Ricci semi-symmetric normal complex contact metric manifolds

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Abstract. In this paper, we obtain the necessary and sufficient conditions a complex almost contact metric manifold to be normal. In addition, we give some new identities for the Riemann curvature and the Ricci curvatures of normal complex contact metric manifolds. Furthermore, we show that a Ricci semi-symmetric normal complex contact metric manifold is Einstein.

Keywords: normal complex contact metric manifold, Ricci semi-symmetric, curvature.

1. Introduction

Kobayashi [18] started studies on complex contact manifolds in 1959. After this Boothby [7], [8] and Wolf [22] presented new results about complex contact manifolds. Further research started again in the early 1980's by Ishihara and Konishi [15], [16] and [17]. They introduced a concept of normality which is called IK-normality in literature [16]. But according to their normality condition complex structure is Kähler. In 1996, Foreman investigated special metrics on complex contact manifolds by studying critical condition of various Riemanian functionals on particular classes of Riemanian metrics called the associated metrics [11]. He studied on classification of three-dimensional complex homogeneous complex contact manifolds, strict normal complex contact manifolds and the Boothby-Wang fibration on complex contact manifolds [12], [13], [14]. In 2000 Korkmaz gave a weaker version of normality in [19] , which must not to be Kähler, and defined the $\mathcal{GH}-$sectional curvature. Blair and present author

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studied energy and corrected energy of vertical distribution for normal complex contact metric manifolds in [5], [21]. Fetcu studied an adapted connection on a strict complex contact manifolds and harmonic maps between complex Sasakian manifolds in [9], [10].

Blair and Molina [2] studied conformal and Bochner curvature tensor of normal complex contact metric manifolds. Korkmaz [19] proved that normality is invariant under $\mathcal{H}$-homothetic deformations and such normal complex contact metric manifolds is called complex $(\kappa,\mu)-$space. In 2012, Blair and Mihai studied on complex $(\kappa,\mu)-$space and they studied on locally symmetric condition of normal complex contact metric manifolds [3], [4].

Our paper is organized as following. In Section 2, some fundamental tools and basic facts are given. Some results on the Riemann curvature are presented in Section 3. Also a new theorem obtained in same section which gives the necessary and sufficient conditions for normality, contains ∇G and ∇H. In addition some results on the Ricci curvature are given in Section 4 and, we proved that a Ricci semi-symmetric normal complex contact metric manifold is Einstein.

2. Preliminaries

Let M be a complex manifold of odd complex dimension $2m+1$ covered by an open covering $\mathcal{A}=\{\mathcal{U}_i\}$ consisting of coordinate neighborhoods. If there is a holomorphic 1-form ω_i in each $\mathcal{U}_i \in \mathcal{A}$ in such a way that for any $\mathcal{U}_i, \mathcal{U}_j \in \mathcal{A}$

$$\omega_i \wedge (d\omega_j)^n \neq 0 \text{ in } \mathcal{U}_i, \quad \text{and} \quad \omega_i = f_{ij}\omega_j,\ \mathcal{U}_i \cap \mathcal{U}_j \neq \emptyset,$$

where f_{ij} is a holomorphic function on $\mathcal{U}_i \cap \mathcal{U}_j \neq \emptyset$ then ω_i is called the complex contact form in $\mathcal{U}_i$, and (M,ω_i) is called a complex contact manifold [18].

$\omega_i = 0$ defines a $2m$-dimensional complex vector subspace $\mathcal{H}_x$ of T_xM [18]. Let $\mathcal{H}$ be the vector bundle over M with fibres $\mathcal{H}_x$ and $\mathcal{V}$ be the line bundle $TM/\mathcal{H}$, from Whitney sum we have $TM = \mathcal{H} \oplus \mathcal{V}$. $\mathcal{H}$ and $\mathcal{V}$ are called horizontal subbundle and vertical subbundle, respectively.

Ishihara and Konishi [15], [16], [17] and Shibuya [20] proved existence of complex almost contact metric structure.

Definition 2.1. *Let (M,J,g) be a Hermitian manifold and $\mathcal{A}=\{\mathcal{U}_i\}$ be open covering of M with coordinate neighbourhoods $\{\mathcal{U}_i\}$. M is called a complex almost contact metric manifold if following two conditions are satisfied:*

1. In each $\mathcal{U}_i$ there exist 1-forms u_i and $v_i = u_i \circ J$, with dual vector fields U_i and $V_i = -JU_i$ and $(1,1)$ tensor fields G_i and $H_i = G_iJ$ such that

$$\begin{aligned} &H_i^2 = G_i^2 = -I + u_i \otimes U_i + v_i \otimes V_i, \\ &G_iJ = -JG_i,\ GU_i = 0, \quad g(X,G_iY) = -g(G_iX,Y). \end{aligned}$$

2.On $\mathcal{U}_i \cap \mathcal{U}_j \neq \emptyset$ we have

$$u_j = au_i - bv_i, v_j = bu_i + av_i, \quad G_j = aG_i - bH_i, H_j = bG_i + aH_i,$$

where a and b are functions on $\mathcal{U}_i \cap \mathcal{U}_j$ with $a^2 + b^2 = 1$.

Also from the second condition on $\mathcal{U}_i \cap \mathcal{U}_j$, we have $U_j = aU_i - bV_i$ and $V_j = bU_i + aV_i$. Since $a^2 + b^2 = 1$ we get $U_i \wedge V_j = U \wedge V$. Thus we have a global vertical distribution $\mathcal{V}$ which is spanned by U and V. In general is assumed that $\mathcal{V}$ is integrable.

