Invo-clean rings associated with central polynomials

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Abstract. Let $R$ be an associative ring with identity and let $C(R)$ be the center of a ring $R$ and let $g(x)$ be a fixed polynomial in $C(R)[x]$. We defined $R$ to be $g(x)$-invo clean if every element in $R$ can be written as a sum of an involution and a root of $g(x)$. In this paper, we investigate conditions on a ring to be $g(x)$-invo clean ring. Some properties and several examples are given.

Keywords: clean rings, $g(x)$-invo clean rings, invo clean rings.

1. Introduction

Let $R$ be an associative ring with identity. Following [6], we define an element $r$ of a ring $R$ to be clean if there is an idempotent $e \in R$ and a unit $u \in R$ such that $r = u + e$. A clean ring is defined to be one in which every element is clean. Clean rings were first introduced by Nicholson [6] as a class of exchange rings.

The invo-clean rings was introduced by Danchev [2]. He defined and completely described the structure of invo-clean rings having identity.

Camillo and Simon [1], defined $g(x)$-clean rings. An element $r \in R$ is called $g(x)$-clean if $r = s + u$ where $g(s) = 0$ and $u$ is a unit of $R$ and $R$ is a $g(x)$-clean ring if every element is $g(x)$-clean. The $(x^2 - x)$-clean rings are precisely the clean rings. In Fan and Yang [3], authors studied more properties of $g(x)$-clean rings. Among many conclusions, they proved that if $g(x) \in (x-a)(x-b)C(R)[x]$, where $a, b \in C(R)$ with $(b - a)$ unit in $R$, then $R$ is a clean ring if and only if $R$ is $(x-a)(x-b)$-clean. For the study of clean rings and their generalizations, we refer to [4], [5], [7].

In this paper, we introduce the notion of $g(x)$-invo clean ring. A ring $R$ is said to be $g(x)$-invo clean ring if any element in $R$ can be written as a sum of

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involution and a root of \( g(x) \). Clearly, invo-clean rings are \((x^2 - x)\)-invo clean rings.

Throughout this paper, we assume that all rings are associative with identity and all modules are unitary. As usual, \( U(R) \) denotes the set of all units of \( R \), \( Inv(R) \) the subset of \( U(R) \) consisting of all involutions (i.e.; \( v \in Inv(R) \) then \( v^2 = 1 \)) of \( R \), \( Id(R) \) the set of all idempotents of \( R \) and \( \text{Nil}(R) \) the set of all nilpotents, \( C(R) \) denotes the center of \( R \) and \( g(x) \) be a fixed polynomial with coefficients in \( C(R) \).

2. \( g(x) \)-Invo clean rings

In this section, we define \( g(x) \) -invo clean rings, we give some properties of \( g(x) \)-invo clean ring and present several examples.

**Definition 2.1.** Let \( R \) be a ring and let \( g(x) \) be a fixed polynomial in \( C(R)[x] \). An element \( r \in R \) is called \( g(x) \)-invo clean if \( r = v + s \) where \( g(s) = 0 \) and \( v \) is an involution of \( R \) i.e., \( v^2 = 1 \). We say that \( R \) is \( g(x) \)-invo clean if every element in \( R \) is \( g(x) \)-invo clean.

Clearly, Every \((x^2 - x)\)-invo clean ring is invo clean.

**Example 2.2.** \( \mathbb{Z}_7 \) is \((x^6 - 1)\)-invo clean ring which is not invo-clean ring.

**Example 2.3.** The ring \( M_2(\mathbb{Z}_2) \) is \((x^3 - x)\) -invo clean ring.

**Proposition 2.4.** Every \( g(x) \) -invo clean ring is \( g(x) \) -clean ring.

**Proof.** Suppose \( R \) is a \( g(x) \) -invo clean ring and let \( r \in R \). Then \( r = v + s \) where \( v \) is involution and \( g(s) = 0 \). But every involution is unit. Thus, \( R \) is \( g(x) \) -clean ring.

The converse of Proposition 2.4 is not true in general. For example, we can see that \( M_2(\mathbb{Z}_2) \) is \((x^6 - 1)\) -clean ring which not \((x^6 - 1)\)-invo clean, since

\[
\begin{bmatrix}
1 & 1 \\
0 & 1 \\
\end{bmatrix}
\]

cannot be written as a sum of involution and a root of \((x^6 - 1) \). \( \square \)

Let \( R \) and \( S \) be rings and \( \Psi : C(R) \to C(S) \) be a ring epimorphism with \( \Psi(1_R) = 1_S \). For \( g(x) = \sum_{i=0}^n a_i x^i \in C(R)[x] \), we let \( g^*(x) = \sum_{i=0}^n \Psi(a_i) x^i \in C(S)[x] \). In particular, If \( g(x) \in \mathbb{Z}[x] \), then \( g^*(x) = g(x) \).

