# On conjugate trigonometrically $\rho$-convex functions

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**Abstract.** The aim of this article is to introduce a definition of conjugate trigonometrically $\rho$-convex functions by using Young's inequality which plays an important role in linking the concept of duality between trigonometrically $\rho$-convex functions, rather the definition given by Fenchel. Furthermore, we show that the integration of any increasing function is trigonometrically $\rho$-convex.

**Keywords:** integral inequalities, supporting functions, trigonometrically $\rho$-convex functions.

## 1. Introduction

In 1908, Phragmén and Lindelöf (see for example [13]) presented that if $F(z)$ is an analytic function inside an angle $D = \{z = re^{\imath\theta} : u < \theta < v\}$, then the function

$$h(\theta) = h_F(\theta) = \limsup_{r\to\infty} \frac{\log|F(re^{i\theta})|}{r^\rho},$$

is called the indicator function of $F(z)$ with respect to the order $0 < \rho < \infty$, and has the property:

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If $0 < \rho(v-u) < \pi$, and $M(\theta)$ is the function defined by,

$$M(\theta) := A\cos\rho\theta + B\sin\rho\theta,$$

(such functions are called sinusoidal or $\rho$-trigonometric) which has the same value of $h(\theta)$ at $\alpha$ and at $\beta$, then for $u \leq \theta \leq v$. We have

$$h(\theta) \leq M(\theta).$$

This property is called a trigonometric $\rho$-convexity.

In [5], Beckenbach and Bing ([4] and [19]) introduced a generalization of the classical convexity by replacing linear functions with another family of continuous functions such that for each pair of points $p_1(x_1, y_1)$ and $p_2(x_2, y_2)$ of the plane there exists exactly one member of the family with a graph joining these points.

In fact, the topic of conjugate convex functions really originate in a paper of Young [23]. This topic attracted some interests [12], [18] and [20], after the work of Fenchel. In [9], [10] Fenchel greatly generalized the whole idea and applied it to the programming problem. Conjugate convex functions have numerous applications mentioned in [11], [17] and [22]. More precisely, in 2014 Gardiner et al. [11] modified an algorithm for computing the convex (Legendre-Fenchel) conjugate of convex piecewise linear-quadratic functions of two variables, to compute its partial conjugate i.e. the conjugate with respect to one of the variables. The structure of the original algorithm is preserved including its time complexity (linear time with some approximation and log-linear time without approximation). Applying twice the partial conjugate (and a variable switching operator) recovers the full conjugate. They presented our partial conjugate algorithm, which was more flexible and simpler than the original full conjugate algorithm. They emphasized the difference with the full conjugate algorithm and illustrate results by computing partial conjugates, partial Moreau envelopes, and partial proximal averages. In 2017, Notarnicola and Notarstefano [17] proposed that a class of distributed optimization algorithms based on proximal gradient methods applied to the dual problem. They showed that, by choosing suitable primal variable copies, the dual problem is itself separable when written in terms of conjugate functions, and the dual variables can be stacked into non-overlapping blocks associated to the computing nodes. In 2018, Rodrigues [22] proposed that a unified optimal control framework that can be used to formulate and solve aircraft performance problems, such as maximum endurance and maximum range, for both propeller-driven airplanes and jet-propelled aircraft. It was proved that such problems have a common mathematical formulation and, under strict convexity assumptions, they had a unique feedback solution for the speed as a function of weight. The feedback solution yields an analytic expression for the optimal speed. For maximum endurance, the solution corresponds to the minimization of the rate of fuel consumption per unit time. For maximum range, the rate of fuel consumption per unit distance is minimized. Moreover, the optimal solution for maximum range was interpreted geometrically using the concept of convex conjugate function and Legendre transformation.

In this paper, we deal with the generalized convex functions in the notion of Beckenbach. For particular choices of the two parameter family $\{M(x)\}$, we consider the following class of generalized convex functions $\{M(x) = A\cos\rho x + B\sin\rho x\}$. This class is called trigonometrically $\rho$-convex functions (see for examples [2]-[3], [7]-[8] and [13]) which have interesting applications in the design of cavitation-free hydrofoils ([1] and [16]) and in the extremum property [2].

