On decomposable MS-algebras

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Abstract. In this paper we give some results on the direct product, subalgebras and homomorphisms of decomposable MS-algebras. We Show how direct products and canonical projections are related. Also, we study homomorphic images of subalgebras of decomposable MS-algebras.

Keywords: direct product, *MS*-algebra, decomposable *MS*-algebra, subalgebra, homomorphism.

1. Introduction

MS-algebras were initiated by T.S. Blyth and J.C. Varlet, see [6], as a generalization of both de Morgan and Stone algebras. In [8], T.S. Blyth and J.C. Varlet described the lattice $\Lambda(\mathbf{MS})$ of subclasses of the class $\mathbf{MS}$ of all MS-algebras. In [3], S. El-Assar and A. Badawy studied many properties of homomomorphisms and subalgebras of MS-algebras from the subclass $\mathbf{K_2}$. In [1], A. Badawy, D. Guffova and M. Haviar introduced and characterized decomposable MS-algebras by means of decomposable MS-triples. In [2], A. Badawy and R. El-Fawal studied many properties of decomposable MS-algebras in terms of decomposable MS-triples as homomorphisms and subalgebras. Also, they solved

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some fill in problems concerning homomorphisms and subalgebras of decomposable MS-algebras.

In this paper we study many properties related to the direct product and subalgebras of decomposable MS-algebras. Also, we reveal the connection between homomorphisms and direct products. We finish with some results on homomorphic images of subalgebras of decomposable MS-algebras.

2. Preliminaries

In this section, we present definitions and main results which are needed through this paper. For basic facts about MS-algebras and related structures we refer the reader to [5], [6], [7], [8], [9] and [10].

An MS-algebra is an algebra $(L; \vee, \wedge, ^\circ, 0, 1)$ of type (2,2,1,0,0) where $(L; \vee, \wedge, 0, 1)$ is a bounded distributive lattice and the unary operation $^\circ$ satisfies:

$$x \leq x^{\circ\circ}, (x \wedge y)^\circ = x^\circ \vee y^\circ, 1^\circ = 0.$$

The following theorem gives the basic properties of MS-algebras.

Theorem 2.1 ([6], [9]). *For any two elements a, b of an MS-algebra L, we have:*

(1) $0^\circ = 1$,

(2) $a \leq b \Rightarrow b^\circ \leq a^\circ$,

(3) $a^{\circ\circ\circ} = a^\circ$,

(4) $(a \vee b)^\circ = a^\circ \wedge b^\circ$,

(5) $(a \vee b)^{\circ\circ} = a^{\circ\circ} \vee b^{\circ\circ}$,

(6) $(a \wedge b)^{\circ\circ} = a^{\circ\circ} \wedge b^{\circ\circ}$.

Lemma 2.2 ([1], [6]). *Let L be an MS-algebra. Then:*

(1) $L^{\circ\circ} = \{x \in L : x = x^{\circ\circ}\}$ *is a de Morgan subalgebra of L,*

(2) $D(L) = \{x \in L : x^\circ = 0\}$ *is a filter (filter of dense elements) of L.*

Definition 2.3 ([4]). Let $L = (L; \vee, \wedge, 0_L, 1_L)$ and $L_1 = (L_1; \vee, \wedge, 0_{L_1}, 1_{L_1})$ be bounded lattices. The map $f : L \to L_1$ is called a (0,1)-lattice homomorphism if:

(1) $f(0_L) = 0_{L_1}$ and $f(1_L) = 1_{L_1}$,

(2) f preserves joins, that is, $f(x \vee y) = f(x) \vee f(y)$ for every $x, y \in L$,

(3) f preserves meets, that is, $f(x \wedge y) = f(x) \wedge f(y)$ for every $x, y \in L$.

Definition 2.4 ([4]). A (0,1)-lattice homomorphism $f : L \to L_1$ of an MS-algebra L into an MS-algebra L_1 is called a homomorphism if $f(x^\circ) = (f(x))^\circ$ for all $x \in L$.

Definition 2.5 ([1]). An MS-algebra L is called decomposable MS-algebra if for every $x \in L$ there exists $d \in D(L)$ such that $x = x^{\circ\circ} \wedge d$.

Definition 2.6 ([2]). A bounded sublattice of a decomposable MS-algebra L is called a subalgebra of L if:

(1) $x^{\circ} \in A, \forall x \in A$,

(2) For every $x \in A$, there exists $d \in D(A)$ such that $x = x^{\circ\circ} \wedge d$.

