Finite group with coincide automizer and central automorphism of subgroups

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Abstract. In this paper, we characterize the finite group G such that $Aut_G(H) = Aut_c(H)$ for every (abelian, non-abelian) subgroup H of G, where $Aut_G(H)$ and $Aut_c(H)$ are automizer and central automorphism of H in G.

Keywords: nilpotent group, automizer, central automorphism.

1. Introduction

All groups considered in this paper are finite.

Let G be a group and H a subgroup of G. The automizer $Aut_G(H)$ of H in G is defined as the group of automorphisms of H induced by conjugation of elements of $N_G(H)$,

$$Aut_G(H) \cong N_G(H)/C_G(H)$$

and we obviously have $Inn(H) \leq Aut_G(H) \leq Aut(H)$. $Aut_G(H)$ is small if $Aut_G(H) = Inn(H)$ and large if $Aut_G(H) = Aut(H)$.

Automizers of some special subgroups had the strong influence toward the group, the best example is probably the well-known Frobenius criterion for p-nilpotency: a finite group is p-nilpotent if and only if the automizers of all its p-subgroups are p-groups. Brandl and Deaconescu [2] gave the structure of finite SANS-groups (Small Automizers for Non-abelian Subgroups) in which the automizers of all non-abelian subgroups are small. Also, Deaconescu and Mazurov obtained the finite groups with large automizers for their non-abelian subgroups in [5], which is called LANS-groups (Large Automizers for Non-abelian Subgroups).

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An automorphism α of G is called a central if $x^{-1}x^{\alpha} \in Z(G)$ for each $x \in G$. The set of all central automorphisms of G, denoted by $Aut_c(G)$, is a normal subgroup of $Aut(G)$. It is easy to see that

$$Aut_c(G) = C_{Aut(G)}(Inn(G)).$$

There have been a number of results on the central automorphism of a finite group, for example: Curran and McCaughan [3, 4] characterized finite p-groups G for which $Aut_c(G) = Inn(G)$ and $Aut_c(G) = Z(Inn(G))$.

From the results above, we found that automizers and central automorphisms have good relationship with Inner automorphism in the group theory. Therefore, it is an interesting topic to investigate the relationship between automizers and central automorphisms in finite groups. In this paper, we focus on the finite group G such that $Aut_G(H) = Aut_c(H)$ for every (abelian, non-abelian) subgroup H of G.

2. Notations and lemmas

The terminology and the notation in this paper are standard as in [6]. For a prime p, $\{a^p | a \in G\}$ will be denoted by $\mho_1(G)$. We use $c(G)$ to denote the nilpotency class of a group G. Following lemmas will be used in the sequel.

Lemma 2.1 ([3])**.** If G is a finite p-group, then $Aut_c(G) = Inn(G)$ if and only if $G' = Z(G)$ and $Z(G)$ is cyclic.

Lemma 2.2 ([1])**.** If finite group G with large automizers of abelian subgroups, then G is isomorphic to either S_3, for $n \leq 3$ or to Q_8.

3. Main theorem

It is well-known that S_3 is the non-abelian group of minimal order and Q_8, D_8 are the non-abelian 2-groups of minimal order. It is easy to see that $S_3 = Aut_{S_3}(S_3) \neq Aut_c(S_3) = 1$ and $K_4 \cong Aut_{Q_8}(Q_8) = Aut_c(Q_8) = Aut_c(D_8)$. However, it holds that $Aut_G(H) = Aut_c(H)$ for every abelian subgroup H of S_3. And it is true that $Aut_G(H) = Aut_c(H)$ for every subgroup H of Q_8. But the result above is not true for D_8. Since D_8 contains an elementary abelian 2-group of order 4. Let H be an elementary abelian 2-group of order 4. Then

$$Aut_{D_8}(H) = N_{D_8}(H)/C_{D_8}(H) = C_{Aut(H)}(Inn(H)) = Aut(H) = S_3,$$

a contradiction. Hence it is interesting to investigate the finite group G such that $Aut_G(H) = Aut_c(H)$ for every non-abelian subgroup H of G.

