positive integers, $\{x_{n_i}\} \to x^*$, $\{y_{n_i}\} \to y^*$ as $i \to \infty$ and $(x_{n_i}, x_{n_{i+1}}) \in \mathcal{E}(\mathcal{J})$, $(y_{n_i}, y_{n_{i+1}}) \in \mathcal{E}(\mathcal{J}^{-1})$ for $n \in N$, implies $\{H(x_{n_i}, y_{n_i})\} \to H(x^*, y^*)$ and $\{H(y_{n_i}, x_{n_i})\} \to H(y^*, x^*)$.

2. Main result

To find our main result, we first define some definitions and lemmas as follows:
First we define property (A) for graph in fuzzy metric space.

Definition 2.1. Let $(\mathcal{K}, M, *)$ be a complete fuzzy metric space endowed with a directed graph $\mathcal{J}$. Then the tuple $(\mathcal{K}, M, *, \mathcal{J})$ has the property (A) if

(i) for any sequence $\{x_n\}_{n\in N}$ in $\mathcal{K}$ such that $\{x_n\} \to x^*$ and $(x_n, x_{n+1}) \in \mathcal{E}(\mathcal{J})$, $(x_n, x^*) \in \mathcal{E}(\mathcal{J})$;

(ii) for any sequence $\{y_n\}_{n\in N}$ in $\mathcal{K}$ such that $\{y_n\} \to y^*$ and $(y_n, y_{n+1}) \in \mathcal{E}(\mathcal{J}^{-1})$, $(y_n, y^*) \in \mathcal{E}(\mathcal{J}^{-1})$.

Next, let us consider $(\mathcal{K}, M, *)$ be a fuzzy metric space endowed with a directed graph $\mathcal{J}$ and $H : \mathcal{K} \times \mathcal{K} \to \mathcal{K}$ and $g : \mathcal{K} \to \mathcal{K}$ be the mappings.
Define the set $(\mathcal{K} \times \mathcal{K})_{Hg}$ as

$$(\mathcal{K}\times\mathcal{K})_{Hg} = \{(x, y) \in \mathcal{K}\times\mathcal{K} : (gx, H(x, y)) \in \mathcal{E}(\mathcal{J}) \text{ and } (gy, H(y, x)) \in \mathcal{E}(\mathcal{J}^{-1})\}.$$

Definition 2.2. The mapping $H : \mathcal{K} \times \mathcal{K} \to \mathcal{K}$ is called a $\mathcal{J} - \gamma$-contraction if

(i) g is edge preserving, i.e., $(gx, gu) \in \mathcal{E}(\mathcal{J})$ and $(gy, gv) \in \mathcal{E}(\mathcal{J}^{-1})$
$\Rightarrow (g(gx), g(gu)) \in \mathcal{E}(\mathcal{J})$ and $(g(gy), g(gv)) \in \mathcal{E}(\mathcal{J}^{-1})$;

(ii) H is g-edge preserving, i.e., $(gx, gu) \in \mathcal{E}(\mathcal{J})$ and
$(gy, gv) \in \mathcal{E}(\mathcal{J}^{-1}) \Rightarrow (H(x, y), H(u, v) \in \mathcal{E}(\mathcal{J}))$ and $(H(y, x), H(v, u) \in \mathcal{E}(\mathcal{J}^{-1})$;

(iii) for all $x, y, u, v \in \mathcal{K}$ such that $(gx, gu) \in \mathcal{E}(\mathcal{J})$ and $(gy, gv) \in \mathcal{E}(\mathcal{J}^{-1})$

$$\begin{aligned}&\gamma(M(H(x,y), H(u,v), t)) * M(H(y,x), H(v,u), t))\\ &\quad \leq k\gamma(M(gx, gu, t) * \gamma(M(gy, gv, t))\end{aligned}$$

where $k \in (0,1)$ is called contraction constant of H.