Definition 2.7 ([2]). A subalgebra of a decomposable MS-algebra L is called a K_2-subalgebra of L if for every $x, y \in A$, the following holds:

(1) $x \wedge x^{\circ} = x^{\circ} \wedge x^{\circ\circ}$,

(2) $x \wedge x^{\circ} \leqslant y \vee y^{\circ}$.

Definition 2.8 ([2]). A subalgebra of a decomposable MS-algebra L is called a Stone subalgebra of L if for every $x \in A$, $x^{\circ} \vee x^{\circ\circ} = 1$

3. Direct products and subalgebras of decomposable *MS*-algebras

We begin by recalling the definition of direct product of MS-algebras.

Definition 3.1. Let $\{L_i,\ i \in\ I_n\}$ be a family of MS-algebras. Then, the direct product $\prod_{i=1}^{n} L_i$ is defined as $\prod_{i=1}^{n} L_i = \{(x_1, x_2, ..., x_n), x_i \in L_i, i \in\ I_n\}$ where the operations $\vee, \wedge$ are defined componentwise and $(x_1, x_2, ..., x_n)^{\circ} = (x_1^{\circ}, x_2^{\circ}, ..., x_n^{\circ})$.

The proof of the following lemma is straightforward.

Lemma 3.2. *Let $\{L_i,\ i \in\ I_n\}$ be a family of MS-algebras. Then:*

1. $(\prod_{i=1}^{n} L_i)^{\circ\circ} = \prod_{i=1}^{n} L_i^{\circ\circ}$,
2. $D(\prod_{i=1}^{n} L_i) = \prod_{i=1}^{n} D(L_i)$.

Theorem 3.3. *Let $\{L_i,\ i \in\ I_n\}$ be a family of MS-algebras. Then, $\prod_{i=1}^{n} L_i$ is decomposable if and only if L_i is decomposable for each $i \in\ I_n$.*

Proof. Suppose that $\prod_{i=1}^{n} L_i$ is decomposable. Let $x_i \in\ L_i,\ i \in\ I_n$. Then,

$$\begin{aligned}
& (x_1, x_2, ..., x_n) \in \prod_{i=1}^{n} L_i \\
\Rightarrow\ & (x_1, x_2, ..., x_n) = (x_1, x_2, ..., x_n)^{\circ\circ} \wedge (d_1, d_2, ..., d_n),\ d_i \in\ D(L_i),\ i \in\ I_n \\
\Rightarrow\ & (x_1, x_2, ..., x_n) = (x_1^{\circ\circ}, x_2^{\circ\circ}, ..., x_n^{\circ\circ}) \wedge (d_1, d_2, ..., d_n) \\
\Rightarrow\ & (x_1, x_2, ..., x_n) = (x_1^{\circ\circ} \wedge d_1, x_2^{\circ\circ} \wedge d_2, ..., x_n^{\circ\circ} \wedge d_n) \\
\Rightarrow\ & x_i = x_i^{\circ\circ} \wedge d_i,\ d_i \in\ D(L_i), \forall\ i \in\ I_n, \\
\Rightarrow\ & L_i \text{ is decomposable}, \forall\ i \in\ I_n.
\end{aligned}$$

Conversely, suppose that L_i is decomposable, $\forall i \in I_n$, and $(x_1, x_2, ..., x_n) \in \prod_{i=1}^{n} L_i$. Then,

$$\begin{aligned}(x_1, x_2, ..., x_n) &= (x_1^{\circ\circ} \wedge d_1, x_2^{\circ\circ} \wedge d_2, ..., x_n^{\circ\circ} \wedge d_n), d_i \in D(L_i) \\ &= (x_1^{\circ\circ}, x_2^{\circ\circ}, ..., x_n^{\circ\circ}) \wedge (d_1, d_2, ..., d_n) \\ &= (x_1, x_2, ..., x_n)^{\circ\circ} \wedge (d_1, d_2, ..., d_n)\end{aligned}$$

Since $(d_1, d_2, ..., d_n) \in \prod_{i=1}^{n} D(L_i) = D(\prod_{i=1}^{n} L_i)$, then $\prod_{i=1}^{n} L_i$ is decomposable. □

Theorem 3.4. *Let A_i be a subalgebra of a decomposable MS-algebra $L_i, i \in I_n$. Then, $\prod_{i=1}^{n} A_i$ is a subalgebra of $\prod_{i=1}^{n} L_i$.*