A complex contact manifold admits a complex almost contact metric structure for which the local contact form ω is $u - iv$ to within a non-vanishing complex-valued function multiple and the local fields G and H are related to du and dv by

$$du(W,T) = g(W,GT) + (\sigma \wedge v)(W,T), \quad dv(W,T) = g(W,HT) - (\sigma \wedge u)(W,T)$$

where $\sigma(W) = g(\nabla_W U, V)$ [16]. With these conditions M is called a *complex almost contact metric manifold.*

Ishihara and Konishi [15], [16] defined two tensor fields $\mathcal{S}$ and $\mathcal{T}$ given by

$$\begin{aligned} \mathcal{S}(W,T) &= [G,G](W,T) + 2g(W,GT)U - 2g(W,HT)V \\ &\quad +2(v(T)HW - v(W)HT) + \sigma(GT)HW \\ &\quad -\sigma(GW)HT + \sigma(W)GHT - \sigma(T)GHW, \end{aligned} \tag{2.1}$$

$$\begin{aligned} \mathcal{T}(W,T) &= [H,H](W,T) - 2g(W,GT)U + 2g(W,HT)V \\ &\quad +2(u(T)GW - u(W)GT) + \sigma(HW)GT \\ &\quad -\sigma(HT)GW + \sigma(W)GHT - \sigma(T)GHW. \end{aligned} \tag{2.2}$$

Here $[G,G]$ is the Nijenhuis torsion of G which is defined following:

$$[G,G](W,T) = (\nabla_{GW} G)T - (\nabla_{GT} G)W - G(\nabla_W G)T + G(\nabla_T G)W.$$

According to Ishihara and Konishi's definition M is normal if $\mathcal{S}(W,T) = \mathcal{T}(W,T) = 0$ for arbitrary vector fields W,T on M. Such manifolds are called IK-normal and an IK-normal complex contact manifold is Kähler. The complex Heisenberg group has not Kähler structure. So it is not IK-normal. For this reason Korkmaz [19] gave an extended definition for normality;

Definition 2.2. *A complex almost contact metric manifold M is called normal if the following conditions are satisfied:*

1. $\mathcal{S}(W,T) = \mathcal{T}(W,T) = 0$, for $W,T \in \mathcal{H}$,

2. $\mathcal{S}(W,U) = \mathcal{T}(W,V) = 0$, for $W \in TM$.

Korkmaz obtained some results on normal complex contact metric manifolds which we list here. For details we refer to reader [1] and [19].

Proposition 2.3. *A complex contact metric manifold is normal if and only if*

$$\begin{aligned} g((\nabla_W G)Z,T) &= \sigma(W)g(HZ,T) + v(W)d\sigma(GT,GZ) - 2v(W)g(HGZ,T) \\ &\quad - u(Z)g(W,T) - v(Z)g(JW,T) + u(T)g(W,Z) \\ &\quad + v(T)g(JW,Z), \end{aligned} \tag{2.3}$$

$$\begin{aligned} g((\nabla_W H)Z,T) &= -\sigma(W)g(GZ,T) - u(W)d\sigma(HT,HZ) - 2u(W)g(HGZ,T) \\ &\quad + u(Z)g(JW,T) - v(Z)g(W,T) - u(T)g(JW,Z) \\ &\quad + v(T)g(W,Z), \end{aligned} \tag{2.4}$$

for arbitrary vector fields W, T on M.

Also from above proposition we have

$$\begin{aligned} g((\nabla_W J)Z,T) &= u(W)(d\sigma(T,GZ) - 2g(HZ,T)) \\ &\quad + v(W)(d\sigma(T,HZ) + 2g(GZ,T)). \end{aligned} \tag{2.5}$$

For W and T horizontal vector fields we have the followings [19];

$$\nabla_W U = -GW + \sigma(W)V, \quad \nabla_W V = -HW - \sigma(W)U, \tag{2.6}$$

$$\nabla_U U = \sigma(U)V, \nabla_U V = -\sigma(U)U, \nabla_V U = \sigma(V)V, \nabla_V V = -\sigma(V)U, \tag{2.7}$$

$$d\sigma(GW,GT) = d\sigma(HW,HT) = d\sigma(T,W) - 2u \wedge v(T,W)d\sigma(U,V), \tag{2.8}$$

$$d\sigma(U,W) = v(W)d\sigma(U,V), \; d\sigma(V,W) = -u(W)d\sigma(U,V). \tag{2.9}$$

$$R(U,V,V,U) = R(V,U,U,V) = -2d\sigma(U,V) \tag{2.10}$$

$$R(W,U)U = W, \quad R(W,V)V = W, \; R(U,V)W = JW \tag{2.11}$$

$$R(W,T)U = 2(g(W,JT) + d\sigma(W,T))V \tag{2.12}$$

$$R(W,T)V = -2(g(W,JT) + d\sigma(W,T))U \tag{2.13}$$

$$R(W,U)V = \sigma(U)GW + (\nabla_U H)W - JW \tag{2.14}$$

$$R(W,V)U = -\sigma(V)HW + (\nabla_V G)W + JW \tag{2.15}$$

$$R(W,U)T = -g(W,T)U - g(JW,T)V + d\sigma(T,W)V, \tag{2.16}$$

$$R(W,V)T = -g(W,T)V + g(JW,T)U - d\sigma(T,W)U \tag{2.17}$$

$$\begin{aligned} g(R(GW,GT)GZ,GY) &= g(R(W,T)Z,W) - 2g(JZ,Y)d\sigma(W,T) \\ &\quad + 2g(HW,T)d\sigma(GZ,Y) + 2g(JW,T)d\sigma(Z,Y) \\ &\quad - 2g(HZ,W)d\sigma(GY,T), \end{aligned} \tag{2.18}$$