**Proposition 2.5.** Let \( \theta : R \to S \) be a ring epimorphism. If \( R \) is \( g(x) \) -invo clean, then \( S \) is \( g^*(x) \) -invo clean.

**Proof.** Let \( g(x) = \sum_{i=0}^n a_i x^i \in C(R)[x] \) and consider \( g^*(x) = \sum_{i=0}^n \theta(a_i) x^i \in C(S)[x] \). For every \( \beta \in S \), there exist \( r \in R \) such that \( \theta(r) = \beta \). Since \( R \) is \( g(x) \) -invo clean, there exists \( s \in R \) and \( v \in Inv(R) \) such that \( r = v + s \) and \( g(s) = 0 \). Then \( \beta = \theta(r) = \theta(v + s) = \theta(v) + \theta(s) \) with \( \theta(v) \in Inv(S) \), and \( g^*(\theta(s)) = \sum_{i=0}^n \theta(a_i) \theta(s)^i = \sum_{i=0}^n \theta(a_i) \theta(s)^i = \sum_{i=0}^n \theta(a_i s^i) = \theta(\sum_{i=0}^n a_i s^i) = \theta(g(s)) = \theta(0) = 0 \). Therefore, \( S \) is \( g^*(x) \) -invo clean. \( \square \)
Proposition 2.6. Let \( R \) be an \( g(x) \)-nil clean with \( n^2 = -2n \) for every \( n \in \text{Nil}(R) \). Then \( R \) is \( g(x) \)-invo clean.

Proof. Suppose \( R \) is a \( g(x) \)-nil clean and let \( r \in R \). Then \( r - 1 = n + s \) where \( n \in \text{Nil}(R) \) and \( g(s) = 0 \). Thus \( r = (1 + n) + s \). Indeed \( 1 + n \) is an involution. Therefore \( R \) is \( g(x) \)-invo clean.

Proposition 2.7. If \( R \) an \( g(x) \)-invo clean ring and \( I \) is an ideal of \( R \), then \( R = R/I \) is \( g^*(x) \)-invo clean.

Proof. Let \( R \) be an \( g(x) \)-invo clean ring and \( \theta : R \to R/I \) defined by \( \theta(r) = r + I \). Then \( \theta \) is an epimorphism. By Proposition 2.5 \( R/I \) is \( g(x) \)-invo clean.

Proposition 2.8. Let \( R_1, R_2, \ldots, R_k \) be rings and \( g(x) \in \mathbb{Z}[x] \). Then \( R = \prod_{i=1}^{k} R_i \) is \( g(x) \)-invo clean if and only if \( R_i \) is \( g(x) \)-invo clean for all \( i \in \{1, 2, \ldots, n\} \).

Proof. \( \Rightarrow \) : For each \( i \in \{1, 2, \ldots, n\} \), \( R_i \) is a homomorphic image of \( \prod_{i=1}^{k} R_i \) under the projection homomorphism. Hence, \( R_i \) is \( g(x) \)-invo clean by Proposition 2.5.

\( \Leftarrow \) : Let \((x_1, x_2, \ldots, x_k) \in \prod_{i=1}^{k} R_i \). For each \( i \), write \( x_i = v_i + s_i \) where \( v_i \in \text{Inv}(R_i) \), \( g(s_i) = 0 \). Let \( v = (v_1, v_2, \ldots, v_k) \) and \( s = (s_1, s_2, \ldots, s_k) \). Then, it is clear that \( v \in \text{Inv}(R) \) and \( g(s) = 0 \). Therefore, \( R \) is \( g(x) \)-invo clean.

Theorem 2.9. Let \( R \) be a ring and let \( R[t] \) be the rings of polynomial in an indeterminate \( t \) with coefficients in \( R \) and let \( f(t) = a_0 + a_1 t + \ldots + a_n t^n \in R[t] \). If \( f(t) \) is an involution then \( a_0 \) is an involution in \( R \) and \( a_1, \ldots, a_n \) are nilpotents.

Proof. Assume \( f(t) \) is a unit then \( a_0 \) is a unit in \( R \) and \( a_1, \ldots, a_n \) are nilpotents. Since \( \text{Inv}(R) \subseteq \mathbb{U}(R) \), the statement holds.

Proposition 2.10. Let \( R \) be a commutative ring, then the ring of polynomials \( R[t] \) is not invo clean (not \((x^2 - x)\)-invo clean).