Lemma 2.1 ([4]). *Suppose that $H : \mathcal{K} \times \mathcal{K} \to \mathcal{K}$ is g-edge preserving and $H(\mathcal{K} \times \mathcal{K}) \subseteq g(x)$. Also, let $\{x_n\}$, $\{y_n\}$, $\{u_n\}$, $\{v_n\}$ be sequences in fuzzy metric space $(\mathcal{K}, M, *)$ endowed with a directed graph $\mathcal{J}$. Then the following statements are true:*

(i) $(gx, gu) \in \mathcal{E}(\mathcal{J})$ *and* $(gy, gu) \in \mathcal{E}(\mathcal{J}^{-1})$
$\Rightarrow (H(x_n, y_n), H(u_n, v_n)) \in \mathcal{E}(\mathcal{J})$ *and* $(H(y_n, x_n), H(v_n, u_n)) \in \mathcal{E}(\mathcal{J}^{-1})$ *for all* $n \in N$

(ii) $(x, y) \in (\mathcal{K} \times \mathcal{K})_{Hg} = (H(x_n, y_n), H(x_{n+1}, y_{n+1})) \in \mathcal{E}(\mathcal{J})$ *and* $(H(y_n, x_n), H(y_{n+1}, x_{n+1})) \in \mathcal{E}(\mathcal{J}^{-1})$ *for all* $n \in N$.

(iii) $(x, y) \in (\mathcal{K} \times \mathcal{K})_{Hg} \Rightarrow (H(x_n, y_n), H(y_n, x_n)) \in (\mathcal{K} \times \mathcal{K})_{Hg}$ *for all* $n \in N$.

Lemma 2.2. *Let $(\mathcal{K}, M, *)$ be a fuzzy metric space endowed with a directed graph $\mathcal{J}$. Let $H : \mathcal{K} \times \mathcal{K} \to \mathcal{K}$ be a $\mathcal{J} - \gamma$-contraction with contraction constant $k \in (0,1)$ and $H(\mathcal{K} \times \mathcal{K}) \subseteq g(x)$. Also suppose that $\{x_n\}$, $\{y_n\}$ be sequences in $\mathcal{K}$. Then, for $(x, y) \in (\mathcal{K} \times \mathcal{K})_{Hg}$, there exists $p(x, y, t) \geq 0$ and $k \in (0,1)$ such that*

$$\gamma(M(gx_n, gx_{n+1}, t) * M(gy_n, gy_{n+1}, t)) \leq k^n \gamma(p(x, y, t)).$$

where

$$p(x, y, t) = (M(gx_0, gx_1, t) * M(gy_0, gy_1, t))$$

Proof. $(x, y) \in (\mathcal{K} \times \mathcal{K})_{Hg}$

$$\begin{aligned}&\Rightarrow \quad (gx, H(x,y)) \in \mathcal{E}(\mathcal{J}) \quad \text{and} \quad (gy, H(y,x)) \in \mathcal{E}(\mathcal{J}^{-1})\\ &\Rightarrow \quad (gx_0, gx_1) \in \mathcal{E}(\mathcal{J}) \quad \text{and} \quad (gy_0, gy_1) \in \mathcal{E}(\mathcal{J}^{-1})\end{aligned}$$

Then, by Lemma 2.1,

$$\begin{aligned}&(H(x_n, y_n), H(x_{n+1}, y_{n+1})) \in \mathcal{E}(\mathcal{J})\\ &\quad \text{and} \quad (H(y_n, x_n), H(y_{n+1}, x_{n+1})) \in \mathcal{E}(\mathcal{J}^{-1}) \quad \text{for all } n \in N\\ &\quad \Rightarrow (gx_n, gx_{n+1}) \in \mathcal{E}(\mathcal{J}) \quad \text{and} \quad (g(y_n), g(y_{n+1})) \in \mathcal{E}(\mathcal{J}^{-1}) \quad \text{for all } n \in N\end{aligned}$$

But H is a $\mathcal{J} - \gamma$-contraction, so

$$\begin{aligned}&\gamma(M(gx_n, gx_{n+1}, t) * M(gy_n, gy_{n+1}, t))\\ &\quad = \gamma(M(H(x_{n-1}, y_{n-1}), H(x_n, y_n), t) * M(H(y_{n-1}, x_{n-1}), H(y_n, x_n), t))\\ &\quad \leq k\gamma(M(gx_{n-1}, gx_n, t) * M(gy_{n-1}, gy_n, t)),\end{aligned}$$

that is,

$$\gamma(M(gx_n, gx_{n+1}, t) * M(gy_n, gy_{n+1}, t)) \\ \leq k\gamma(M(gx_{n-1}, gx_n, t) * M(gy_{n-1}, gy_n, t)) \tag{2.1}$$