$$\begin{aligned} g(R(HW,HT)HZ,HY) &= g(R(W,T)Z,Y) - 2g(JZ,W)d\sigma(W,T) \\ &\quad - 2g(GW,T)d\sigma(HZ,Y) + 2g(JW,T)d\sigma(Z,Y) \\ &\quad + 2g(GZ,Y)d\sigma(HW,T). \end{aligned} \tag{2.19}$$

On the other hand, in [11] we have

$$d\sigma(W,T) = 2g(JW,T) + g((\nabla_U J)GW,T). \tag{2.20}$$

For an arbitrary vector field W on M we can write

$$W = W_0 + u(W)U + v(W)V, \quad W_0 \in \mathcal{H}. \tag{2.21}$$

3. Curvature properties and normality

In this section we obtain some results on the Riemannian curvature of normal complex contact metric manifolds. In addition we give a new expression for normality by covariant derivation of G and H structure tensors. Firstly we have some useful results which are listed following.

Theorem 3.1. *Let M be a normal complex contact metric manifold. For X, Y, Z, T horizontal vector fields we have*

$$g(R(GX, GY)GZ, GT) = g(R(HX, HY)HZ, HT) = g(R(X, Y)Z, T). \tag{3.1}$$

Proof. By using (2.20) in (2.18) we have

$$\begin{aligned}
&- 2g(JZ, T)d\sigma(X, Y) + 2g(HX, Y)d\sigma(GZ, T)\\
&+ 2g(JX, Y)d\sigma(Z, T) - 2g(HZ, T)d\sigma(GX, Y)\\
&= -2g(JZ, T)\left(2g(JX, Y) + g((\bigtriangledown_U J)GX, Y)\right)\\
&+ 2g(HX, Y)\left(2g(JGZ, T) + g((\bigtriangledown_U J)G^2 Z, T)\right)\\
&+ 2g(JX, Y)\left(2g(JZ, T) + g((\bigtriangledown_U J)GZ, T)\right)\\
&- 2g(HZ, T)(2g(JGX, Y) + g((\bigtriangledown_U J)G^2 X, Y)).
\end{aligned}$$

Since $JG = -H$ and for X horizontal vector field $G^2X = -X$ we have

$$\begin{aligned}
&- 2g(JZ, T)d\sigma(X, Y) + 2g(HX, Y)d\sigma(GZ, T)\\
&+ 2g(JX, Y)d\sigma(Z, T) - 2g(HZ, T)d\sigma(GX, Y))\\
&= -2g(JZ, T)g((\bigtriangledown_U J)GX, Y) - 2g(HX, Y)g((\bigtriangledown_U J)GZ, T)\\
&+ 2g(JX, Y)g((\bigtriangledown_U J)GZ, T) + 2g(HZ, T)g((\bigtriangledown_U J)GX, Y)).
\end{aligned}$$

From (2.5) and by simply computation we get

$$\begin{aligned}
&-2g(JZ, T)d\sigma(X, Y) + 2g(HX, Y)d\sigma(GZ, T)\\
&+2g(JX, Y)d\sigma(Z, T) - 2g(HZ, T)d\sigma(GX, Y) = 0.
\end{aligned}$$

Considering (2.18) from last equation we obtain (3.1). By the same way we can easily show that $g(R(HX, HY)HZ, HT) = g(R(X, Y)Z, T)$. So, the proof is completed. □

Curvature identities for normal complex contact metric manifolds were computed by Korkmaz [19] for horizontal vector fields. From (2.21) and by direct computation we obtain the following Lemma which presents the curvature identities for general vector fields.

Lemma 3.2. *Let M be a normal complex contact metric manifold and W, T be two arbitrary vector fields on M. Then the Riemannian curvature of a normal* *complex contact metric manifold satisfies following equations.*