The objective of the present paper is to define a conjugate trigonometrically $\rho$-convex functions defined on the real line $\mathbb{R}$. We shall be interested in real finite functions on a finite or infinite interval $I$ such that $I \subset \mathbb{R}$ and an interior $I^o$ of $I$.

## 2. Definitions and preliminary results

In this section, we present the basic definitions and results which are used later, see for details [2], [13]-[15] and [21].

**Definition 2.1** ([14], (see for example [2], [13], [15])). *A function $f : I \to \mathbb{R}$ is said to be* ***Trigonometrically $\rho$-Convex Function*** *if for any arbitrary closed subinterval $[u, v]$ of $I$ such that $0 < \rho(v-u) < \pi$, the graph of $f(x)$ for $x \in [u, v]$ lies nowhere above the $\rho$-trigonometric function, determined by the equation*

$$M(x) = M(x; u, v, f) = A\cos\rho x + B\sin\rho x,$$

*where $A$ and $B$ are chosen such that $M(u) = f(u)$, and $M(v) = f(v)$.*

*Equivalently, if for all $x \in [u, v]$*

$$f(x) \le M(x) = \frac{f(u)\sin\rho(v-x) + f(v)\sin\rho(x-u)}{\sin\rho(v-u)}. \tag{2.1}$$

The trigonometrically $\rho$-convex functions possess a number of properties analogous to those of convex functions.

For example: If $f : I \to \mathbb{R}$ is trigonometrically $\rho$-convex function, then for any $u, v \in I$ such that $0 < \rho(v-u) < \pi$, the inequality

$$f(x) \ge M(x; u, v, f),$$

holds outside the interval $[u, v]$.

**Definition 2.2** ([4], [5]). *A function $T_u(x) = A\cos\rho x + B\sin\rho x$, is said to be* ***supporting function*** *for $f(x)$ at the point $u \in I$, if*

$$T_u(u) = f(u), \text{ and } T_u(x) \le f(x), \quad \forall x \in I. \tag{2.2}$$

**Theorem 2.1** ([2])**.** *A function $f : I \to \mathbb{R}$ is trigonometrically $\rho$-convex function on $I$ if and only if there exists a supporting function for $f(x)$ at each point $x \in I$.*

**Remark 2.1.** [[2]] If $f : I \to \mathbb{R}$ is differentiable trigonometrically $\rho$-convex function, then the supporting function for $f(x)$ at the point $u \in I$ has the form

$$T_u(x) = f(u)\cos\rho(x-u) + \frac{f'(u)}{\rho}\sin\rho(x-u), \quad \forall x \in I. \tag{2.3}$$

**Remark 2.2.** [[2]] For a trigonometrically $\rho$-convex function $f : I \to \mathbb{R}$, if $f(x)$ is not differentiable at the point $u$ then the supporting function has the form

$$T_u(x) = f(u)\cos\rho(x-u) + K_{u,f}\sin\rho(x-u), \quad \forall x \in I, \tag{2.4}$$

where $K_{u,f} \in [\frac{f'_-(u)}{\rho}, \frac{f'_+(u)}{\rho}]$.

**Theorem 2.2** ([13]-[15])**.** *A trigonometrically $\rho$-convex function $f : I \to \mathbb{R}$ has finite right and left derivatives $f'_+(x)$, $f'_-(x)$ at every point $x \in I$ and $f'_-(x) \leq f'_+(x)$.*

**Theorem 2.3** ([14])**.** *Let $f : I \to \mathbb{R}$ be a two times continuously differentiable function. Then $f$ is trigonometrically $\rho$-convex on $I$ if and only if $f''(x) + \rho^2 f(x) \geq 0$, $\forall x \in I$.*

**Property 2.1** ([2])**.** *A necessary and sufficient condition for the function $f(x)$ to be a trigonometrically $\rho$-convex in $I$ is that the function*

$$\varphi(x) = f'(x) + \rho^2 \int_w^x f(t)dt, \quad w \in I$$

*is non-decreasing in $I$.*

**Property 2.2** ([15], [21])**.** *If a trigonometrically $\rho$-convex function $f : I \to \mathbb{R}$ is bounded, i.e, $|f(x)| < k$ for $x \in I$, then it is a continuous function of $x \in I$, and in each closed subinterval $J$ of $I$, it satisfies a Lipschitz condition, that is*

$$|f(x) - f(y)| \leq k|x-y|, \; for\ some\ k\ and\ \forall x,\ y \in J. \tag{2.5}$$