Theorem 3.1. Let G be a non-abelian group. Then $Aut_G(H) = Aut_c(H)$ for every non-abelian subgroup H of G if and only if

(i) G is a p-group, $G' = Z(G)$ and $Z(G)$ is cyclic.

(ii) $G = P \times C_2$, where P is a p-group such that $p \neq 2$, $P' = Z(P)$ and $Z(P)$ is cyclic.

Proof. It is easy to check that $Aut_G(H) = Aut_c(H)$ for every non-abelian subgroup H of G if G satisfying (i), (ii). Conversely, proof can proceed by following steps.

Step 1. $P' = Z(P)$ and $Z(P)$ is cyclic for $P \in Syl_p(G)$.

It is easy to see that $Inn(G) = Aut_G(G) = Aut_c(G)$ for non-abelian group G, then $Inn(G)$ is abelian and G is nilpotent group of class 2.

Since G is a non-abelian group, there exists at least one non-abelian Sylow subgroup of G. For $p \in \pi(G)$. Let $P \in Syl_p(G)$ and P is non-abelian. Since G is nilpotent, P is normal in G. Then

(∗)

$$Inn(P) = P/C_P(P) \cong G/C_G(P) = Aut_G(P) = Aut_c(P) = C_{Aut(P)}(Inn(P))$$

By Lemma 2.1, $P' = Z(P)$ and $Z(P)$ is cyclic.

Case 1. If G is a p-group, then $G = P$, $P' = Z(P)$ and $Z(P)$ is cyclic, as required.

Case 2. If G is not a p-group, there exists prime $q \neq p \in \pi(G)$. Let $Q \in Syl_q(G)$.

Step 2. Q is abelian.

If Q is non-abelian, then $|Q| \geq q^3$. There exists normal subgroup Q_1 of order q in Q, which is also normal in G. Then

$$\begin{aligned} Inn(P) &= P/Z(P) \cong (P \times Q_1)/(Z(P) \times Z(Q_1)) \\ &\cong G/C_G(P \times Q_1) = Aut_G(P \times Q_1) \\ &= Aut_c(P \times Q_1) = C_{Aut(P \times Q_1)}(Inn(P \times Q_1)) \\ &= C_{Aut(P) \times Aut(Q_1)}(Inn(P)). \end{aligned}$$

By (∗), $Aut(Q_1) = 1$, so $Q_1 = C_2$, and Q is a 2-group.

Now we choose a normal subgroup P_1 of order p in P, which is also normal in G. We consider non-abelian group $P_1 \times Q$, by the similar argument,

$$\begin{aligned} Inn(Q) &= Q/Z(Q) \cong (Q \times P_1)/(Z(Q) \times Z(P_1)) \cong G/C_G(Q \times P_1) \\ &= Aut_G(Q \times P_1) = Aut_c(Q \times P_1) = C_{Aut(Q \times P_1)}(Inn(Q \times P_1)) \\ &= C_{Aut(Q) \times Aut(P_1)}(Inn(Q)). \end{aligned}$$

By (∗), $Aut(P_1) = 1$, so $P_1 = C_2$, and P is a 2-group, a contradiction.

Step 3. $p \neq 2$, and $G = P \times C_2$.

Since Q is abelian, we consider non-abelian subgroup $H = P \times Q$. Hence

$$\begin{aligned} Inn(P) &= P/Z(P) \cong (P \times Q)/(Z(P) \times Z(Q)) \\ &\cong G/C_G(H) = Aut_G(H) = Aut_c(H) \\ &= C_{Aut(H)}(Inn(H)) = C_{Aut(P \times Q)}(Inn(P \times Q)) = C_{Aut(P \times Q)}(Inn(P)) \\ &= C_{Aut(P) \times Aut(Q)}(Inn(P)). \end{aligned}$$

By $(*)$, $Aut(Q) = 1$, so $Q = 1$ or $Q = C_2$.