$$R(W,U)U = W_0 - 2d\sigma(U,V)v(W)V, \tag{3.2}$$
$$R(W,V)V = W_0 - 2d\sigma(U,V)u(W)U,$$
$$R(W,U)V = \sigma(U)GW_0 + (\nabla_U H)W_0 - JW_0 + 2d\sigma(U,V)v(W)U, \tag{3.3}$$
$$R(W,V)U = -\sigma(V)HW_0 + (\nabla_V G)W_0 + JW_0 + 2d\sigma(U,V)u(W)V, \tag{3.4}$$
$$R(U,V)W = JW_0 + (u(W)V - v(W)U)2d\sigma(U,V), \tag{3.5}$$
$$\begin{aligned} R(W,T)U = &-u(W)T_0 + v(W)(\sigma(V)HT_0 + (\nabla_V G)T_0 + JT_0) \\ &+ u(T)W_0 + v(T)(-\sigma(V)HW_0 + (\nabla_V G)W_0 + JW_0) \\ &+ [2(g(W_0,JT_0) + d\sigma(W_0,T_0)) + 2d\sigma(U,V)u \wedge v(W,T)]V, \end{aligned} \tag{3.6}$$
$$\begin{aligned} R(W,T)V = &-u(W)(\sigma(U)GT_0 + (\nabla_U H)T_0 - JT_0) - v(W)T_0 \\ &u(T)(-\sigma(U)GW_0 + (\nabla_U H)W_0 - JW_0) + v(T)W_0 \\ &+ [-2(g(W_0,JT_0) + d\sigma(W_0,T_0)) - 2d\sigma(U,V)u \wedge v(W,T)]U, \end{aligned} \tag{3.7}$$
$$\begin{aligned} R(W,U)T = &\, u(T)W_0 - v(W)JT_0 + v(T)(\sigma(U)GW_0 + (\nabla_U H)W_0 \\ &- JW_0) + [-g(W_0,T_0) - 2d\sigma(U,V)v(W)v(T)]U \\ &+ [d\sigma(T_0,W_0) - g(JW_0,T_0) - 2d\sigma(U,V)v(W)u(T))]V, \end{aligned} \tag{3.8}$$
$$\begin{aligned} R(W,V)T = &\, u(W)JT_0 + v(T)W_0 + u(T)(-\sigma(U)HW_0 + (\nabla_V G)W_0 \\ &+ JW_0) + [-g(W_0,T_0) + u(W)u(T)2d\sigma(U,V)]V \\ &[-d\sigma(T_0,W_0) + g(JW_0,T_0) - 2d\sigma(U,V)u(W)v(T)]U. \end{aligned} \tag{3.9}$$

Proposition 3.3. *Let M be a normal complex contact metric manifold . Then for arbitrary vector fields Z and T on M we have*

$$d\sigma(Z,T) = 2g(JZ_0,T_0) + g((\nabla_U J)GZ_0,T_0) + d\sigma(U,V)u \wedge v(Z,T). \tag{3.10}$$

Proof. For vector fields Z and T we have

$$\begin{aligned} 2d\sigma(Z,T) &= Zg(\nabla_T U,V) - Tg(\nabla_Z U,V) - g\left(\nabla_{[Z,T]}U,V\right) \\ &= g(\nabla_Z \nabla_T U,V) + g(\nabla_T U,\nabla_Z V) - g(\nabla_T \nabla_Z U,V) \\ &\quad -g(\nabla_Z U,\nabla_T V) - g\left(\nabla_{[Z,T]}U,V\right) \\ &= g(R(Z,T)U,V) + g(\nabla_T U,\nabla_Z V) - g(\nabla_Z U,\nabla_T V) \end{aligned}$$

from (2.6) and since $HG = -GH = J + u \otimes V - v \otimes U$ we have

$$2d\sigma(Z,T) = g(R(Z,T)U,V) + 2g(JZ,T) + 2u \wedge v(Z,T).$$

In addition from (3.6) we get

$$g(R(Z,T)U,V) = 2(g(Z_0,JT_0) + d\sigma(Z_0,T_0)) + 2d\sigma(U,V)u \wedge v(Z,T)$$

and since $g(JZ,T) = g(JZ_0,T_0) - u \wedge v(Z,T)$ we obtain (3.10). □

Ishihara and Konishi gave an expression of covariant derivative of G and H for IK-normal complex contact metric manifolds. Korkmaz [19] gave a weaker definition for normality of a complex contact metric manifold and obtained Proposition 2.1. In following theorem by using this result we give a new expression for the covariant derivative of G and H such as Ishihara and Konishi's result. Our result is necessary and sufficient condition for normality of complex contact metric manifolds in the sense of Korkmaz's definition.

Theorem 3.4. *A complex contact metric manifold is normal M if and only if*

$$\begin{aligned}(\nabla_W G)T &= \sigma(W)HT - 2v(W)JT - u(T)W \\ &\quad - v(T)JW + v(W)\left(2JT_0 - (\nabla_U J)\,GT_0\right) \\ &\quad + g(W,T)U + g(JW,T)V - d\sigma(U,V)v(W)\left(u(T)V - v(T)U\right),\end{aligned} \tag{3.11}$$

and

$$\begin{aligned}(\nabla_W H)T &= -\sigma(W)GT + 2u(W)JT + u(T)JW \\ &\quad - v(T)W + u(W)\left(-2JT_0 - (\nabla_U J)\,GT_0\right) \\ &\quad - g(JW,T)U + g(W,T)V + d\sigma(U,V)u(W)\left(u(T)V - v(T)U\right).\end{aligned} \tag{3.12}$$

Proof. Suppose that M is a normal complex contact metric manifold. Then from (2.3) and (2.4) we have

$$\begin{aligned} g\left((\nabla_W G)T, Z\right) &= g\left(\sigma(W)HT - 2v(W)JT - u(T)\,W - v(T)\,JW\right. \\ &\quad \left. + g(W,T)\,U + g(W,JT)\,V, Z\right) + v(W)\,d\sigma(GZ,GT)\,. \end{aligned}$$

Since $u \wedge v(T,Z) = g\left(u(T)\,V - v(T)U, Z\right)$ and from (2.8) we get

$$\begin{aligned} g\left((\nabla_W G)T, Z\right) &= g\left(\sigma(W)HT - 2v(W)JT - u(T)\,W - v(T)\,JW\right. \\ &\quad \left. + g(W,T)\,U + g(W,JT)\,V, Z\right) \\ &\quad v(W)\left[d\sigma(T,Z) - 2d\sigma(U,V)\,g\left(u(T)V - v(T)\,U, Z\right)\right]. \end{aligned}$$

From (3.10) we can write

$$d\sigma(T,Z) = g(2JT_0 + (\triangledown_U J)\,GT_0 + d\sigma(U,V)\left(u(T)V - v(T)\,U\right), Z)\,.$$

By using this equation we obtain (3.11). Similarly one can get (3.12).

