Some results on $K$-frames

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Abstract. In this paper we present some results on $K$-frames when $K \in B(H)$ is an injective closed range operator. Also we give a condition on $K$-frames $\{f_n\}_{n \in \mathbb{N}}$ and $\{g_n\}_{n \in \mathbb{N}}$ so that $\{f_n + g_n\}_{n \in \mathbb{N}}$ is again a $K$-frame for $H$. Finally, Schatten class operators are also discussed in terms of $K$-frames.

Keywords: $K$-frames, Schatten class operators.

1. Introduction

Frames in Hilbert spaces were introduced by R.J. Duffin and A.C. Schaffer. Later Daubechies, Grossmann and Meyer gave a strong place to frames in harmonic analysis. Frame theory plays an important role in signal processing, sampling theory, coding and communications and so on. Frames were introduced as a better replacement to orthonormal basis. We refer [2] for an introduction to frame theory.

$K$-frames were introduced by L. Gavruta, to study atomic systems with respect to bounded linear operators. $K$-frames are more general than classical frames. In $K$-frames the lower bound only holds for the elements in the range of $K$.

Some basic definitions and results related to frames and $K$-frames are contained in section 2. In section 3 we have included some new results on $K$-frames. Section 4 contains our main results relating $K$-frames and operators in Schatten classes.

Throughout this paper, $H$ is a separable Hilbert space and we denote by $B(H)$, the space of all linear bounded operators on $H$. For $K \in B(H)$, we denote $R(K)$ the range of $K$. Also, $GL(H)$ denote the set of all bounded linear operators which have bounded inverses.

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2. Preliminaries

For a separable Hilbert space $H$, a sequence $\{f_n\}_{n \in \mathbb{N}} \subset H$ is said to be a frame ([2]) for $H$ if there exist $A, B > 0$ such that

$$A\|x\|^2 \leq \sum_{n=1}^{\infty} |\langle x, f_n \rangle|^2 \leq B\|x\|^2,$$

for all $x \in H$. If $A = B$, we say that $\{f_n\}_{n \in \mathbb{N}}$ is a tight frame in $H$. Let $K \in B(H)$. We say that $\{f_n\}_{n \in \mathbb{N}} \subset H$ is a $K$-frame ([3]) for $H$ if there exist constants $A, B > 0$ such that

$$A\|Kx\|^2 \leq \sum_{n=1}^{\infty} |\langle x, f_n \rangle|^2 \leq B\|x\|^2,$$

for all $x \in H$.

If $\{f_n\}_{n \in \mathbb{N}} \subset H$ is an ordinary frame for $H$, then $\{Kf_n\}_{n \in \mathbb{N}}$ is a frame for $K^*H$ and hence $\{KK^*f_n\}_{n \in \mathbb{N}}$ is a $K$-frame for $H$.

3. $K$-frames

In this section we present our results on $K$-frames.

Theorem 3.1. Let $K \in B(H)$ be an injective and closed range operator. If $\{f_n\}_{n \in \mathbb{N}}$ is a frame for $R(K)$, then $\{K^*f_n\}_{n \in \mathbb{N}}$ is a frame for $H$ and hence $\{KK^*f_n\}_{n \in \mathbb{N}}$ is a $K$-frame for $H$.

Proof. Let $\{f_n\}_{n \in \mathbb{N}}$ be a frame for $R(K)$. Then there exist constants $A, B > 0$ such that, for all $x \in R(K),

$$A\|x\|^2 \leq \sum_{n=1}^{\infty} |\langle x, f_n \rangle|^2 \leq B\|x\|^2.$$

Also, by our assumption, there exists $c > 0$ such that $c\|x\|^2 \leq \|Kx\|^2$, for all $x \in H$. For $x \in H, Kx \in R(K)$, and we get

$$A\|Kx\|^2 \leq \sum_{n=1}^{\infty} |\langle Kx, f_n \rangle|^2 \leq B\|Kx\|^2.$$
Therefore,

\[ Ac \|x\|^2 \leq A\|Kx\|^2 \leq \sum_{n=1}^{n=\infty} |\langle Kx, f_n \rangle|^2 \leq B\|Kx\|^2 \leq B\alpha^2 \|x\|^2, \]

for all \( x \in H \) and for some \( \alpha > 0 \), i.e.