Proof. Clearly, $\prod_{i=1}^{n} A_i$ is a bounded sublattice of $\prod_{i=1}^{n} L_i$. Let $(x_1, x_2, ...x_n) \in \prod_{i=1}^{n} A_i$. Then, $(x_1, x_2, ..., x_n)^{\circ} = (x_1^{\circ}, x_2^{\circ}, ..., x_n^{\circ}) \in \prod_{i=1}^{n} A_i$ (as $x_i^{\circ} \in A_i$). Assuming that $x_i = x_i^{\circ\circ} \wedge d_i$, $d_i \in D(A_i)$, we get

$$(x_1, x_2, ..., x_n) = (x_1^{\circ\circ} \wedge d_1, x_2^{\circ\circ} \wedge d_2, ..., x_n^{\circ\circ} \wedge d_n) = (x_1^{\circ\circ}, x_2^{\circ\circ}, ..., x_n^{\circ\circ}) \wedge (d_1, d_2, ..., d_n).$$

Since $(d_1, d_2, ..., d_n) \in D(\prod_{i=1}^{n} A_i)$, then $\prod_{i=1}^{n} A_i$ is a subalgebra of $\prod_{i=1}^{n} L_i$. □

Corollary 3.5. $(\prod_{i=1}^{n} L_i)^{\circ\circ}$ *is a subalgebra of* $\prod_{i=1}^{n} L_i$.

Proof. Since $(\prod_{i=1}^{n} L_i)^{\circ\circ} = \prod_{i=1}^{n} L_i^{\circ\circ}$ and $L_i^{\circ\circ}$ is a subalgebra of L_i, then $(\prod_{i=1}^{n} L_i)^{\circ\circ}$ is a subalgebra of $\prod_{i=1}^{n} L_i$. □

Lemma 3.6. *Let A_i be a K_2-subalgebra of a decomposable MS-algebra L_i, $i \in I_n$. Then, $\prod_{i=1}^{n} A_i$ is a K_2-subalgebra of $\prod_{i=1}^{n} L_i$.*

Proof. By Theorem 3.4, $\prod_{i=1}^{n} A_i$ is a subalgebra of $\prod_{i=1}^{n} L_i$.

Let $(x_1, x_2, ..., x_n) \in \prod_{i=1}^{n} A_i$. Then,

$$\begin{aligned}(x_1, x_2, ..., x_n) \wedge (x_1, x_2, ..., x_n)^{\circ} &= (x_1 \wedge x_1^{\circ}, x_2 \wedge x_2^{\circ}, ..., x_n \wedge x_n^{\circ}) \\ &= (x_1^{\circ} \wedge x_1^{\circ\circ}, x_2^{\circ} \wedge x_2^{\circ\circ}, ..., x_n^{\circ} \wedge x_n^{\circ\circ}) \\ &= (x_1, x_2, ..., x_n)^{\circ} \wedge (x_1, x_2, ..., x_n)^{\circ\circ}.\end{aligned}$$

Moreover,

$$\begin{aligned}&(x_1, x_2, ..., x_n) \wedge (x_1, x_2, ..., x_n)^{\circ} \\ =\ & (x_1 \wedge x_1^{\circ}, x_2 \wedge x_2^{\circ}, ..., x_n \wedge x_n^{\circ}) \\ \leq\ & (y_1 \vee y_1^{\circ}, y_2 \vee y_2^{\circ}, ..., y_n \vee y_n^{\circ}), \quad \forall\ y_i \in A_i \\ =\ & (y_1, y_2, ..., y_n) \vee (y_1, y_2, ..., y_n)^{\circ}, \quad \forall\ (y_1, y_2, ..., y_n) \in \prod_{i=1}^{n} A_i.\end{aligned}$$

Hence, $\prod_{i=1}^{n} A_i$ is a K_2-subalgebra of $\prod_{i=1}^{n} L_i$. □

Lemma 3.7. *Let S_i be a Stone subalgebra of a decomposable MS-algebra L_i, $i \in I_n$. Then, $\prod_{i=1}^n S_i$ is a Stone subalgebra of $\prod_{i=1}^n A_i$.*

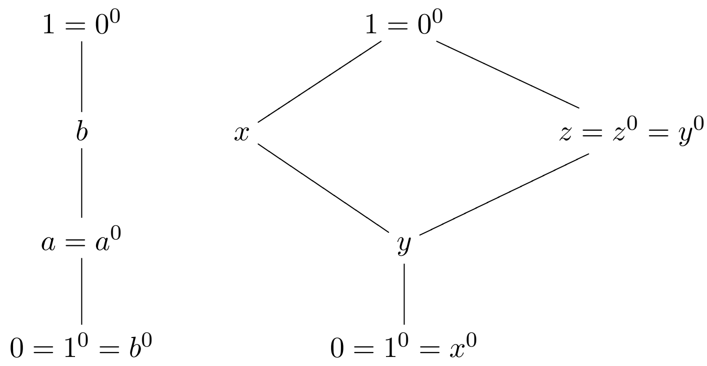
Proof. We need to verify the Stone identity. Namely, $z^\circ \vee z^{\circ\circ} = 1$, $\forall z \in \prod_{i=1}^n S_i$. Let $(x_1, x_2, ..., x_n) \in \prod_{i=1}^n S_i$. Then,