Conversely suppose that (3.11) and (3.12) hold. For arbitrary vector field W and from (2.1), (2.2) we have

$$\begin{aligned}\mathcal{S}(W,U) &= (\nabla_{GW}G)U - G(\nabla_W G)U + G(\nabla_U G)W - \sigma(U)\,GHW, \\ \mathcal{T}(W,V) &= (\nabla_{HW}H)V - H(\nabla_W H)V + H(\nabla_V H)W - \sigma(V)\,GHW.\end{aligned}$$

From (3.11) and (3.12) we get $S(W,U) = T(W,V) = 0$.

Now let W and T be two horizontal vector fields. Then from (2.1) and (2.2) we have

$$\begin{aligned}
\mathcal{S}(W,T) &= (\nabla_{GW}G)T-(\nabla_{GT}G)W-G(\nabla_W G)T+G(\nabla_T G)W\\
&\quad +2g(W,GT)U-2g(W,HT)V+\sigma(GT)HW\\
&\quad -\sigma(GW)HT+\sigma(W)GHT-\sigma(T)GHW,\\
\mathcal{T}(W,T) &= (\nabla_{HW}H)T-(\nabla_{HT}H)W-H(\nabla_W H)T+H(\nabla_T H)W\\
&\quad -2g(W,GT)U+2g(W,HT)V+\sigma(HW)GT\\
&\quad -\sigma(HT)GW+\sigma(W)GHT-\sigma(T)GHW.
\end{aligned}$$

By applying (3.11) and (3.12) we get

$$\begin{aligned}
\mathcal{S}(W,T) &= \sigma(GW)HT-\sigma(GT)HW-2g(W,GT)U\\
&\quad +2g(W,HT)V-\sigma(W)GHT+\sigma(T)GHW\\
&\quad +2g(W,GT)U-2g(W,HT)V+\sigma(GT)HW\\
&\quad -\sigma(GW)HT+\sigma(W)GHT-\sigma(T)GHW=0,\\
\mathcal{T}(W,T) &= -\sigma(HW)GT+\sigma(HT)GW+\sigma(W)HGT\\
&\quad -\sigma(T)HGW+2g(W,GT)U-2g(W,HT)V\\
&\quad -2g(W,GT)U+2g(W,HT)V+\sigma(HW)GT\\
&\quad -\sigma(HT)GW+\sigma(W)GHT-\sigma(T)GHW=0.
\end{aligned}$$

Therefore M is normal. □

By using (2.5), (3.10), (3.11) and (3.12) following corollary is obtained.

Corollary 3.5. *Let M be a normal complex contact metric manifold and W,T be two arbitrary vector fields on M. Then we have*

$$\begin{aligned}
(\nabla_W J)T &= -2u\,(W)\,HT+2v(W)GT+u(W)\,(2HT_0+(\nabla_U J)\,T_0)\\
&\quad +v(W)\,(-2GT_0+(\nabla_U J)\,JT_0)\,.
\end{aligned}$$

4. Ricci semi-symmetric normal complex contact metric manifold

In this section we studied the Ricci curvature of normal complex contact metric manifolds. We obtain some useful results for future works and apply all curvature results to complex Heisenberg group. Finally we examine the Ricci semi-symmetric normal complex contact metric manifolds.

Let us choose a local orthonormal basis of the form $\{E_i, GE_i, HE_i, JE_i, U, V : 1\le i\le n\}$ for a $(2n+1)-$ complex dimensional normal complex contact metric manifold M. Then the Ricci curvature of M has the form

$$\begin{aligned}
Ric(W,T) &= \sum_{i=1}^{n}[g(R(E_iW)T,E_i)+g(R(GE_i,W)T,GE_i)\\
&\quad +g(R(HW_i,W)T,HE_i)+g(R(JE_i,W)T,JE_i)]\\
&\quad +g(R(U,W)T,U)+g(R(V,W)T,V).
\end{aligned} \tag{4.1}$$

We obtain useful relations for Ricci curvature at the next results.

Lemma 4.1. *Let M be a normal complex contact metric manifold and W, T be horizontal vector fields on M. Then we have*

$$Ric\,(GW, GT) = Ric\,(HW, HT) = Ric\,(W, T)$$
$$Ric\,(GW, T) = -Ric\,(W, GT)\,, \quad Ric\,(HW, T) = -Ric\,(W, HT)\,.$$

Proof. By (4.1) we can write

$$\begin{aligned} Ric(GW, GT) \;&=\; \sum_{i=1}^{n}[g(R(E_iGW)GT, E_i) + g(R(GE_i, GW)GT, GE_i) \\ &\quad +g(R(HE_i, GW)GT, HE_i) + g(R(JE_i, GW)GT, JE_i)] \\ &\quad +g(R(U, GW)GT, U) + g(R(V, GW)GT, V). \end{aligned} \tag{4.2}$$