\[ E\|x\|^2 \leq \sum_{n=1}^{n=\infty} |\langle x, K^* f_n \rangle|^2 \leq F\|x\|^2, \]

for all \( x \in H \) where \( E = Ac > 0, F = B\alpha^2 > 0 \). Therefore, \( \{K^* f_n\}_{n \in N} \) is a frame for \( H \) and hence \( \{KK^* f_n\}_{n \in N} \) is a \( K \)-frame for \( H \).

**Corollary 3.2.** Let \( K \in B(H) \) be an injective and closed range operator and \( \{f_n\}_{n \in N} \subset H \) be such that \( \{(K^{-1})^* f_n\}_{n \in N} \) is a frame for \( R(K) \). Then \( \{f_n\}_{n \in N} \) is a frame for \( H \).

**Theorem 3.3.** Suppose \( \{f_n\}_{n \in N} \) is a \( K \)-frame for \( H \) where \( K^* \) is an injective and closed range operator. Then there exist constants \( A, B > 0 \) such that

\[ A\|K^*x\|^2 \leq \sum_{n=1}^{n=\infty} |\langle x, f_n \rangle|^2 \leq B\|K^*x\|^2, \]

for all \( x \in H \).

**Proof.** Since \( \{f_n\}_{n \in N} \) is a \( K \)-frame for \( H \), there exist constants \( C, D > 0 \) such that

\[ C\|K^*x\|^2 \leq \sum_{n=1}^{n=\infty} |\langle x, f_n \rangle|^2 \leq D\|x\|^2, \]

for all \( x \in H \). Since \( K^* \in B(H) \) is an injective and closed range operator, there exist \( d > 0 \) such that

\[ d\|x\|^2 \leq \|K^*x\|^2, \]

for all \( x \in H \). Therefore, for all \( x \in H \),

\[ C\|K^*x\|^2 \leq \sum_{n=1}^{n=\infty} |\langle x, f_n \rangle|^2 \leq D\|x\|^2 \leq \frac{D}{d}\|K^*x\|^2, \]

for all \( x \in H \) there exist \( A = C, B = D/d > 0 \) such that

\[ A\|K^*x\|^2 \leq \sum_{n=1}^{n=\infty} |\langle x, f_n \rangle|^2 \leq B\|K^*x\|^2. \]

**Corollary 3.4.** Suppose \( \{f_n\}_{n \in N} \) is a \( K \)-frame for \( H \) where \( K^* \) is an injective and closed range operator. Then \( \{f_n\}_{n \in N} \) is a frame for \( H \).
Definition 3.5. A sequence \( \{f_n\}_{n \in \mathbb{N}} \subset H \) is said to be a \( 2K \)-frame for \( H \) if there exist \( A, B > 0 \) such that
\[
A \|K^* x\|^2 \leq \sum_{n=1}^{\infty} |\langle x, f_n \rangle|^2 \leq B \|K^* x\|^2,
\]
for all \( x \in H \).

Theorem 3.6. Let \( \{f_n\}_{n \in \mathbb{N}} \) be a \( K \)-frame for \( H \) with bounds \( A_1, B_1 \) and \( \{g_n\}_{n \in \mathbb{N}} \) be a \( 2K \)-frame for \( H \) with bounds \( A_2, B_2 \) such that \( 0 < B_2 < A_1 \). Then \( \{f_n + g_n\}_{n \in \mathbb{N}} \) is a \( K \)-frame for \( H \) with frame bounds \( A_1 - B_2 \) and \( B_1 + B_2 \|K^*\|^2 \).