$$\begin{aligned}(x_1, x_2, ..., x_n)^\circ \vee (x_1, x_2, ..., x_n)^{\circ\circ} &= (x_1^\circ \vee x_1^{\circ\circ}, x_2^\circ \vee x_2^{\circ\circ}, ..., x_n^\circ \vee x_n^{\circ\circ}) \\ &= (1_1, 1_2, ..., 1_n),\end{aligned}$$

where 1_i is the greatest element of S_i. Thus, $\prod_{i=1}^n S_i$ is a Stone subalgebra of $\prod_{i=1}^n L_i$. □

The following example shows that the converse of Theorem 3.4, lemma 3.6 and lemma 3.7 is not true, respectively.

Example 3.8. *Consider the following two decomposable MS-algebras:*



L_1 L_2

$A = \{(1,1), (0,0)\}$ *is a subalgebra (respectively a K_2-subalgebra, a Stone subalgebra) of $L_1 \times L_2$ while it can not be written as a product of two subalgebras (respectively K_2-subalgebras, Stone subalgebras) of L_1 and L_2.*

Lemma 3.9. *Let $\{A_i,\ i \in I_n\}$ be a family of subalgebras of a decomposable MS-algebra L. Then:*

1. *$\bigcap_{i=1}^n A_i$ is a subalgebra of L,*
2. *$\bigcup_{i=1}^n A_i$ is not necessarily a subalgebra of L.*

Proof. 1. Clearly, $\bigcap_{i=1}^n A_i$ is a bounded sublattice of L. Let $x \in \bigcap_{i=1}^n A_i$. Then, $x \in A_i$, $\forall i \in I_n$. Consequently, $x^\circ \in A_i$, $\forall i \in I_n$. Hence, $x^\circ \in \bigcap_{i=1}^n A_i$. Moreover, we have $x = x^{\circ\circ} \wedge d_i$, $d_i \in D(A_i)$, $i \in I_n$. As $d_i \in A_i$, then $\bigvee_{i=1}^n d_i \in A_i$, $\forall i \in I_n$. Also, $(\bigvee_{i=1}^n d_i)^\circ = \bigwedge_{i=1}^n d_i^\circ = 0$. Then, $\bigvee_{i=1}^n d_i \in \bigcap_{i=1}^n D(A_i) = D(\bigcap_{i=1}^n A_i)$. Now, we can write $x = x^{\circ\circ} \vee d$ where $d = \bigvee_{i=1}^n d_i \in D(\bigcap_{i=1}^n A_i)$. Hence, $\bigcap_{i=1}^n A_i$ is a subalgebra of L.

2. Consider L_2 of example 3.8, we observe that $A_1 = \{1, 0, x\}$ and $A_2 = \{1, 0, z\}$ are subalgebras of L_2 while $A_1 \cup A_2 = \{1, 0, x, z\}$ is not a subalgebra of L_2 (as $x \wedge z = y \notin A_1 \cup A_2$). □

4. Direct products and homomorphisms of decomposable MS-algebras

Theorem 4.1. *Let $\{\varphi_i : A_i \to B_i, i \in I_n\}$ be a family of homomorphisms between MS-algebras. Define $\varphi : \prod_{i=1}^n A_i \to \prod_{i=1}^n B_i$, by $\varphi(a_1, a_2, ..., a_n) = (\varphi_1(a_1), \varphi_2(a_2), ..., \varphi_n(a_n))$. Then:*

1. *φ is a homomorphism,*
2. *φ is one to one if and only if each φ_i is one to one,*
3. *φ is onto if and only if each φ_i is onto,*
4. *$ker\varphi = \prod_{i=1}^n ker\varphi_i$,*
5. *$\varphi(\prod_{i=1}^n A_i) = \prod_{i=1}^n \varphi_i(A_i)$.*

Proof.