From (3.1) we have

$$g(R(W_i, GW)GT, E_i) = g(R(GE_iGGW)GGT, GE_i) = (g(R(GE_iW)T, GE_i),$$
$$g(R(GE_i, GW)GT, GE_i) = g(R(E_i, W)T, E_i),$$
$$g(R(HE_i, GW)GT, HE_i) = g(R(GJE_i, GW)GT, GJE_i) = g(R(JE_i, W)T, JE_i),$$
$$g(R(JE_i, GW)GT, JE_i) = g(R(-GHE_i, GW)GT, -GHE_i)$$
$$= g(R(HE_i, W)T, HW_i).$$

From (2.11) we have $g(R(U, GW)GT, U) = g(R(U, W)T, U), g(R(V, GW)GT, V) = g(W, T) = g(R(V, W)T, V)$. Using these equations in (4.2) we get $Ric(GW, GT) = Ric(W, T)$. Similarly, it can be easily show that $Ric(HW, HT) = Ric(W, T)$. In addition from (2.11) , (3.1) and (4.1) we obtain $Ric\,(GW, T) = -Ric\,(W, GT)\,.$ Similarly, one can show that $Ric\,(HW, T) = -Ric\,(W, HT)\,.$ □

Lemma 4.2. *Let M be a normal complex contact metric manifold. For any W horizontal vector filed on M the Ricci curvature tensor satisfies*

$$Ric(W, U) = Ric(W, V) = 0, \tag{4.3}$$
$$Ric(U, U) = Ric(V, V) = 4n - 2d\sigma(U, V), \quad Ric(U, V) = 0. \tag{4.4}$$

Proof. Since E_i, W are horizontal vector fields on M from (2.12) and (2.13) we get

$$\begin{aligned} g(R(E_iW)U, E_i) \;&=\; g(2(g(E_i, JW) + d\sigma(E_i, W))V, E_i) \\ &=\; 2(g(E_i, JW) + d\sigma(E_i, W))g(V, E_i) = 0. \end{aligned}$$

By the same way, we get

$$g(R(GE_i, W)U, GE_i) = g(R(HE_i, W)U, HE_i) = g(R(JE_i, W)U, JE_i) = 0.$$

On the other hand using (2.11),(2.14) and (2.15) we have

$$g(R(U,W)U,U) = -g(R(W,U)U,U) = -g(W,U) = 0,$$
$$g(R(V,W)U,V) = -g(-\sigma(V)HW + (\nabla_V G)W + JW, V) = 0.$$

Thus from (4.1) we get (4.3) and by following same steps we obtain (4.4). □

Also by using curvature properties and from (2.21) we get following corollaries.

Corollary 4.3. *For arbitrary W vector field on a normal complex contact metric manifold M we have*

$$Ric(W,U){=}(4n-2d\sigma(U,V))u(W), Ric(W,V){=}(4n-2d\sigma(U,V))v(W). \tag{4.5}$$

Corollary 4.4. *Let M be a normal complex contact metric manifold. Assume that W and T be two arbitrary vector fields on M provided W_0 and T_0 are the horizontal part of W and T, respectively. Then the Ricci curvature tensor satisfies*

$$Ric(W,T) = Ric(W_0,T_0) + (4n-2d\sigma(U,V))(u(W)u(T) + v(W)v(T)). \tag{4.6}$$

Corollary 4.5. *Let M be a normal complex contact metric manifold and W, T be two arbitrary vector fields on M. Then the Ricci curvature tensor satisfies*

$$\begin{aligned} Ric(W,T) &= Ric(GW,GT) + (4n-2d\sigma(U,V))\left(u(W)u(T) + v(W)v(T)\right), \\ Ric(W,T) &= Ric(HW,HT) + (4n-2d\sigma(U,V))\left(u(W)u(T) + v(W)v(T)\right). \end{aligned}$$

Corollary 4.6. *On a normal complex contact metric manifold M, for Q the Ricci operator we have $QG = GQ$, $QH = HQ$.*

The well known example of complex contact metric manifolds is Iwasawa manifold. We compute the Riemann, Ricci and scalar curvatures of Iwasawa manifold.

Example 4.7. The closed subgroup $H_{\mathbb{C}}$ of $GL(3,\mathbb{C})$ is presented by

$$H_{\mathbb{C}} = \left\{ \begin{pmatrix} 1 & b_{12} & b_{13} \\ 0 & 1 & b_{23} \\ 0 & 0 & 1 \end{pmatrix} : b_{12}, b_{13}, b_{23} \in \mathbb{C} \right\} \simeq \mathbb{C}^3$$

is called the complex Heisenberg group. Baikoussis, Blair and Gouli-Andreou defined the following complex contact metric structure on $H_{\mathbb{C}}$ in [6]. Let z_1, z_2, z_3 be the coordinates on $H_{\mathbb{C}} \simeq \mathbb{C}^3$ defined by $z_1(B) = b_{23}, z_2(B) = b_{12}, z_3(B) = b_{13}$ for B in $H_{\mathbb{C}}$. Here $H_{\mathbb{C}} \simeq \mathbb{C}^3$ and $\theta = \frac{1}{2}(dz_3 - z_2 dz_1)$ is global, so the structure tensors may be taken globally. With J denoting the standard almost complex structure on $\mathbb{C}^3$, we may give a complex almost contact structure to $H_{\mathbb{C}}$ as follows. Since θ is holomorphic, set $\theta = u + iv$, $v = u \circ J$; also set $4\frac{\partial}{\partial z_3} =$