Proof. Let \( t \) be an invo clean, then we may write \( t = a_0 + a_1 t + \ldots + a_n t^n + e \) where \( e \in \text{Id}(R[t]) = \text{Id}(R) \) and \( a_0 \in \text{Inv}(R), a_1, \ldots, a_n \in \text{Nil}(R) \). Hence, \( 1 = a_1 \in J(R) \). Which a contradiction. Hence, \( R[t] \) is not invo-clean.

Let \( R \) be a commutative ring and \( M \) an \( R \)-module. The idealization \( R(M) \) of \( R \) and \( M \) is the ring \( R(M) = R \oplus M \) with multiplication \( (r_1, m_1)(r_2, m_2) = (r_1 r_2, r_1 m_2 + r_2 m_1) \). Note that if \((r, m) \in R(M) \), then \((r, m)^k = (r^k, kr^{k-1}m) \) for any \( k \in \mathbb{N} \).

Lemma 2.11. Let \( R \) be a commutative ring with \( \text{char}(R) = 2 \) and \( M \) an \( R \)-module. Then \((v, m) \) is an involution in \( R(M) \) if and only if \( v \) is involution in \( R \).
Proof. \( \Rightarrow \): Let \((v, m) \in R(M)\) then \((v, m)^2 = (v^2, 2vm) = (1, 0)\). So, \(v^2 = 1\). Thus, \(v\) is involution.

\(\Leftarrow\): Let \(v\) be an involution, and \((v, m) \in R(M)\). Then \((v, m)^2 = (1, 0)\). Hence, \((v, m)\) is an involution of \(R(M)\).

We recall that \(R\) logically embeds into \(R(M)\) via \(r \rightarrow (r, 0)\). Therefore any polynomial \(g(x) = \sum_{i=0}^{n} a_i x^i \in R[x]\) can be written as \(g(x) = \sum_{i=0}^{n} (a_i, 0) x^i \in R(M)[x]\) and conversely.

**Proposition 2.12.** Suppose \(R\) is a commutative ring with \(Char(R) = 2\) and \(M\) an \(R\)-module. So the idealization \(R(M)\) of \(R\) and \(M\) is \(g(x)\)-invo clean if and only if \(R\) is \(g(x)\)-invo clean.

**Proof.** \(\Rightarrow\): Since \(R \cong R(M)/(0 \oplus M)\) is a homomorphic image of \(R(M)\). Hence \(R\) is \(g(x)\)-invo clean by Proposition 2.5.

\(\Leftarrow\): Let \(g(x) = \sum_{i=0}^{n} a_i x^i \in R[x]\) and \(r \in R\). Write \(r = v + s\) where \(v \in Inv(R)\) and \(g(s) = 0\). Then for \(m \in M, (r, m) = (v, m) + (s, 0)\) where \((v, m) \in Inv(R(M))\) and

\[
g(s, 0) = a_0 (1, 0) + a_1 (s, 0) + a_2 (s, 0)^2 + \ldots + a_n (s, 0)^n
\]

\[
= a_0 (1, 0) + a_1 (s, 0) + a_2 (s^2, 0) + \ldots + a_n (s^n, 0)
\]

\[
= (a_0 + a_1 s + a_2 s^2 + \ldots + a_n s^n, 0) = (g(s), 0) = (0, 0).
\]

Therefore, \(R(M)\) is \(g(x)\)-invo clean.

\(\Box\)

3. \((x^2 + cx + d)\)-invo clean rings

We consider some types of \((x^2 + cx + d)\) -invo clean rings.

**Theorem 3.1.** Let \(R\) be a ring and \(a, b \in C(R)\) and \(g(x) \in (x-a)(x-b)\) where \(b-a \in Inv(R)\). Then \(R\) is invo-clean if and only if \(R\) is \((x-a)(x-b)\)-invo clean.

**Proof.** \(\Rightarrow\): Since \(R\) is invo-clean and \(r \in R\) then \(\frac{r-a}{b-a} = v + e\) where \(v \in Inv(R)\) and \(e \in Id(R)\) then \(r = v (b-a) + e (b-a) + a, b-a \in C(R)\) and \(C(R)\) is a subring of \(R\). Since \((e (b-a) + a) (e (b-a) + a-b) = (eb - ea)(eb - ca + a-b) =\)

\[
e^2b^2 - e^2ba + eab - eb^2 - e^2ab + e^2a^2 - e^2a^2 - ea^2 + eab = 0,
\]

it follows \(e (b-a) + a\) is root of \((x-a)(x-b)\). Since \(v (b-a) \in Inv(R)\) by \((v (b-a))^2 = v (b-a) v (b-a) = v^2 (b-a)^2 = 1.1 = 1\), it follow that \(v (b-a) \in Inv(R)\). Then \(R\) is \((x-a)(x-b)\)-invo clean.