The relationship between a convex function and its conjugate is at the heart of much recent research. The basic idea can be traced back to Young's Inequality

**Theorem 2.4** ([6], Young's Inequality)**.** *Suppose that $g : [0,\infty) \to [0,\infty)$ be strictly increasing and continuous function with $g(0) = 0$ and $g(t) \to \infty$ as $t \to \infty$ (under these circumstances, $g$ has an inverse function $g^{-1}$, which has the same properties as $g$). Then, for any $x \geq 0$, $y \geq 0$*

$$xy \leq \int_0^x g(t)dt + \int_0^y g^{-1}(t)dt. \tag{2.6}$$

We can now state the main result on the operation of conjugacy:

**Theorem 2.5** ([21])**.** *Let $f : I \to \mathbb{R}$ be a convex and closed function, then $f^* : I^* \to \mathbb{R}$ is denote the conjugate function and defined by*

$$f^*(y) = \sup_{x \in I}[xy - f(x)],$$

*it is convex and closed with the domain $I^* = \{y \in \mathbb{R} : f^*(y) < \infty\}$ and*

(a) $xy \leq f(x) + f^*(y)$ *for all* $x \in I$ $y \in I^*$,

(b) $xy = f(x) + f^*(y)$ *if and only if* $y \in \partial f(x)$,

(c) $\partial(f^*) = (\partial f)^{-1}$,

(d) $f^{**} = f$,

*where* $\partial f(x) = \{y \in \mathbb{R}; y$ *is the slope of a support line for* $f$ *at* $x\}$.

## 3. Main results

The purpose of the present section is to show that the integration of increasing function is trigonometrically $\rho$-convex. Moreover, we show Young's inequality in the class of trigonometrically $\rho$-convex. Furthermore, we introduce a definition of conjugate trigonometrically $\rho$-convex. The relationship between trigonometrically $\rho$-convex and its conjugate is revealed through Theorem 3.2.

**Theorem 3.1.** *Let $g : I \to [0, \infty)$ be an increasing function, and $c \in I^o$. Then, $\int_c^x g(t)dt$ is trigonometrically $\rho$-convex function for all $x \in I$.*

**Proof.** Put,

$$f(x) = \int_c^x g(t)dt$$

let $u, v \in I$ such that $0 < \rho(v - u) < \pi$, $x = \lambda u + \mu v$, where $\lambda + \mu = 1$ and $\lambda, \mu \in [0, 1]$,

$$\begin{aligned}
f(x) &= \int_c^x g(t)dt \\
&= \frac{\sin\rho(v - x + x - u)}{\sin\rho(v - u)} \int_c^x g(t)dt \\
&= \frac{\sin\rho(v - x)\cos\rho(x - u) + \cos\rho(v - x)\sin\rho(x - u)}{\sin\rho(v - u)} \int_c^x g(t)dt \\
&= \frac{\sin\rho(v-x)\cos\rho(x-u)}{\sin\rho(v-u)} \int_c^x g(t)dt + \frac{\cos\rho(v - x)\sin\rho(x - u)}{\sin\rho(v - u)} \int_c^x g(t)dt \\
&= \frac{\sin\rho(v - x)\cos\rho(x - u)}{\sin\rho(v - u)}[\int_c^u g(t)dt + \int_u^x g(t)dt] \\
&\quad + \frac{\cos\rho(v - x)\sin\rho(x - u)}{\sin\rho(v - u)}[\int_c^v g(t)dt - \int_x^v g(t)dt].
\end{aligned}$$

Since

$$\cos\rho(x-u) \le 1,\ \cos\rho(v-x) \le 1, \tag{3.1}$$

we get that

$$f(x) \le \frac{\sin\rho(v-x)\int_c^u g(t)dt + \sin\rho(x-u)\int_c^v g(t)dt}{\sin\rho(v-u)}$$
$$+\frac{\sin\rho(v-x)\cos\rho(x-u)}{\sin\rho(v-u)}\int_u^x g(t)dt - \frac{\cos\rho(v-x)\sin\rho(x-u)}{\sin\rho(v-u)}\int_x^v g(t)dt.$$