From (2.1)we can get for all $n \geq 1$, $t > 0$,

$$\begin{aligned} &\gamma(M(gx_n, gx_{n+1}, t) * M(gy_n, gy_{n+1}, t)) \\ &\leq k\gamma(M(gx_{n-1}, gx_n, t) * M(gy_{n-1}, gy_n, t)) \\ &\leq k^2\gamma(M(gx_{n-2}, gx_{n-1}, t) * M(gy_{n-2}, gy_{n-1}, t)) \\ &\leq k^3\gamma(M(gx_{n-3}, gx_{n-2}, t) * M(gy_{n-3}, gy_{n-2}, t)) \\ &\vdots \\ &\leq k^n\gamma(M(gx_0, gx_1, t) * M(gy_0, gy_1, t)) \end{aligned} \tag{2.2}$$

From the definition of γ-function we have

$$\gamma(M(gx_n, gx_{n+1}) * M(gy_n, gy_{n+1}, t)) \geq k^n\gamma(p(x, y, t)),$$

where

$$p(x, y, t) = (M(gx_0, gx_1, t) * M(gy_0, gy_1, t)) \tag{2.3}$$

Hence the lemma is proved. □

Lemma 2.3. *Let $(\mathcal{K}, M, *)$ be fuzzy metric space endowed with a directed graph $\mathcal{J}$. Let $H : \mathcal{K} \times \mathcal{K} \to \mathcal{K}$ be a $\mathcal{J} - \gamma$-contraction with contraction constant $k \in (0, 1)$ and $H(\mathcal{K} \times \mathcal{K}) \subseteq g(\mathcal{K})$.*

If the mapping H satisfies the conditions:

(*i*) *There exists x_0 and y_0 in $\mathcal{K}$ such that*

$$\prod_{i=1}^{l}(M(gx_0, H(x_0, y_0), t_i) * M(gy_0, H(y_0, x_0), t)) \neq 0, \text{ for all } l \in N,$$

(*ii*) $r * s > 0 \Rightarrow \gamma(r * s) \leq \gamma(r) + \gamma(s)$ *for all* $r, s \in \{M(gx_0, H(x_0, y_0), t) * M(qy_0, H(y_0, x_0), t)$ *for all* $x_0, y_0 \in \mathcal{K}$, $t > 0\}$,

(*iii*) $\{\gamma(M(gx_0, H(x_0, y_0), t_i) * M(gy_0, H(y_0, x_0), t_i)) : i \in N\}$ *is bounded for all x_0 and y_0 in $\mathcal{K}$ and any sequence $\{t_i\}_i \subset (0, \infty)$,*

Also, suppose that $\{x_n\}$, $\{y_n\}$ be sequences in $\mathcal{K}$. Then, for $(x, y) \in (\mathcal{K} \times \mathcal{K})_{Hg}$, there exist $x^*, y^* \in \mathcal{K}$ such that $\{gx_n\} \to x^*$ and $\{gy_n\} \to y^*$, as $n \to \infty$.

Proof. Let for any $n, m \in N$, $n > m$, $t > 0$ and let $\{a_i\}_{i \in N}$ be a strictly decreasing sequence of positive numbers such that $\sum_{i=1}^{\infty} a_i = 1$. From (2.3) and using the property of γ, we have

$$M(gx_n, gx_{n+1}, t) * M(gy_n, gy_{n+1}, t) \geq M(gx_0, gx_1, t) * M(gy_0, gy_1, t). \tag{2.4}$$

From (2.4) and condition (i) given in Lemma 2.3 we have

$$\begin{aligned} &M(gx_m, gx_n, t) * M(gy_m, gy_n, t) \\ &\geq \left(M(gx_m, gx_m, t - \sum_{i=m}^{n-1} a_i t) * M(gx_m, gx_n, \sum_{i=m}^{n-1} a_i t) \right) \\ &\quad * \left(M\left(gy_m, gy_m, \sum_{i=m}^{n-1} a_i t\right) * M\left(gy_m, gy_n, \sum_{i=m}^{n-1} a_i t\right)\right) \\ &= \left(1 * M\left(gx_m, gx_n, \sum_{i=m}^{n-1} a_i t\right)\right) * \left(1 * M\left(gy_m, gy_n, \sum_{i=m}^{n-1} a_i t\right)\right) \\ &\geq \prod_{i=m}^{n-1} (M(gx_i, gx_{i+1}, a_i t) * M(gy_i, gy_{i+1}, a_i t)) \\ &\geq \prod_{i=m}^{n-1} (M(gx_0, gx_1, a_i t) * M(gy_0, gy_1, a_i t)) \end{aligned} \tag{2.5}$$