Proof. By definition of \( K \)-frame and \( 2K \)-frame, we have
\[
A_1 \|K^* x\|^2 \leq \sum_{n=1}^{\infty} |\langle x, f_n \rangle|^2 \leq B_1 \|x\|^2
\]
and
\[
A_2 \|K^* x\|^2 \leq \sum_{n=1}^{\infty} |\langle x, g_n \rangle|^2 \leq B_2 \|K^* x\|^2,
\]
for all \( x \in H \). Consider,
\[
\sum_{n=1}^{\infty} |\langle x, f_n + g_n \rangle|^2 \leq \sum_{n=1}^{\infty} |\langle x, f_n \rangle|^2 + \sum_{n=1}^{\infty} |\langle x, g_n \rangle|^2 \leq B_1 \|x\|^2 + B_2 \|K^* x\|^2 \leq (B_1 + B_2 \|K^*\|^2) \|x\|^2,
\]
for all \( x \in H \).

Consider,
\[
\sum_{n=1}^{\infty} |\langle x, f_n \rangle|^2 = \sum_{n=1}^{\infty} |\langle x, f_n + g_n - g_n \rangle|^2 \leq \sum_{n=1}^{\infty} |\langle x, f_n + g_n \rangle|^2 + \sum_{n=1}^{\infty} |\langle x, g_n \rangle|^2.
\]
This implies that,
\[
A_1 \|K^* x\|^2 \leq \sum_{n=1}^{\infty} |\langle x, f_n + g_n \rangle|^2 + B_2 \|K^* x\|^2
\]
i.e. \( \sum_{n=1}^{\infty} |\langle x, f_n + g_n \rangle|^2 \geq (A_1 - B_2) \|K^* x\|^2 \)
where \( A_1 - B_2 > 0 \). This completes the proof. \( \square \)
4. *K*-frames and operators in Schatten classes

**Definition 4.1** ([7]). Let $T$ be a compact operator on a separable Hilbert space $H$. Given $0 < p < \infty$, we define the **Schatten $p$-class** of $H$, denoted by $S_p(H)$ or simply $S_p$, to be the space of all compact operators $T$ on $H$ with its singular value sequence $\{\lambda_n\}$ belonging to $l^p$. $S_p(H)$ is a two sided ideal in $B(H)$.

Following two theorems by H. Bingyang, L.H. Khoi and K. Zhu gives a characterization for Schatten $p$-class operators in terms of frames.

**Theorem 4.2** ([1]). Suppose $T$ is a compact operator on $H$ and $2 \leq p < \infty$. Then the following conditions are equivalent:

(a) $T \in S_p$;

(b) $\|Te_n\|_{n \in \mathbb{N}} \in l^p$ for every orthonormal basis $\{e_n\}_{n \in \mathbb{N}}$ in $H$;

(c) $\|Tf_n\|_{n \in \mathbb{N}} \in l^p$ for every frame $\{f_n\}_{n \in \mathbb{N}}$ in $H$.

**Theorem 4.3** ([1]). Suppose $T$ is a compact operator on $H$ and $0 \leq p \leq 2$. Then the following conditions are equivalent:

(a) $T \in S_p$;

(b) $\|Te_n\|_{n \in \mathbb{N}} \in l^p$ for some orthonormal basis $\{e_n\}_{n \in \mathbb{N}}$ in $H$;

(c) $\|Tf_n\|_{n \in \mathbb{N}} \in l^p$ for some frame $\{f_n\}_{n \in \mathbb{N}}$ in $H$.

At first we focus on the case where $2 \leq p < \infty$.

**Theorem 4.4.** Suppose $T$ is a compact operator on $H$ and $K \in B(H)$. If $T$ is in the Schatten class $S_p$, then $\|Tf_n\|_{n \in \mathbb{N}} \in l^p$ for every $K$-frame $\{f_n\}_{n \in \mathbb{N}}$ in $H$, where $2 \leq p < \infty$.