Take

$$\lambda = \frac{\sin\rho(v-x)\cos\rho(x-u)}{\sin\rho(v-u)} \ \ \&\ \ \mu = \frac{\cos\rho(v-x)\sin\rho(x-u)}{\sin\rho(v-u)}.$$

We obtain

$$f(x) \le \frac{\sin\rho(v-x)\int_c^u g(t)dt + \sin\rho(x-u)\int_c^v g(t)dt}{\sin\rho(v-u)} + \lambda\int_u^x g(t)dt - \mu\int_x^v g(t)dt.$$

Since $g$ is increasing, then $g(t) \le g(x)$ for all $t \in [u,x]$, and $-g(t) \le -g(x)$ for all $t \in [x,v]$, we conclude that

$$\begin{aligned}
f(x) &\le \frac{\sin\rho(v-x)\int_c^u g(t)dt + \sin\rho(x-u)\int_c^v g(t)dt}{\sin\rho(v-u)} \\
&\quad +\lambda\int_u^x g(x)dt - \mu\int_x^v g(x)dt \\
&= \frac{\sin\rho(v-x)\int_c^u g(t)dt + \sin\rho(x-u)\int_c^v g(t)dt}{\sin\rho(v-u)} \\
&\quad +\lambda g(x)[x-u] - \mu g(x)[v-x] \\
&= \frac{\sin\rho(v-x)\int_c^u g(t)dt + \sin\rho(x-u)\int_c^v g(t)dt}{\sin\rho(v-u)} \\
&\quad +g(x)[(\lambda+\mu)x - (\lambda u + \mu v)] \\
&= \frac{\sin\rho(v-x)\int_c^u g(t)dt + \sin\rho(x-u)\int_c^v g(t)dt}{\sin\rho(v-u)}.
\end{aligned}$$

Applying Definition 2.1, then $\int_c^x g(t)dt$ is trigonometrically $\rho$-convex function $\forall x \in I$. □

**Example 3.1.** Let $g(x) = \sin\rho x$, $\forall x \in [0, \frac{\pi}{2\rho}]$. Then, $f(x) = \int_0^x \sin\rho t dt$ is trigonometrically $\rho$-convex function $\forall x \in [0, \frac{\pi}{2\rho}]$.

As

$$f(x) = \int_0^x \sin\rho t dt = \frac{1}{\rho}(1-\cos\rho x), \quad \forall x \in [0, \frac{\pi}{2\rho}]. \tag{3.2}$$

Differentiate $f(x)=\int_0^x \sin\rho t dt$ with respect to $x$, implies

$$\begin{aligned} f^{'}(x) &= \sin\rho x, \\ f^{''}(x) &= \rho\cos\rho x. \end{aligned} \tag{3.3}$$

From equations (3.2), (3.3) implies

$$f^{''}(x)+\rho^2 f(x)=\rho\cos\rho x+\rho^2\frac{1}{\rho}(1-\cos\rho x)=\rho,$$

since $0<\rho<\infty$, then $f^{''}(x)+\rho^2 f(x)=\rho>0,\ \forall x\in[0,\frac{\pi}{2\rho}]$. By using Theorem 2.3, then $f(x)$ trigonometrically $\rho$-convex function.

**Proposition 3.1.** *Suppose that $g:[0,\frac{\pi}{2\rho}]\to[0,\infty)$ be strictly increasing and continuous function with $g(0)=0$, $g(x)\to\infty$ as $x\to\infty$.*

*If we take,*

$$f(x)=\int_0^x g(t)dt, \quad f^*(m)=\int_0^m g^{-1}(t)dt. \tag{3.4}$$

*Then $f$ and $f^*$ are both trigonometrically $\rho$-convex functions for all $x\in[0,\frac{\pi}{2\rho}]$, $m\in[0,\infty)$, and satisfy*

$$m\sin(\rho x)\le f(x)+f^*(m),\ \ \forall x\in[0,\frac{\pi}{2\rho}]. \tag{3.5}$$

**Proof.** Since $g$ is strictly increasing and continuous function with $g(0)=0$, then $g$ has an inverse function $g^{-1}$, which has the same properties as $g$ for all $x\in[0,\frac{\pi}{2\rho}]$, and by using Theorem 3.1, then $f$ *and* $f^*$ are both trigonometrically $\rho$-convex functions for any $x\in[0,\frac{\pi}{2\rho}]$, $m\in[0,\infty)$.