By (2.5) and the condition (ii) of Lemma 2.3 , we have

$$\begin{aligned} &\gamma(M(gx_m, gx_n, t) * M(gy_m, gy_n, t)) \\ &\leq \gamma\left(\prod_{i=m}^{n-1} (M(gx_i, gx_{i+1}, a_i t) * M(gy_i, gy_{i+1}, a_i t))\right) \\ &\leq \sum_{i=m}^{n-1} \gamma\left(M(gx_i, gx_{i+1}, a_i t) * M(gy_i, gy_{i+1}, a_i t)\right). \end{aligned} \tag{2.6}$$

From (2.2) and (2.6), we have

$$\begin{aligned} &\gamma(M(gx_m, gx_n, t) * M(gy_m, gy_n, t)) \\ &\leq \sum_{i=m}^{n-1} k^i \gamma(M(gx_0, gx_1, a_i t) * M(gy_0, gy_1, a_i t)). \end{aligned} \tag{2.7}$$

Here the sequence $\gamma(M(gx_0, gx_1, a_i t) * M(gy_0, gy_1, a_i t))$ for all $i \in N$, is increasing and by condition (iii) of the Lemma 2.3, we find the convergence of the series $\sum_{i=m}^{n-1} k^i \gamma(M(gx_0, gx_1, a_i t) * M(gy_0, gy_1, a_i t))$.

For given $\epsilon > 0$ there exists $n_0 \in N$ such that

$$\sum_{i=m}^{n-1} k^i \gamma(M(gx_0, gx, a_i t) * M(gy_0, gy_1, a_i t)) < \epsilon \text{ for all } n, m \geq n_0,\ n > m. \tag{2.8}$$

From (2.7) and (2.8) we have

$$\gamma(M(gx_m, gx_n, t) * M(gy_m, gy_n, t)) \leq \epsilon.$$

So, by Lemma 2.2, we conclude that $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences. Also $(\mathcal{K}, M, *)$ is complete, therefore there exists $x^*, y^* \in \mathcal{K}$ such that

$$\lim_{n\to\infty} gx_n = x^* \quad \text{and} \quad \lim_{n\to\infty} gy_n = y^* .$$

Theorem 2.1. *Suppose that $(\mathcal{K}, M, *)$ be a complete fuzzy metric space endowed with a directed graph $\mathcal{J}$. Let $H : \mathcal{K} \times \mathcal{K} \to \mathcal{K}$ be a $\mathcal{J} - \gamma$-contraction with contraction constant $k \in (0, 1)$ and $H(\mathcal{K} \times \mathcal{K}) \subseteq g(\mathcal{K})$. Let g be $\mathcal{J}$-continuous and commutes with H. Also, we assume, either*

(i) H is $\mathcal{J}$-continuous, or

(ii) the four tuple $(\mathcal{K}, M, *, \mathcal{J})$ has the Property (A). Then $C(Hg) \neq \phi$ iff $(\mathcal{K} \times \mathcal{K})_{Hg} \neq \phi$.

$C(Hg)$ denotes the set of coupled coincidence points.

Proof. Suppose that $C(Hg) \neq \phi$.

Then these exists some $(x^*, y^*) \in C(Hg)$, i.e. $gx^* = H(x^*, y^*)$ and $gy^* = H(y^*, x^*)$. So,

$$\begin{aligned}
&(gx^*, H(x^*, y^*)) = (gx^*, gx^*) \in \Delta \subseteq \mathcal{E}(\mathcal{J}) \\
&\text{and } (gy^*, H(y^*, x^*)) = (gy^*, gy^*) \in \Delta \subseteq \mathcal{E}(\mathcal{J}^{-1}). \\
&\Rightarrow (x^*, y^*) \in (\mathcal{K} \times \mathcal{K})_{Hg}. \\
&\Rightarrow (\mathcal{K} \times \mathcal{K})_{Hg} \neq \phi.
\end{aligned}$$

Next, let us assume $(\mathcal{K} \times \mathcal{K})_{Hg} \neq \phi$.

Then there exists $(x_0, y_0) \in (\mathcal{K} \times \mathcal{K})_{Hg}$, i.e., $(gx_0, H(x_0, y_0)) \in \mathcal{E}(\mathcal{J})$ and $(gy_0, H(y_0, x_0)) \in \mathcal{E}(\mathcal{J}^{-1})$.