(1) Let $(a_1, a_2, ..., a_n)$, $(b_1, b_2, ..., b_n) \in \prod_{i=1}^n A_i$. Then,

$$\begin{aligned} & \varphi\big((a_1, a_2, ..., a_n) \vee (b_1, b_2, ..., b_n)\big) \\ = \; & \varphi(a_1 \vee b_1, a_2 \vee b_2, ..., a_n \vee b_n) \\ = \; & (\varphi_1(a_1 \vee b_1), \varphi_2(a_2 \vee b_2), ..., \varphi_n(a_n \vee b_n)) \\ = \; & (\varphi_1(a_1) \vee \varphi_1(b_1), \varphi_2(a_2) \vee \varphi_2(b_2), ..., \varphi_n(a_n) \vee \varphi_n(b_n)) \\ = \; & (\varphi_1(a_1), \varphi_2(a_2), ..., \varphi_n(a_n)) \vee (\varphi_1(b_1), \varphi_2(b_2), ..., \varphi_n(b_n)) \\ = \; & \varphi(a_1, a_2, ..., a_n) \vee \varphi(b_1, b_2, ..., b_n). \end{aligned}$$

Similarly, we can show that

$$\varphi\big((a_1, a_2, ..., a_n) \wedge (b_1, b_2, ..., b_n)\big) = \varphi(a_1, a_2, ..., a_n) \wedge \varphi(b_1, b_2, ..., b_n).$$

Moreover,

$$\begin{aligned} \varphi(a_1, a_2, ..., a_n)^\circ &= \varphi(a_1^\circ, a_2^\circ, ..., a_n^\circ) \\ &= (\varphi_1(a_1^\circ), \varphi_2(a_2^\circ), ..., \varphi_n(a_n^\circ)) \\ &= (\varphi_1(a_1)^\circ, \varphi_2(a_2)^\circ, ..., \varphi_n(a_n)^\circ) \\ &= (\varphi_1(a_1), \varphi_2(a_2), ..., \varphi_n(a_n))^\circ \\ &= (\varphi(a_1, a_2, ..., a_n))^\circ. \end{aligned}$$

Hence, φ is a homomorphism from $\prod_{i=1}^n A_i$ into $\prod_{i=1}^n B_i$.

(2) Let φ be one to one and suppose that $\varphi_i(a_i) = \varphi_i(b_i), i \in I_n$. Then,

$$\begin{aligned} \varphi(a_1, a_2, ..., a_n) &= (\varphi_1(a_1), \varphi_2(a_2), ..., \varphi_n(a_n)) \\ &= (\varphi_1(b_1), \varphi_2(b_2), ..., \varphi_n(b_n)) \\ &= \varphi(b_1, b_2, ..., b_n). \end{aligned}$$

This gives $(a_1, a_2, ..., a_n) = (b_1, b_2, ..., b_n)$. So, $a_i = b_i$, $\forall i \in I_n$. Hence, each φ_i is one to one. Conversely, assume φ_i is one to one for each i and $\varphi(a_1, a_2, ..., a_n) = \varphi(b_1, b_2, ..., b_n)$. Then, $(\varphi_1(a_1), \varphi_2(a_2), ..., \varphi_n(a_n)) = (\varphi_1(b_1), \varphi_2(b_2), ..., \varphi_n(b_n))$. Thus, $\varphi_i(a_i) = \varphi_i(b_i)$ $\forall i$. Hence, φ is one to one.

(3) Let φ be onto and $b_i \in B_i$, $\forall i$. Then, $(b_1, b_2, ..., b_n) \in \prod_{i=1}^n B_i$. As φ is onto, there exists $(a_1, a_2, ..., a_n) \in \prod_{i=1}^n A_i$ such that $\varphi(a_1, a_2, ..., a_n) = (b_1, b_2, ..., b_n)$. Equivalently, $(\varphi_1(a_1), \varphi_2(a_2), ..., \varphi_n(a_n)) = (b_1, b_2, ..., b_n)$. That is, $\varphi_i(a_i) = b_i$, $\forall i$. Hence, each φ_i is onto. Conversely, let φ_i be onto for each i and $(b_1, b_2, ..., b_n) \in \prod_{i=1}^n B_i$. Since $b_i \in B_i$ and φ_i is onto, then there exists $a_i \in A_i$ such that $b_i = \varphi_i(a_i)$, $\forall i$. So, $(b_1, b_2, ..., b_n) = (\varphi_1(a_1), \varphi_2(a_2), ..., \varphi_n(a_n))$. Consequently, φ is onto.