Now, we prove the inequality (3.5).

Since $\sin\rho x$ is trigonometrically $\rho$-convex function and from Property 2.2, $|\sin\rho x-\sin\rho 0|\le k|\rho x-0|$. Take $k=\frac{1}{\rho}$, then

$$\sin\rho x\le x,\ \ \forall x\in[0,\frac{\pi}{2\rho}]. \tag{3.6}$$

From inequality (3.6) and Theorem 2.4, implies

$$m\sin\rho x\le mx\le\int_0^x g(t)dt+\int_0^m g^{-1}(t)dt,\ \ \forall x\in[0,\frac{\pi}{2\rho}].$$

Then, $m\sin(\rho x)\le f(x)+f^*(m),\ \ \forall x\in[0,\frac{\pi}{2\rho}]$. □

**Example 3.2.** Let $g(x)=\sin\rho x$ for all $x\in[0,\frac{\pi}{2\rho}]$. Then, $f(x)=\int_0^x g(t)dt$ and $f^*(m)=\int_0^m g^{-1}(t)dt$ are trigonometrically $\rho$-convex functions, and satisfy

$$m\sin(\rho x)\le f(x)+f^*(m),\ \ \forall x\in[0,\frac{\pi}{2\rho}].$$

As from Example 3.1, $f(x)$ is trigonometrically $\rho$-convex function. Since $g(x) = \sin \rho x$, we observe that $g^{-1}(m) = \frac{1}{\rho} \arcsin m, \ \ \forall m \in [0,1]$, then

$$\begin{aligned} f^*(m) &= \frac{1}{\rho}\int_0^m \arcsin t dt \\ &= \frac{1}{\rho}[m \arcsin m + \sqrt{1-m^2} - 1] \qquad (3.7) \\ f^{*'}(m) &= \frac{1}{\rho} \arcsin m \\ f^{*''}(m) &= \frac{1}{\rho}\frac{1}{\sqrt{1-t^2}}, \end{aligned}$$

using Theorem 2.3, we observe that

$$f^{*''}(m) + \rho^2 f^*(m) = \frac{1}{\rho}\frac{1}{\sqrt{1-t^2}} + \rho \int_0^m \arcsin t dt \geq 0,$$

then $f^*(m)$ is trigonometrically $\rho$-convex function for all $m \in [0,1]$.
Now, we Check the inequality (3.5).

Define

$$h(x,m) = \int_0^x \sin \rho t dt + \int_0^m \frac{1}{\rho} \arcsin t dt - xm,$$

for $x \in [0, \frac{\pi}{2\rho}]$ and $m$ in $[0,1]$,

$$\begin{aligned} h(x,m) - h(x, \sin \rho x) &= \int_0^m \frac{1}{\rho} \arcsin t dt - \int_0^{\sin \rho x} \frac{1}{\rho} \arcsin t dt - xm + x \sin \rho x \\ &= \int_{\sin \rho x}^m [\frac{1}{\rho} \arcsin t - x] dt \\ &= \int_m^{\sin \rho x} [x - \frac{1}{\rho} \arcsin t] dt. \end{aligned}$$

The first case is $m \geq \sin \rho x$, we have $\arcsin m \geq \arcsin t \geq \arcsin \sin \rho x = \rho x$, $\forall t \in [\sin \rho x, m]$.

Consequently,

$$h(x,m) - h(x, \sin \rho x) = \int_{\sin \rho x}^m [\frac{1}{\rho} \arcsin t - x] dt \geq 0. \qquad (3.8)$$

The second case is $m \leq \sin \rho x$, we have $\arcsin m \leq \arcsin t \leq \arcsin \sin \rho x = \rho x$ $\forall t \in [m, \sin \rho x]$.
Consequently,

$$h(x,m) - h(x, \sin \rho x) = \int_m^{\sin \rho x} [x - \frac{1}{\rho} \arcsin t] dt \geq 0. \qquad (3.9)$$

Using equations (3.2) and (3.7), we obtain that

$$\begin{aligned} h(x,\sin\rho x) &= \frac{1}{\rho}[1-\cos\rho x] + \frac{1}{\rho}[\sin\rho x \arcsin\sin\rho x + \sqrt{1-\sin^2\rho x} - 1] \\ &\quad - x\sin\rho x = 0. \end{aligned} \tag{3.10}$$

From equations (3.8), (3.9), (3.10) and by using inequality (3.6), we conclude that $f(x) + f^*(m) \geq mx \geq m\sin\rho x$, $\forall x \in [0, \frac{\pi}{2\rho}]$.