Then by Lemma 2.1, we have a sequence $\{n_i\}_{i\in N}$ of positive integers such that

$$(H(x_{n_i}, y_{n_i}), H(x_{n_i+1}, y_{n_i+1})) \in \mathcal{E}(\mathcal{J}) \text{ and } (H(y_{n_i}, x_{n_i}), H(y_{n_i+1}, x_{n_i+1})) \in \mathcal{E}(\mathcal{J}^{-1}).$$

Also, $H(\mathcal{K} \times \mathcal{K}) \subseteq g(x)$. Therefore

$$(gx_{n_i+1}, gx_{n_i+2}) \in \mathcal{E}(\mathcal{J}) \quad \text{and} \quad (gy_{n_i+1}, y_{n_i+2}) \in \mathcal{E}(\mathcal{J}^{-1}). \tag{2.9}$$

Also, from Lemma 2.3

$$\lim_{n\to\infty} gx_{n_i} = x^* \quad \text{and} \quad \lim_{n\to\infty} gy_{n_i} = y^*. \tag{2.10}$$

But g is $\mathcal{J}$-continuous

$$\Rightarrow \quad \lim_{n\to\infty} g(gx_{n_i}) = gx^* \quad \text{and} \quad \lim_{n\to\infty} g(gy_{n_i}) = gy^*.$$

Also, since H and g are commutative

$$g(g(x_{n_i+1})) = g(H(x_{n_i}, y_{n_i})) \ \text{ and } \ g(g(y_{n_i+1})) = g(H(y_{n_i}, x_{n_i}))$$

implies

$$g(gx_{n_i+1}) = H(gx_{n_i}, gy_{n_i}) \ \text{ and } \ g(gy_{n_i+1}) = H(gy_{n_i}, gx_{n_i}). \tag{2.11}$$

Finally, we show that

$$gx^* = H(x^*, y^*) \ \text{ and } \ gy^* = H(y^*, x^*).$$

Let H be $\mathcal{J}$-continuous.

Then, from (2.11), we have

$$\lim_{n\to\infty} g(gx_{n_i+1}) = \lim_{n\to\infty} H(gx_{n_i}, gy_{n_i}) \quad \text{gives} \quad gx^* = H(x^*, y^*)$$
$$\text{and} \quad \lim_{n\to\infty} g(gy_{n_i+1}) = \lim_{n\to\infty} H(gy_{n_i}, gx_{n_i}),$$

implies $\quad gy^* = H(y^*, x^*)$.

Thus, (x^*, y^*) is a coupled coincidence point of the mapping H and g, i.e. $C(Hg) \neq \phi$.

Next, we assume that Property (A) holds.

From (2.9) and (2.10), we have $\{gx_{n_i}\} \to x^*$ as $i \to \infty$ and $(gx_{n_i}, gx_{n_i+1}) \in \mathcal{E}(\mathcal{J})$ and $\{gy_{n_i}\} \to y^*$ as $i \to \infty$ and $(gy_{n_i}, gy_{n_i+1}) \in \mathcal{E}(\mathcal{J}^{-1})$. Therefore, using property (A),

$$(gx_{n_i}, x^*) \in \mathcal{E}(\mathcal{J}) \quad \text{and} \quad (gy_{n_i}, y^*) \in \mathcal{E}(\mathcal{J}^{-1}).$$

Therefore,

$$\begin{aligned}
&M(gx^*, H(x^*, y^*), t) * M(gy^*, H(y^*, x^*), t)\\
&\geq (M(gx^*, g(gx_{n_i+1}), t/2) * M(g(gx_{n_i+1}), H(x^*, y^*), t/2))\\
&* (M(gy^*, g(gy_{n_i+1}), t/2) * M(g(gy_{n_i+1}), H(y^*, x^*), t/2))\\
&= (M(gx^*, g(gx_{n_i+1}), t/2) * M(H(gx_{n_i}, gy_{n_i}), H(x^*, y^*), t/2))\\
&* (M(gy^*, g(gy_{n_i+1}), t/2) * M(H(gy_{n_i}, gx_{n_i}), H(y^*, x^*), t/2))\\
&= (1 * M(H(gx_{n_i}, gy_{n_i}), H(x^*, y^*), t/2))\\
&\cdot (1 * M(H(gy_{n_i}, gx_{n_i}), H(y^*, x^*), t/2)).
\end{aligned}$$