\(\Leftarrow\): Let \(r \in R\). Since \(R\) is \((x-a)(x-b)\)-invo clean, \(r (b-a) + a = v + e\) where \(e\) is root of \((x-a)(x-b)\) and \(v \in Inv(R)\). Thus, \(r = \frac{e-a}{b-a} + \frac{v}{b-a}\). Clearly, \(\frac{v}{b-a} \in Inv(R)\) and \(\frac{e-a}{b-a}\) is an idempotent since \(\left(\frac{e-a}{b-a}\right)^2 = \frac{e-a}{b-a}\). Hence \(R\) is invo-clean.

\(\Box\)

**Corollary 3.2.** Let \(R\) be a ring. Then \(R\) is invo-clean if and only if \(R\) is \((x^2 + x)\)-invo clean.
Proof. In the previous Theorem 3.1 but \( a = 0 \) and \( b = -1 \).

\[ \square \]

Proposition 3.3. Let \( R \) be a ring with \( 2 \in \text{Inv}(R) \) and \( k \in \mathbb{N} \). Then the following are equivalent:

1. \( R \) is invo clean
2. \( R \) is \((x^2 - 2x)\)-invo clean
3. \( R \) is \((x^2 + 2x)\)-invo clean
4. \( R \) is \((x^2 - 2^{2k} x)\)-invo clean
5. \( R \) is \((x^2 + 2^{2k} x)\)-invo clean
6. \( R \) is \((x^2 - 1)\)-invo clean
7. \( R \) is For every \( r \in R, r \) can be expressed as \( r = v + s \) with \( v, s \in \text{Inv}(R) \).

Proof. (1) \( \Rightarrow \) (2) Since \( R \) is invo clean and \( r \in R, \frac{r}{2} = v + s \) with \( v \in \text{Inv}(R) \) and \( s^2 = s \), then \( r = 2v + 2s \) with \( 2v \in \text{Inv}(R) \) and \((2s)^2 - 2 \cdot 2s = 4s^2 - 4s = 0 \). Hence, \( R \) is \((x^2 - 2x)\)-invo clean.

(2) \( \Rightarrow \) (1) Since \( R \) is \((x^2 - 2x)\)-invo clean, \( 2r = v + s \) where \( v \in \text{Inv}(R) \) and \( s \) is a root of \((x^2 - 2x)\). Then, \( r = \frac{v}{2} + \frac{s}{2} \), where \( \frac{v}{2} \) is an invo of \( R \) and \((\frac{1}{2})^2 = (\frac{(s)(s-2+2)}{(2)^2}) = \frac{s^2}{(2)^2} = \frac{1}{2} \). So, \( R \) is invo clean. Correspondingly, we may prove \( (3) \Rightarrow (1) \).

(2) \( \Rightarrow \) (3) \( R \) is \((x^2 - 2x)\)-invo clean and let \( r \in R \), \(-r = v + s \) such that \( v \in \text{Inv}(R) \) and \( s^2 - 2s = 0 \). Then, \( r = (-v) + (-s) \) with \(-v \in \text{Inv}(R) \) and \((-s)^2 + 2 \cdot (-s) = s^2 - 2s = 0 \). Thus, \( R \) is \((x^2 + 2x)\)-invo clean.

(1) \( \iff \) (4) By Theorem 3.1, let \( a = 0 \) and \( b = 2^{2k} \). Then, \( R \) is \((x^2 - 2^{2k} x)\)-invo clean.

(1) \( \Rightarrow \) (5) Can be proved by (1) \( \iff \) (4) and (2) \( \Rightarrow \) (3).

(1) \( \Rightarrow \) (6) Since \( R \) is invo clean and \( r \in R \) then \( r = v + s \) where \( v, s \in \text{Inv}(R) \) and \( s^2 = s \). Then \( s \) is a root of \( x^2 - 1 \) by (7). Then \((x^2 - 1)\)-invo clean.

(7) \( \Rightarrow \) (6) Let \( r \in R \) we write \( r = v + s \) with \( v, s \in \text{Inv}(R) \) and \( s^2 = 1 \), then \( s \) is a root of \( x^2 - 1 \) and \( v \in \text{Inv}(R) \). Then, \((x^2 - 1)\) is invo clean ring.

(6) \( \Rightarrow \) (7) If \( R \) is \((x^2 - 1)\)-invo clean, then for every \( r \in R \) there exist \( v, s \in \text{Inv}(R) \) such that \( r = v + s \).

\[ \square \]

References


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