**Lemma 3.1.** *Let $f_\alpha : I \to \mathbb{R}$ be an arbitrary family of trigonometrically $\rho$-convex functions and*

$$f(x) = \sup_\alpha (f_\alpha(x)) \tag{3.11}$$

*if $J = \{x \in I : f(x) < \infty\}$ is nonempty, then $f : J \to \mathbb{R}$ is trigonometrically $\rho$-convex function.*

**Proof.** Let $x \in [a,b] \subseteq J \subseteq I$ such that $0 < \rho(b-a) < \pi$, since $f_\alpha(x)$ is trigonometrically $\rho$-convex function for all $\alpha$ and equation (3.11). Then,

$$\begin{aligned} f(x) &= \sup_\alpha (f_\alpha(x)) \\ &\leq \sup_\alpha [\frac{f_\alpha(a)\sin\rho(b-x) + f_\alpha(b)\sin\rho(x-a)}{\sin\rho(b-a)}] \\ &\leq \frac{\sup_\alpha[f_\alpha(a)]\sin\rho(b-x) + \sup_\alpha[f_\alpha(b)]\sin\rho(x-a)}{\sin\rho(b-a)} \\ &= \frac{f(a)\sin\rho(b-x) + f(b)\sin\rho(x-a)}{\sin\rho(b-a)}. \end{aligned}$$

From Definition 2.1, implies $f(x) = \sup_\alpha(f_\alpha(x))$ is trigonometrically $\rho$-convex function. $\square$

**Definition 3.1.** *If $f : [0, \frac{\pi}{\rho}] \to \mathbb{R}$ is trigonometrically $\rho$-convex function, then $f^* : I^* \to \mathbb{R}$ is the* ***conjugate of trigonometrically $\rho$-convex function*** *and defined by*

$$f^*(m) := \sup_x [m\sin\rho x - f(x)], \tag{3.12}$$

*with domain $I^* = \{m \in \mathbb{R} : f^*(m) < \infty\}$ such that $m\sin\rho x \geq f(x)$, for all $m$ in $I^*$.*

**Example 3.3.** Let $f(x) = \sin\rho x$ $\forall x \in [0, \frac{\pi}{\rho}]$ be trigonometrically $\rho$-convex function such that $m \geq 1$. Then, its conjugate $f^*(m) = m - 1$, is trigonometrically $\rho$-convex function

As from Definition 3.1,

$$f^*(m) = \sup_x[m\sin\rho x - \sin\rho x], = (m-1)\sup_x[\sin\rho x], = m-1,$$

then $f^{*'}(m) = 1, f^{*''}(m) = 0$. From $m \geq 1$ and $0 < \rho < \infty$. Then

$$f^{*''}(m) + \rho^2 f^*(m) = \rho^2(m-1) \geq 0.$$

By using Theorem 2.3, then $f^*(m) = m-1$ is trigonometrically $\rho$-convex function.

**Theorem 3.2.** *If $f : [0, \frac{\pi}{\rho}] \to \mathbb{R}$ is trigonometrically $\rho$-convex function. Its conjugate $f^* : I^* \to \mathbb{R}$ such that $m\sin\rho x \geq f(x),\ \ \forall m \in I^*$. Then*

(c1) *$f^*(m)$ is trigonometrically $\rho$-convex function.*

(c2) *$m\sin\rho x \leq f(x) + f^*(m)$.*

(c3) *If $f$ is differentiable then*
*$m\sin\rho x = f(x) + f^*(m)$ if and only if $\rho m\cos\rho x = f'(x)$.*