Now, taking the limit $n \to \infty$,

$$M(gx^*, H(x^*, y^*), t) = 1 \quad \text{gives} \quad gx^* = H(x^*, y^*).$$

Also, $M(gy^*, H(y^*, x^*)t) = 1 \quad$ implies $\quad gy^* = H(y^*, x^*)$. $\square$

Theorem 2.2. *Suppose that the hypotheses of Theorem 2.1 hold. Beside, let for every* $(x, y), (x^*, y^*) \in \mathcal{K} \times \mathcal{K}$ *there exist* $(u, v) \in \mathcal{K} \times \mathcal{K}$ *such that*
$(H(x, y), H(u, v)) \in \mathcal{E}(\mathcal{J}), (H(y, x), H(v, u)) \in \mathcal{E}(\mathcal{J}^{-1})$ *and*
$(H(x^*, y^*), H(u, v)) \in \mathcal{E}(\mathcal{J}), (H(y^*, x^*), H(v, u)) \in \mathcal{E}(\mathcal{J}^{-1})$.

Then H *and* g *have a unique coupled common fixed point.*

Proof. Let (x, y) and (x^*, y^*) be coupled coincidence points, i.e.,

$$gx = H(x, y),\ gy = H(y, x) \quad \text{and} \tag{2.12}$$
$$gx^* = H(x^*, y^*),\ gy^* = H(y^*, x^*). \tag{2.13}$$

By hypothesis, we have

$$(H(x, y), H(u, v)) \in \mathcal{E}(\mathcal{J}) \ \text{ and } \ (H(y, x), H(v, u)) \in \mathcal{E}(\mathcal{J}^{-1}) \tag{2.14}$$
$$(H(x^*, y^*), H(u, v)) \in \mathcal{E}(\mathcal{J}) \ \text{ and } \ (H(y^*, x^*), H(v, u)) \in \mathcal{E}(\mathcal{J}^{-1}) \tag{2.15}$$

Set $H(u_n, v_n) = gu_{n+1}$, $u = u_0$ and $H(v_n, u_n) = gv_{n+1}$, $v = v_0$.

Then, using (2.12), (2.13), (2.14) and (2.15) we get

$$\begin{aligned}
&(gx, gu_1) \in \mathcal{E}(\mathcal{J}), \quad (gy, gv_1) \in \mathcal{E}(\mathcal{J}),\\
&(gx^*, gu_1) \in \mathcal{E}(\mathcal{J}), \quad (gy^*, gv_1) \in \mathcal{E}(\mathcal{J}^{-1}).
\end{aligned}$$

But H is g-edge preserving, so

$$\begin{aligned}
&(H(x, y), H(u_1, v_1)) \in \mathcal{E}(\mathcal{J}),\ (H(y, x), H(v_1, u_1)) \in \mathcal{E}(\mathcal{J}^{-1}) \text{ and}\\
&(H(x^*, y^*), H(u_1, v_1)) \in \mathcal{E}(\mathcal{J}),\ (H(y^*, x^*), H(v_1, u_1)) \in \mathcal{E}(\mathcal{J}^{-1})
\end{aligned}$$

this implies $(gx, gu_2) \in \mathcal{E}(\mathcal{J})$, $(gy, gv_2) \in \mathcal{E}(\mathcal{J}^{-1})$ and $(gx^*, gu_2) \in \mathcal{E}(\mathcal{J})$, $(gy^*, gv_2) \in \mathcal{E}(\mathcal{J}^{-1})$. Using the g-edge preserving property of H repeatedly, for all $n \geq 1$, one can obtain

$$\begin{aligned}
&(gx, gu_n) \in \mathcal{E}(\mathcal{J}), \quad (gy, gv_n) \in \mathcal{E}(\mathcal{J}^{-1}) \quad \text{and}\\
&(gx^*, gu_n) \in \mathcal{E}(\mathcal{J}), \quad (gy^*, gv_n) \in \mathcal{E}(\mathcal{J}^{-1}).
\end{aligned}$$