$U+iV$. Then $u(W)=g(U,W)$ and $v(W)=g(V,W)$. Since we will work in real coordinates, G and H are given by

$$G=\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_2 & y_2 & 0 & 0 \\ 0 & 0 & y_2 & -x_2 & 0 & 0 \end{bmatrix},$$

$$H=\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -y_2 & x_2 & 0 & 0 \\ 0 & 0 & x_2 & y_2 & 0 & 0 \end{bmatrix}.$$

Then relative to the coordinates $(x_1,y_1,x_2,y_2,x_3,y_3)$ the Hermitian metric (matrix) is given by

$$g=\frac{1}{4}\begin{bmatrix} 1+x_2^2+y_2^2 & 0 & 0 & 0 & x_2 & y_2 \\ 0 & 1+x_2^2+y_2^2 & 0 & 0 & y_2 & x_2 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ x_2 & y_2 & 0 & 0 & 1 & 0 \\ y_2 & x_2 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Let $\{e_1;e_1^*;e_2;e_2^*;e_3;e_3^*\}$ be an orthonormal basis where

$$e_1=2\left(\frac{\partial}{\partial x_1}+x_2\frac{\partial}{\partial x_3}+y_2\frac{\partial}{\partial y_3}\right)\ ,\ \ e_1^*=2\left(\frac{\partial}{\partial y_1}+y_2\frac{\partial}{\partial x_3}+x_2\frac{\partial}{\partial y_3}\right), \tag{4.7}$$
$$e_2=2\frac{\partial}{\partial x_2}\ \ ,\ e_2^*=2\frac{\partial}{\partial y_2}\ ,\ e_3=U=2\frac{\partial}{\partial x_3}\ \ ,\ \ e_3^*=V=2\frac{\partial}{\partial y_3}.$$

Furthermore we have [21]

$$\begin{aligned} Ge_1 &= -e_2,\ Ge_1^*=e_2^*,\ Ge_2=e_1,\ Ge_2^*=-e_1^*, \\ He_1 &= -e_2^*,\ He_1^*=-e_2,\ He_2=e_1^*,\ He_2^*=-e_1, \\ Je_1 &= -e_1^*,\ Je_1^*=e_1,\ Je_2=-e_2^*,\ Je_2^*=-e_2. \end{aligned}$$

Let $\bigtriangledown$ be the Levi-Civita connection with respect to metric g. Then from (4.7) we have

$$[e_1,e_2]=-2e_3,\ \ [e_1,e_2^*]=-2e_3^*,\ \ [e_1^*,e_2]=-2e_3^*,\ \ [e_1^*,e_2^*]=2e_3 \tag{4.8}$$

and the other Lie brackets are zero [21]. In addition we have

$$2g(\bigtriangledown_{e_i}e_j,e_k)=g\left[e_i,e_j\right],e_k+g\left(\left[e_k,e_i\right],e_j\right)-g\left(\left[e_j,e_k\right],e_i\right)$$

and from that we obtain

$$\nabla_{e_j} e_j = \nabla_{e_j} e_j = \nabla_{e_j} e_{j^*} = \nabla_{e_j^*} e_j^* = 0, \tag{4.9}$$

where $j = 1, 2, 3$. From (4.8) and (4.9) we need only list following

$$\begin{array}{rclr}
\nabla_{e_2} e_3 & = & \nabla_{e_2^*} e_3^* = -e_1, & \nabla_{e_2^*} e_3 = -\nabla_{e_2} e_3^* = e_1^*, \\
\nabla_{e_1} e_3 & = & \nabla_{e_1^*} e_3^* = e_2, & \nabla_{e_1} e_3^* = -\nabla_{e_1^*} e_3 = e_2^*, \\
-\nabla_{e_1} e_2 & = & \nabla_{e_1^*} e_2^* = e_3, & \nabla_{e_1} e_2^* = \nabla_{e_1^*} e_2 = -e_3^*.
\end{array}$$

Now, let

$$\Gamma = \left\{ \left(\begin{array}{ccc} 1 & \gamma_2 & \gamma_3 \\ 0 & 1 & \gamma_1 \\ 0 & 0 & 1 \end{array} \right) \middle| \gamma_k = m_k + i n_k,\ m_k,\ n_k \in \mathbb{Z} \right\}.$$

Γ is subgroup of $H_{\mathbb{C}} \simeq \mathbb{C}^3$, the 1-form $dz_3 - z_2 dz_1$ is invariant under the action on Γ and with $\xi = U \wedge V$, hence the quotient $H_{\mathbb{C}}/\Gamma$ is a compact complex contact manifold with a global complex contact form. $H_{\mathbb{C}}/\Gamma$ is known the *Iwasawa manifold.*

It is known that with the help of the above results , it can be easily verified that

$$\begin{array}{l}
R(e_1, e_1^*)e_1 = 0, R(e_1, e_1^*)e_1^* = 0, R(e_1, e_1^*)e_2 = -2e_2^*, R(e_1, e_1^*)e_2^* = 2e_2, \\
R(e_1, e_2)e_1 = 3e_2, R(e_1, e_2)e_1^* = -e_2^*, R(e_1, e_2)e_2 = -3e_1, R(e_1, e_2)e_2^* = e_1^*, \\
R(e_1, e_2^*)e_1 = 3e_2, R(e_1, e_2^*)e_1^* = 0, R(e_1, e_2^*)e_2 = -e_2^*, R(e_1, e_2^*)e_2^* = -3e_1, \\
R(e_1^*, e_2)e_1 = e_2^*, R(e_1^*, e_2)e_1^* = 3e_2, R(e_1^*, e_2)e_2 = -3e_1^*, R(e_1^*, e_2)e_2^* = 3e_1 \\
R(e_1^*, e_2^*)e_1 = -e_2, R(e_1^*, e_2^*)e_1^* = 3e_2^*, R(e_1^*, e_2^*)e_2 = e_1, R(e_1^*, e_2^*)e_2^* = -3e_1^*, \\
R(e_2, e_2^*)e_1 = -2e_1^*, R(e_2, e_2^*)e_1^* = 2e_1, R(e_2, e_2^*)e_2 = 0, R(e_2, e_2^*)e_2^* = 0 \ .
\end{array}$$