**Proof.** Suppose $T \in S_p, 2 \leq p < \infty$.

Let $\{f_n\}_{n \in \mathbb{N}}$ be a $K$-frame for $H$ and $\{e_n\}_{n \in \mathbb{N}}$ be an orthonormal basis for $H$. Then $\{h_n\}_{n \in \mathbb{N}} = \{f_n\}_{n \in \mathbb{N}} \cup \{e_n\}_{n \in \mathbb{N}}$ is a frame for $H$ and $\|T_h_n\|_{n \in \mathbb{N}} \in l^p, 2 \leq p < \infty$. Therefore $\|Tf_n\|_{n \in \mathbb{N}} \in l^p, 2 \leq p < \infty$ and the result is proved.

**Theorem 4.5.** Suppose $T$ is a compact operator on $H$ and $K \in B(H)$. If $\|Tf_n\|_{n \in \mathbb{N}} \in l^p$ for every $K$-frame $\{f_n\}_{n \in \mathbb{N}}$ in $H$, then $\|TKe_n\|_{n \in \mathbb{N}} \in l^p$ for every orthonormal basis $\{e_n\}_{n \in \mathbb{N}}$ in $H$, where $2 \leq p < \infty$.

**Proof.** Let $\{e_n\}_{n \in \mathbb{N}}$ be an orthonormal basis for $H$. Then $\{Ke_n\}_{n \in \mathbb{N}}$ is a $K$-frame for $H$. Therefore by our assumption $\|TKe_n\|_{n \in \mathbb{N}} \in l^p, 2 \leq p < \infty$. Hence $\|TKe_n\|_{n \in \mathbb{N}} \in l^p$ for every orthonormal basis $\{e_n\}_{n \in \mathbb{N}}$ in $H$.

**Theorem 4.6.** Suppose $T$ is a compact operator on $H$ and $K \in GL(H)$ and $2 \leq p < \infty$. Then the following are equivalent:
(a) $T$ is in the Schatten class $S_p$;

(b) $\|Tf_n\|_{n \in N} \in l^p$ for every $K$-frame $\{f_n\}_{n \in N}$ in $H$.

**Proof.** Clearly, (a) implies (b) holds by Theorem 4.4. Now suppose (b) holds. Then $\|TK e_n\|_{n \in N} \in l^p$ for every orthonormal basis $\{e_n\}_{n \in N}$ in $H$. This implies that $TK \in S_p$. Using the fact that $S_p$ is a two-sided ideal in $B(H)$, $TKK^{-1} \in S_p$, i.e. $T \in S_p$. This completes the proof. \qed

Now we move onto the case where $0 < p \leq 2$.

**Theorem 4.7.** Let $T$ be a compact operator on $H$ and $K \in B(H)$. Suppose $\|Te_n\|_{n \in N} \in l^p$ for some orthonormal basis $\{e_n\}_{n \in N} \subset H$. Then $\|Tf_n\|_{n \in N} \in l^p$ for some $K$-frame $\{f_n\}_{n \in N}$ for $H$, where $0 < p \leq 2$.

**Proof.** Suppose $\|Te_n\|_{n \in N} \in l^p$ for some orthonormal basis $\{e_n\}_{n \in N} \subset H$. Then $T \in S_p$, which implies that $TK \in S_p$ for any $K \in B(H)$. By Theorem 4.3, $\|TK e_n\|_{n \in N} \in l^p$ for some orthonormal basis $\{e_n\}_{n \in N}$ in $H$. Now take $f_n = Ke_n$, so that $\{f_n\}_{n \in N}$ is a $K$-frame for $H$ and hence the theorem holds. \qed

**Theorem 4.8.** Let $T$ be a compact operator on $H$ and $K \in B(H)$, where $K^*$ is an injective closed range operator. If $\|Tf_n\|_{n \in N} \in l^p$ for some $K$-frame $\{f_n\}_{n \in N}$ for $H$, then $T \in S_p$, where $0 < p \leq 2$.

**Proof.** By Corollary 3.4, if $K^*$ is an injective closed range operator, then every $K$-frame is a frame and then applying Theorem 4.3, we get $T \in S_p$. \qed

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**References**


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