Therefore

$$\begin{aligned}
&\gamma(M(gx, gu^*, t) * M(gy, gy^*, t))\\
&\leq \gamma((M(gx, gu_{n+1}, t/2) * M(gu_{n+1}, gx^*, t/2))\\
&* (M(gy, gv_{n+1}, t/2) * M(gv_{n+1}, gy^*, t/2)))\\
&\leq \gamma((M(gx, gu_{n+1}, t/2) * M(gu_{n+1}, gx^*, t/2))\\
&+ \gamma(M(gy, gv_{n+1}, t/2) * M(gv_{n+1}, gy^*, t/2))\\
&\leq k^n \gamma(p(x, y, t)). \qquad \text{(by Lemma (2.2))}
\end{aligned}$$

Letting $n \to \infty$, we have

$$\gamma(M(gx, gx^*t) * M(gy, gy^*, t)) = 0$$

$$\text{implies} \quad M(gx, gx^*, t) = 1 \text{ and } M(gy, gy^*, t) = 1, \text{this gives}$$

$$gx = gx^* \text{ and } gy = gy^*. \tag{2.16}$$

Let $gx = gx^* = r$ and $gy = gy^* = s$.

Then using the commutativity of H and g, (2.16) gives
$g(gx) = g(H(x, y)) = H(gx, gy)$ and $g(gy) = g(H(y, x)) = H(gy, gx)$,
$gr = H(r, s)$ and $gs = H(s, r)$.
Thus, (r, s) is a coupled coincidence point.
Now, the same for (x, y) as (r, s),

$$gx = gr \quad \text{and} \quad gy = gs \quad \text{this gives}$$
$$r = gr \quad \text{and} \quad s = gs.$$

Thus, $r = gr = H(r, s)$ and $s = gs = H(s, r)$. So, (r, s) is a coupled common fixed point of H and g.
Finally, we prove that the coupled fixed point of H and g is unique.

Let us suppose that (a, b) is another coupled common fixed point of H and g. Then

$$a = ga = H(a, b) \quad \text{and} \quad b = gb = H(b, a). \tag{2.17}$$

But, from (2.17) we get

$$ga = gr = r \quad \text{and} \quad gb = gs = s, \tag{2.18}$$

implies $a = r$ and $b = s$.

Hence the coupled common fixed point of H and g is unique.

Corollary 2.1. *Let $(\mathcal{K}, M, ^*)$ be a complete fuzzy metric space. Let $P : \mathcal{K} \times \mathcal{K}$ be a $\mathcal{J} - \gamma$-contraction and $\gamma : (0, 1] \to [0, \infty]$ satisfies the properties (W1) and (W2). If the mapping P satisfies the conditions:*

(i) *There exists x_0 and y_0 in $\mathcal{K}$ such that*

$$\sum_{i=1}^{m} \{M(x_0, P(x_0, y_0), t_i) * M(y_0, P(y_0, x_0), t_i)\} \neq 0$$

for all $m \in N$;

(ii) *$r^*s > 0$ implies $\gamma(r^*s) \leq \gamma(r) + \gamma(s)$ for all $r, s \in \{M(x_0, P(x_0, y_0), t) * M(y_0, P(y_0, x_0), t)$ for all $x_0, y_0 \in \mathcal{K}, t > 0\}$;*

(iii) *$\{\gamma(M(x_0, P(x_0, y_0), t_i) * M(y_0, P(y_0, x_0), t_i)) : i \in N\}$ is bounded for all x_0 and y_0 in $\mathcal{K}$ and any sequence $\{t_i\}_i \subset (0, \infty)$;*

(iv) $\left(\frac{1}{M(P(x,y),P(u,v),t)*M(P(y,x),P(v,u),t)}\right) \leq k\left(\frac{1}{M(x,u,t)*M(y,v,t)}\right)$
for all $x, y, u, v \in \mathcal{K}$, $t > 0$;

(v) $P(\mathcal{K} \times \mathcal{K}) \subseteq \mathcal{K}$;

(vi) P *is* $\mathcal{J}$*-continuous or*
the tuple $(\mathcal{K}, M, *, \mathcal{J})$ *has the Property (A);*

(vii) *For every* $(x, y), (x^*, y^*) \in \mathcal{K} \times \mathcal{K}$, *there exist* $(u, u) \in \mathcal{K} \times \mathcal{K}$ *such that*
$(P(x, y), P(u, v)) \in \mathcal{E}(\mathcal{J})$, $(P(y, x), P(v, u)) \in \mathcal{E}(\mathcal{J}^{-1})$ *and*
$(P(x^*, y^*), P(u, u)) \in \mathcal{E}(\mathcal{J})$,$(P(y^*, x^*), P(v, u)) \in \mathcal{E}(\mathcal{J}^{-1})$.