(4)
$$\begin{aligned} (a_1, a_2, ..., a_n) \in ker\varphi &\Leftrightarrow \varphi(a_1, a_2, ..., a_n) = (0_1, 0_2, ..., 0_n) \\ &\Leftrightarrow (\varphi_1(a_1), \varphi_2(a_2), ..., \varphi_n(a_n)) = (0_1, 0_2, ..., 0_n) \\ &\Leftrightarrow \varphi_i(a_i) = 0_i,\ \forall i \in I_n \\ &\Leftrightarrow a_i \in ker\varphi_i\ \forall i \in I_n \\ &\Leftrightarrow (a_1, a_2, ..., a_n) \in \prod_{i=1}^n ker\varphi_i. \end{aligned}$$

(5)
$$\begin{aligned} &(b_1, b_2, ..., b_n) \in \varphi(\prod_n^{i=1} A_i) \\ &\Leftrightarrow (b_1, b_2, ..., b_n) = \varphi((a_1, a_2, ..., a_n)), (a_1, a_2, ..., a_n) \in \prod_n^{i=1} A_i \\ &\Leftrightarrow \varphi_i(a_i) = b_i, a_i \in A_i \\ &\Leftrightarrow (b_1, b_2, ..., b_n) \in \prod_{i=1}^n \varphi_i(A_i). \end{aligned}$$

□

Theorem 4.2. *Let $\{A_i, i \in I_n\}$ be a family of MS-algebras. Then, the map $\varphi_k : \prod_{i=1}^n A_i \to A_k$ defined by $\varphi_k(a_1, a_2, ..., a_k, ..., a_n) = a_k$ is an epimorphism for each $k \in I_n$.*

Proof. Let $(a_1, a_2, ..., a_k, ..., a_n) = (b_1, b_2, ..., b_k, ..., b_n)$. Then, $a_i = b_i$ $\forall i \in I_n$. Therefore, $\varphi_k(a_1, a_2, ..., a_k, ..., a_n) = a_k = b_k = \varphi_k(b_1, b_2, ..., b_k, ..., b_n)$. So, φ_k is well defined, $\forall k \in I_n$. Now, suppose that $(a_1, a_2, ..., a_k, ..., a_n)$, $(b_1, b_2, ..., b_k, ..., b_n) \in \prod_{i=1}^{n} A_i$. Then, $\varphi_k\big((a_1, a_2, ..., a_k, ..., a_n) \vee (b_1, b_2, ..., b_k, ..., b_n)\big) = a_k \vee b_k = \varphi_k(a_1, a_2, ..., a_k, ..., a_n) \vee \varphi_k(b_1, b_2, ..., b_k, ..., b_n)$. Similarly, φ_k preserves the meet operation. Besides,

$$\varphi_k(a_1, a_2, ..., a_k, ..., a_n)^\circ = a_k^\circ = (\varphi_k(a_1, a_2, ..., a_k, ..., a_n))^\circ$$

Finally, if $c_k \in A_K$, then $(0_1, 0_2, ..., c_k, ..., 0_n) \in \prod_{i=1}^{n} A_i$ with $\varphi_k(0_1, 0_2, ..., c_k, ..., 0_n) = c_k$. Thus, φ_k is onto and hence φ_k is an epimorphism . □

The previous maps ($\varphi_k s$) are called the canonical projections of the direct product.

Theorem 4.3. *Let $\{L_i, i \in I_n\}$ be a family of MS-algebras. Then there exists a unique (up to isomorphism) MS-algebra L, together with a family of homomorphisms $\{\varphi_i : L \to L_i, i \in I_n\}$, with the following property:*

For any MS-algebra M and any family of homomorphisms $\{f_i : M \to L_i, i \in I_n\}$, there exists a unique homomorphism $f : M \to L$ such that $\varphi_i \circ f = f_i$, $\forall i \in I_n$.

Proof. Let $L = \prod_{i=1}^{n} L_i$ and $\{\varphi_i : L \to L_i, i \in I_n\}$ be the family of canonical projections. Define $f : M \to L$ by $f(a) = (f_1(a), f_2(a), ..., f_n(a)), \forall a \in M$. For any $a, b \in M$, we have

$$\begin{aligned} f\big(a \vee b\big) &= (f_1(a \vee b), f_2(a \vee b), ..., f_n(a \vee b)) \\ &= (f_1(a) \vee f_1(b), f_2(a) \vee f_2(b), ..., f_n(a) \vee f_n(b)) \\ &= (f_1(a), f_2(a), ..., f_n(a)) \vee (f_1(b), f_2(b), ..., f_n(b)) \\ &= f(a) \vee f(b). \end{aligned}$$

Similarly, $f\big(a \wedge b\big) = f(a) \wedge f(b)$. Also,

$$f(a^\circ) = (f_1(a^\circ), f_2(a^\circ), ..., f_n(a^\circ)) = ((f_1(a))^\circ, (f_2(a))^\circ, ..., (f_n(a))^\circ) = ((f(a))^\circ.$$

Thus, f is a homomorphism. Moreover,

$$(\varphi_i \circ f)(a) = \varphi_i(f(a)) = \varphi_i(f_1(a), f_2(a), ..., f_n(a)) = f_i(a), \ \forall a \in M.$$