(c4) *For every*

$$g : [0, \frac{\pi}{\rho}] \to \mathbb{R}, m\sin\rho x \geq g(x),\ \forall m \in I^*,\ \text{and } f \leq g \text{ on } [0, \frac{\pi}{\rho}],$$
$$\text{implies } f^* \geq g^* \text{ on } I^*. \tag{3.13}$$

**Proof.** We first prove that $I^* \neq \emptyset$. For if $I$ is single point $x_o$, $f(x)$ is trigonometrically $\rho$-convex function and from Theorem 2.1, Remark 2.1, then $T_{x_o}(x) = f(x_o)\cos\rho(x-x_o) + K_{x_o,f}\sin\rho(x-x_o)$ supports $f$ for each $K_{x_o,f} \in \mathbb{R}$. Otherwise, we choose any interior point $x_o$, choose $K_{x_o,f} \in \frac{1}{\rho}[f'_-(x_o), f'_+(x_o)]$, and again from Theorem 2.1, Remark 2.2 then also $T_{x_o}(x)$ is supporting function for $f(x)$ at $x_o$. In either case then choose $K_{x_o,f}$ such that $T_{x_o}(x) \leq f(x),\ \forall x \in [0, \frac{\pi}{\rho}]$,

$$f(x_o)\cos\rho(x-x_o) + K_{x_o,f}\sin\rho(x-x_o) \leq f(x), \tag{3.14}$$

implies

$$f(x_o)\cos\rho x\cos\rho x_o + f(x_o)\sin\rho x\sin\rho x_o + K_{x_o,f}\sin\rho x\cos\rho x_o$$
$$- K_{x_o,f}\cos\rho x\sin\rho x_o \leq f(x),$$

hence

$$(K_{x_o,f}\cos\rho x_o + f(x_o)\sin\rho x_o)\sin\rho x - f(x) \leq (K_{x_o,f}\sin\rho x_o$$
$$- f(x_o)\cos\rho x_o)\cos\rho x,\ \ \forall x \in [0, \frac{\pi}{\rho}]. \tag{3.15}$$

Let $m = K_{x_o,f} \cos \rho x_o + f(x_o) \sin \rho x_o$, $A = K_{x_o,f} \sin \rho x_o - f(x_o) \cos \rho x_o$, implies $m \sin \rho x - f(x) \leq A \cos \rho x$. If $A$ positive, then we get

$$m \sin \rho x - f(x) \leq A \quad : \cos \rho x \leq 1 \ \ \forall x \in [0, \frac{\pi}{\rho}].$$

Otherwise, if $A$ negative, then we get

$$m \sin \rho x - f(x) \leq -A \quad : \cos \rho x \geq -1 \ \ \forall x \in [0, \frac{\pi}{\rho}].$$

In either case, we have $f^*(m) = \sup_x [m \sin \rho x - f(x)] < \infty$.

Then $I^* \neq \emptyset$.

(c1) Let $g_x(m) = m \sin \rho x - f(x)$, suppose that $u, v \in I^* : 0 < \rho(v-u) < \pi$ and let $m \in (u, v)$, $m = \lambda u + \mu v : \lambda + \mu = 1$ and $\lambda, \mu \in [0, 1]$,

$$\begin{aligned} g_x(m) &= (\lambda u + \mu v) \sin \rho x - f(x)(\lambda + \mu) \\ &= \lambda(u \sin \rho x - f(x)) + \mu(v \sin \rho x - f(x)). \end{aligned}$$

Take

$$\lambda = \frac{\sin \rho(v-m) \cos \rho(m-u)}{\sin \rho(v-u)}, \quad \mu = \frac{\sin \rho(m-u) \cos \rho(v-m)}{\sin \rho(v-u)},$$

hence

$$\begin{aligned} g_x(m) = {} & \frac{\sin \rho(v-m) \cos \rho(m-u)}{\sin \rho(v-u)}(u \sin \rho x - f(x)) \\ & + \frac{\sin \rho(m-u) \cos \rho(v-m)}{\sin \rho(v-u)}(v \sin \rho x - f(x)). \end{aligned}$$