Then P *has a coupled fixed point in* $\mathcal{K}$.
Putting $\gamma(t) = \frac{1}{t} - 1$, *and then proof follows by Theorem2.2.*

References

[1] T.G. Bhaskar, V. Lakshmikantham, *Fixed point theorems in partially ordered metric spaces and applications*, Nonlinear Anal., Theory Methods Appl., 65(2006), 1379-1393.

[2] C. Chifu, G. Petrusel, *New results on coupled fixed point theory in metric spaces endowed with a directed graph*, Fixed Point Theory Appl., 151 (2014).

[3] B.S. Choudhury, P. Das, *A new contraction mapping principle in partially ordered fuzzy metric space*, Annals of Fuzzy Mathematics and Informatics, 8 (2014), 889-901.

[4] D. Eshi, P.K. Das, P. Debnath, *Coupled coincidence and coupled common fixed point theorems on metric space with graph*, Fixed Point Theory and Applications, 37 (2016).

[5] R. Espinola, W.A. Kirk, *Fixed point theorems in R-trees with applications to graph theory*, Topol. Appl., 153 (2006), 1046-1055.

[6] A. George, P. Veeramani, *On some result in fuzzy metric space*, Fuzzy Sets and Systems, 64 (1994), 395-399.

[7] J. Jachymski, *The contraction principle for mappings on a metric space with a graph*, Proc. Am. Math. Soc., 136 (2008), 1359-1373.

[8] I. Kramosil, J. Michalek, *Fuzzy metrics and statistical metric spaces*, Kybernetika, 11 (1975), 336-344.

[9] V. Lakshmikantham, L. Ćiric, *Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces*, Nonlinear Anal., Theory Methods Appl., 70 (2009), 4341-4349.

[10] R.K. Saini, V. Gupta, S.B. Singh, *Fuzzy version of some fixed points theorems on expansion type maps in fuzzy metric space*, Thai Journal of Mathematics, 5 (2007), 245-252.

[11] B. Schweizer, A. Sklar, *Statistical metric spaces*, Pacif. J. Math., 10 (1960), 313-334.

[12] W. Shatanawi, *Fixed and common fixed point theorems in frame of quasi metric spaces under contraction condition based on ultra distance functions*, Nonlinear Analysis: Modelling and Control, 23 (2018), 724-748.

[13] W. Shatanawi, K. Abodayeh, A. Mukheimer, *Some fixed point theorems in extended b-metric spaces*, U.P.B. Sci. Bull., Series A, 80 (2018).

[14] W. Shatanawi, *Common fixed points for mappings under contractive conditions of* (α, β, ψ) *admissibility type*, Mathematics, 6 (2018).

[15] S. Shukla, *Fixed point theorems of G-fuzzy contractions in fuzzy metric space endowed with a graph*, Communication in Mathematics, 22 (2014), 1-12.

[16] D. Wardowski, *Fuzzy contractive mappings and fixed point in fuzzy metric space*, Fuzzy Sets and Systems, 222 (2013), 108-114.

[17] V. Gupta, A. Kanwar, $\mathcal{V}$*-fuzzy metric spaces and related fixed point theorems*, Fixed Point Theory and Applications, 15(2016).

[18] V. Gupta, H. Aydi, N. Mani, *Some fixed point theorems for symmetric Hausdorff function on Hausdorff spaces*, Appl. Math. Inf. Sci., 9 (2015), 833-839.

[19] V. Gupta, W. Shatanawi, M. Verma, *Existence of fixed points for* J*-fuzzy contractions in fuzzy metric spaces endowed with graph*, J. Anal, 2018.

[20] V. Gupta, N. Mani, *Existence and uniqueness of fixed point in fuzzy metric spaces and its applications*, Advances in Intelligent Systems and Computing, 236 (2014), 217-224.

[21] V. Gupta, R.K. Saini, A. Kanwar, *Some common coupled fixed point results on modified intuitionistic fuzzy metric spaces*, Procedia Computer Science, 79 (2016), 32-40.

[22] L.A. Zadeh, *Fuzzy sets*, Information and Control, 89 (1965), 338-353.

[23] X.H. Zhu, J. Xiao, *Note on coupled fixed point theorems for contractions in fuzzy metric space*, Nonlinear Anal., 74 (2011), 5475-5479.

Accepted: 9.01.2019