Hence, $\varphi_i \circ f = f_i$, $\forall i \in I_n$. To prove the uniqueness of f, let $g : M \to L$ be another homomorphism such that $\varphi_i \circ g = f_i$, $\forall i \in I_n$. This implies that $(\varphi_i \circ f)(a) = f_i(a) = (\varphi_i \circ g)(a)$, $\forall a \in M$, Assume that $g(a) = (a_1, a_2, ..., a_n)$, $\forall a \in M$. Then,

$$\begin{aligned} a_i &= \varphi_i\big(a_1, a_2, ..., a_n\big) = \varphi_i(g(a)) \\ &= \varphi_i(f(a)) = \varphi_i\big(f_1(a), f_2(a), ..., f_n(a)\big) = f_i(a) \ \forall i \in I_n. \end{aligned}$$

Therefore, $f(a) = (a_1, a_2, ..., a_n) = g(a)$, $\forall a \in M$. So, $f = g$ and f is unique. It remains to prove the uniqueness of L. Suppose that L_1 is an MS-algebra which has the same property as L with the family of homomorphisms $\{\psi_i : L_1 \to L_i,\ i \in I_n\}$. Apply the property to L and L_1, we get unique homomorphisms $\alpha : L_1 \to L$ and $\beta : L \to L_1$ with $\varphi_i \circ \alpha = \psi_i$ and $\psi_i \circ \beta = \varphi_i$, $\forall i \in I_n$. Consequently, $\alpha \circ \beta : L \to L_1$ is a unique homomorphism with $\varphi_i \circ (\alpha \circ \beta) = \varphi_i\ \forall i \in I_n$. Since the identity map $id_L : L \to L$ is also a homomorphism with $\varphi_i \circ id_L = \varphi_i\ \forall i \in I_n$, then $\alpha \circ \beta = id_L$. Similarly, $\beta \circ \alpha = id_{L_1}$. This shows that β is an isomorphism and L is unique up to isomorphism. □

Noting that the proofs of the previous three theorems do not rely on the decomposability of the MS-algebras, we conclude that they hold for decomposable MS-algebras.

Theorem 4.4. *Let $\varphi : L_1 \to L_2$ be a homomorphism between decomposable MS-algebras L_1 and L_2. If A is a subalgebra of L_1, then $\varphi(A)$ is a subalgebra of L_2.*

Proof. Let $b_1, b_2 \in \varphi(A)$. Then, there exist $a_1, a_2 \in A$ with $\varphi(a_1) = b_1$, $\varphi(a_2) = b_2$. So, $\varphi(a_1 \vee a_2) = b_1 \vee b_2$. As $a_1 \vee a_2 \in A$, then $b_1 \vee b_2 \in \varphi(A)$. A similar argument shows that $b_1 \wedge b_2 \in \varphi(A)$. Now, let $b \in \varphi(A)$. Then, $b = \varphi(a)$, for some $a \in A$. So, $b^\circ = \varphi(a^\circ)$. Since $a^\circ \in A$, then $b^\circ \in \varphi(A)$. Writing $a = a^{\circ\circ} \wedge d, d \in D(A)$, we get

$$b = \varphi(a) = \varphi(a^{\circ\circ} \wedge d) = \varphi(a^{\circ\circ}) \wedge \varphi(d) = (\varphi(a))^{\circ\circ} \wedge \varphi(d) = b^{\circ\circ} \wedge \varphi(d).$$

We note that $(\varphi(d))^\circ = \varphi(d^\circ) = \varphi(0_1) = 0_2$. So, $\varphi(d) \in D(\varphi(A))$. Hence, $\varphi(A)$ is a subalgebra of L_2. □

Theorem 4.5. *Let $\varphi : L_1 \to L_2$ be a monomorphism. If B is a subalgebra of L_2, then $\varphi^{-1}(B)$ is a subalgebra of L_1.*

Proof. Let $a_1, a_2 \in \varphi^{-1}(B)$. Then, there exist $b_1, b_2 \in B$ with $\varphi(a_1) = b_1$ and $\varphi(a_2) = b_2$. So, $\varphi(a_1 \vee a_2) = b_1 \vee b_2$ and $\varphi(a_1 \wedge a_2) = b_1 \wedge b_2$. As $b_1 \vee b_2, b_1 \wedge b_2 \in B$, then $a_1 \vee a_2, a_1 \wedge a_2 \in \varphi^{-1}(B)$. Now, let $a \in \varphi^{-1}(B)$, then $a = \varphi^{-1}(b)$ for some $b \in B$. So, $\varphi(a) = b$. Then, $\varphi(a^\circ) = b^\circ$. As $b^\circ \in B$, then $a^\circ \in \varphi^{-1}(B)$. Assuming that $b = b^{\circ\circ} \wedge e,\ e \in D(B)$, we get

$$\begin{aligned} a = \varphi^{-1}(b) &= \varphi^{-1}(b^{\circ\circ} \wedge e),\ e \in D(B). \\ &= \varphi^{-1}(b^{\circ\circ}) \wedge \varphi^{-1}(e) \\ &= (\varphi^{-1}(b))^{\circ\circ} \wedge \varphi^{-1}(e)) = a^{\circ\circ} \wedge \varphi^{-1}(e). \end{aligned}$$