From (2.7) and since $\sigma = 0$ [19] $R(e_3, e_3^*)e_3^* = 0$ and we have $R(W, U)U = W$ and $R(W, V)V = W$ for $W \in \mathcal{H}$. Similarly from (2.14) and (2.15) we get

$$\begin{array}{l}
R(e_1, e_3)e_3^* = -e_1^*, R(e_1^*, e_3^*)e_3 = 3e_1, R(e_2, e_3)e_3^* = -e_2^*, R(e_2^*, e_3^*)e_3 = e_2, \\
R(e_1, e_3^*)e_3 = e_1^*, R(e_1^*, e_3^*)e_3 = -e_1, R(e_2, e_3^*)e_3 = e_2^*, R(e_2^*, e_3^*)e_3 = -3e_2,
\end{array}$$

and from (2.12), (2.13) we get

$$\begin{array}{c}
R(e_1, e_1^*)e_3 = R(e_1, e_2)e_3 = R(e_1, e_2^*)e_3 = 0 \\
R(e_1^*, e_2)e_3 = R(e_1^*, e_2^*)e_3 = R(e_2, e_2^*)e_3 = 0
\end{array}$$

and

$$\begin{array}{c}
R(e_1, e_1^*)e_3^* = R(e_1, e_2)e_3^* = R(e_1, e_2^*)e_3^* = 0 \\
R(e_1^*, e_2)e_3^* = R(e_1^*, e_2^*)e_3^* = R(e_2, e_2^*)e_3^* = 0 \quad .
\end{array}$$

Using these equations and from (4.3) and (4.4) the Ricci curvature is obtained

$$Ric\,(e_i, e_i) = Ric\,(e_i^*, e_i^*) = 4,\ i = 1, 2 \text{ ve } Ric\,(e_3, e_3) = Ric\,(e_3^*, e_3^*)$$
$$Ric\,(e_i, e_j) = Ric\left(e_i^*, e_j^*\right) = 0,\ j = 1, 2, 3.$$

By direct computation the scalar curvature of Iwasava manifold is obtained as $\tau = -8$. Furthermore from curvature equalities the sectional curvature is

$$\begin{aligned} k(e_1, e_3) &= k(e_1^*, e_3) = k(e_2, e_3) = k(e_2^*, e_3) = 1, \\ k(e_1, e_3^*) &= k(e_1^*, e_3^*) = k(e_2, e_3^*) = k(e_2^*, e_3^*) = 1 \end{aligned}$$

and since $\sigma = 0$ we get $k(e_3, e_3^*) = 0$. In addition can be easily verified that

$$\begin{aligned} k(e_1, e_1^*) &= k(e_1, e_2^*) = k(e_1^*, e_2) = k(e_2, e_2^*) = 0 \\ k(e_1, e_2) &= 3 \text{ and } k(e_1^*, e_2^*) = 1. \end{aligned}$$

Definition 4.8. *A complex contact metric manifold M is said to be Ricci semi-symmetric if it satisfies the condition*

$$R(X, Y).Ric = 0 \tag{4.10}$$

for all X, Y vector fields on M.

Theorem 4.9. *A Ricci semi-symmetric normal complex contact metric manifold is Einstein.*

Proof. Let us consider a semi-symmetric complex contact metric manifold. Then for arbitrary vector fields X, Y, W and T on M and from (4.10) we have

$$Ric(R(X, Y)W, T) + Ric(W, R(X, Y)T) = 0. \tag{4.11}$$

By setting $Y = W = U$ and $X = X_0$, $T = T_0$, X_0, $T_0 \in \mathcal{H}$ in (4.11) we have

$$Ric(R(X_0, U)U, T_0) + Ric(U, R(X_0, U)T_0) = 0.$$

From (480) and (2.16) we get

$$Ric(X_0, T_0) + Ric(U, -g(X_0, T_0)U - g(JX_0, T_0)V + d\sigma(T_0, X_0)V) = 0$$

and

$$Ric(X_0, T_0) - g(X_0, T_0)Ric(U, U) - g(JX_0, T_0)Ric(U, V) + d\sigma(T_0, X_0)Ric(U, V) = 0.$$

From (4.4) we obtain

$$Ric(X_0, T_0) = (4n - 2d\sigma(U, V))\, g(X_0, T_0). \tag{4.12}$$

By similar way taking $Y = W = V$ and $X = X_0$, $T = T_0$, X_0, $T_0 \in \mathcal{H}$ in (4.11) and from (2.11) and (2.17) we have

$$Ric(X_0, T_0) - g(X_0, T_0)Ric(V, V) + g(JX_0, T_0)Ric(U, V) - d\sigma(T_0, X_0)Ric(U, V) = 0.$$

By using (4.4) we get (4.12). So, the manifold is Einstein. □

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