Since $u, v \in I^*$ implies $u \sin \rho x - f(x) \geq 0$ and $v \sin \rho x - f(x) \geq 0$, and from $\cos \rho(m-u) \leq 1$ and $\cos \rho(v-m) \leq 1$. Then

$$\begin{aligned} g_x(m) &\leq \frac{\sin \rho(v-m)}{\sin \rho(v-u)}(u \sin \rho x - f(x)) + \frac{\sin \rho(m-u)}{\sin \rho(v-u)}(v \sin \rho x - f(x)) \\ &= \frac{(u \sin \rho x - f(x)) \sin \rho(v-m) + (v \sin \rho x - f(x)) \sin \rho(m-u)}{\sin \rho(v-u)} \\ &= \frac{g_x(u) \sin \rho(v-m) + g_x(v) \sin \rho(m-u)}{\sin \rho(v-u)}. \end{aligned}$$

By using Definition 2.1, then $g_x(x)$ is a trigonometrically $\rho$-convex function, and by the Lemma 3.1, $f^*(m) = \sup_x(g_x(m))$ is a trigonometrically $\rho$-convex function.

(c2) Since equation (3.12), implies

$$m \sin \rho x - f(x) \leq f^*(m), \tag{3.16}$$

then $m\sin\rho x \le f(x) + f^*(m)$.

(c3) To prove the necessity, by differentiate $m\sin\rho x = f(x) + f^*(m)$ with respect to $x$ implies, $\rho m\cos\rho x = f^{'}(x)$.

The sufficiency, let

$$\rho m\cos\rho x = f^{'}(x), \tag{3.17}$$

implies $\int_0^x \rho m\cos\rho t dt = \int_0^x f^{'}(t)dt$. Hence,

$$m\sin\rho x = f(x) - f(0), \tag{3.18}$$

since $m \in I^*$, then

$$-f(0) = m\sin\rho x - f(x) \ge 0. \tag{3.19}$$

From Remark 2.1, then the supporting function for $f(x)$ at the point $0 \in [0, \frac{\pi}{\rho}]$, has the form

$$T_0(x) = f(0)\cos\rho x + \frac{f^{'}(0)}{\rho}\sin\rho x \le f(x), \quad \forall x \in [0, \frac{\pi}{\rho}]$$

from equation(3.17) at $x = 0$ implies $m = \frac{f^{'}(0)}{\rho}$, and we get $m\sin\rho x - f(x) \le -f(0)\cos\rho x$, hence

$$\begin{aligned}
\sup_x[m\sin\rho x - f(x)] &\le -f(0)\cos\rho x \\
f^*(m) = \sup_x[m\sin\rho x - f(x)] &\le -f(0)\cos\rho x \\
f^*(m) &\le -f(0)\cos\rho x. \qquad (3.20)
\end{aligned}$$

From equation (3.19), inequality (3.20) and $\cos\rho x \le 1$, implies

$$f^*(m) \le m\sin\rho x - f(x) \tag{3.21}$$

from inequalities(3.16), and(3.21), then

$$m\sin\rho x - f(x) = f^*(m). \tag{3.22}$$

(c4) Since $f \le g$ implies $f(x) \le g(x), \forall x \in [0, \frac{\pi}{\rho}]$, then

$$\begin{aligned}
-g(x) &\le -f(x) \\
m\sin\rho x - g(x) &\le m\sin\rho x - f(x) \\
\sup_x[m\sin\rho x - g(x)] &\le m\sin\rho x - f(x), \qquad (3.23)
\end{aligned}$$

from Definition 3.1, $m\sin\rho x \ge g(x)$ and inequality (3.23) implies

$$\begin{aligned}
g^*(m) &= \sup_x[m\sin\rho x - g(x)] \\
&\le m\sin\rho x - f(x) \\
&\le \sup_x[m\sin\rho x - f(x)],
\end{aligned}$$

from Definition 3.1, and $m \sin \rho x \geq f(x)$ then

$$g^*(m) \leq f^*(m), \quad \forall m \in I^*, \tag{3.24}$$

then $f^* \geq g^*$ *on* $I^*$.

□

**Remark 3.1.** For a trigonometrically $\rho$-convex function $f : [0, \frac{\pi}{2\rho}] \to \mathbb{R}$ if $\inf f(x) \neq -\infty$, then the domain of its conjugate $I^* = \mathbb{R}$, where

$$f^*(m) = \sup_x [m \sin \rho x - f(x)] \leq m - \inf f(x).$$

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