Now, we prove that $\varphi^{-1}(e) \in D(\varphi^{-1}(B))$. Let $d = \varphi^{-1}(e)$. Then, $\varphi(d) = e$. This gives $\varphi(d^\circ) = e^\circ = 0_2$. Therefore, $d^\circ = \varphi^{-1}(0_2) = 0_1$. So, $d \in D(\varphi^{-1}(B))$. Hence, $a = a^{\circ\circ} \wedge d$, $d \in D(\varphi^{-1}(B))$. Hence, $\varphi^{-1}(B)$ is a subalgebra of L_1. □

Theorem 4.6. *Let L_1 and L_2 be two MS-algebras. Then, L_1 can be embedded into $L_1 \times L_2$ if and only if there exists a homomorphism from L_1 to L_2.*

Proof. Assume that L_1 can be embedded into $L_1 \times L_2$. Then, there exists a monomorphism $\varphi : L_1 \to L_1 \times L_2$. Let $\varphi(a) = (a_1, a_2)$, $\forall a \in L_1$. Define $f : L_1 \to L_2$ by $f(a) = a_2$. Then, $f(a \vee b) = a_2 \vee b_2 = f(a) \vee f(b)$. Similarly, $f(a \wedge b) = f(a) \wedge f(b)$ Also, $f(a^\circ) = a_2^\circ = (f(a))^\circ$. Hence, f is a homomorphism.

Conversely, assume that there exists a homomorphism $f : L_1 \to L_2$. Define $\phi : L_1 \to L_1 \times L_2$ by $\phi(a) = (a, f(a))$. Then,

$$\begin{aligned} \phi(a \vee b) &= (a \vee b, f(a \vee b)) = (a \vee b, f(a) \vee f(b)) = (a, f(a)) \vee (b, f(b)) \\ &= \phi(a) \vee \phi(b), \ \forall a, b \in L_1. \end{aligned}$$

Analogously, $\phi(a \wedge b) = \phi(a) \wedge \phi(b)$. Also, $\phi(a^\circ) = (a^\circ, f(a^\circ)) = (a, f(a))^\circ = (\phi(a))^\circ$. Assume $\phi(a) = \phi(b)$, then $(a, f(a)) = (b, f(b))$. This gives $a = b$. Hence, ϕ is an embedding. □

References

[1] A. Badawy, D. Guffova and M. Haviar, *Triple construction of decomposable MS-algebras*, Acta Univ. Palacki Olomuc., Fac. Rer. Nat., Mathematica, 51 (2012), 53-65.

[2] A. Badawy and Ragaa El-Fawal, *Homomorphisms and subalgebras of decomposable MS-algebra*, Journal of the Egyption Mathematical Society, 25 (2017), 119-124.

[3] S. El-Assar and A. Badawy, *Homomorphisms and subalgebras of MS-algebra*, Quter Univ. Sci. J., 15 (1995), 279-289.

[4] T.S. Blyth, *Lattices and ordered algebraic structures*, Springer-Verlag, London, 2005.

[5] T.S. Blyth and J.C. Varlet, *Sur la construction de certaines MS-algebres*, Portugaliae Math., 39 (1980), 489-496.

[6] T.S. Blyth and J.C. Varlet, *On a common abstraction of de Morgan algebras and Stone algebras*, Proc. Roy. Soc. Edinburugh, 94A (1983), 301-308.

[7] T.S. Blyth and J.C. Varlet, *corrigendum sur la construction de certaines MS-algebres*, Portugaliae Math., 42 (1983), 469-471.

[8] T.S. Blyth and J.C. Varlet, *Subvarieties of the class of MS-algebras*, Proc. Roy. Soc. Edinburgh, 95A (1983), 157-169.

[9] T.S. Blyth and J.C. Varlet, *Ockham algebras*, London, Oxford University Press, 1994.

[10] M. Haviar, *On certain construction of MS-algebras*. Portugaliae Math., 51 (1994), 71-83.

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