If $p = 2$, then $G = P$, a contradiction. Hence $p \neq 2$, and $G = P \times C_2$, $P' = Z(P)$ and $Z(P)$ is cyclic, as required.

Zassenhaus proved in [7] that a finite group G is abelian if and only if $N_G(H) = C_G(H)$ for all abelian subgroups $H$of G. Translated into automizer terminology, the elegant result: a finite group is abelian if and only if the automizers of all its abelian subgroups are small. Later on, Bechtell, Deaconescu and Silberberg [1] classified the finite group with large automizers of abelian subgroups, which is called LAAS-groups.

It is easy to see that $Aut_c(H) = Aut(H)$ for every abelian subgroup H of G, that is, LAAS-groups is equivalent to $Aut_G(H) = Aut(H)$ for every abelian subgroup H of G. By Lemma 2.2, we can get the following theorem.

Theorem 3.2. Let G be a group. Then $Aut_G(H) = Aut_c(H)$ for every abelian subgroup H of G if and only if $G = S_n$, for $n \leq 3$ or Q_8.

Finally, we can easy to see that $G = C_2$, or Q_8 if $Aut_G(H) = Aut_c(H)$ for every abelian and non-abelian subgroups H of G by Theorem 3.1 and 3.2 above. Independent on the result above, we classify the finite group G such that $Aut_G(H) = Aut_c(H)$ for every subgroup H of G by the elementary way as follows.

Theorem 3.3. Let G be a group. Then $Aut_G(H) = Aut_c(H)$ for every subgroup H of G if and only if $G = C_2$, or Q_8.

Proof. It is easy to check that $Aut_G(H) = Aut_c(H)$ for every subgroup H of G if $G = C_2$, or $G = Q_8$. Conversely, proof can proceed by following steps.

Step 1. G is nilpotent and $c(G) = 2$.

Since $Aut_G(H) = Aut_c(H)$ for every subgroup H of G,

$$N_G(H)/C_G(H) = C_{Aut(H)}(Inn(H)).$$

Then it is clear that $Inn(H)$ is abelian by $Inn(H) \leq Aut_G(H)$. Hence $H' \leq Z(H)$ for every subgroup H of G, so G is nilpotent and $c(G) = 2$.

Step 2. $Z(G) = C_2$.

Consider $H = Z(G)$. Then $Aut_G(Z(G)) = Aut_c(Z(G))$, it is easy to see that

$$1 = N_G(Z(G))/C_G(Z(G)) = C_{Aut(Z(G))}(Inn(Z(G))) = Aut(Z(G)).$$

Hence $Z(G) = C_2$ by G is nilpotent.

Step 3. $G' = 1$ or $G' = C_2$.

Let $H = G'$. Then $Aut_G(G') = Aut_c(G')$, that is,

$$N_G(G')/C_G(G') = C_{Aut(G')}(Inn(G')).$$

Since $G' \leq Z(G)$, $1 = N_G(G')/C_G(G') = C_{Aut(G')}(Inn(G')) = Aut(G')$. Hence $G' = 1$ or $G' = C_2$.

Step 4. G is a 2-group.

If $G' = 1$, then $Z(G) = C_2 = G$.

If $G' = C_2 = Z(G)$, then we assume that G is not a 2-group. Let $P_2 \in Syl_2(G)$. Since G is nilpotent, P_2 is normal in G. Then $G = P_2 \times K$, and K is Hall $2'$-subgroup of G. By $G' = P_2' \times K' = C_2$, then $P_2' = C_2$, and K is abelian. Then $Z(G) = Z(P_2) \times Z(K) = Z(P_2) \times K$. By

$$Aut_G(Z(P_2)) = Aut_c(Z(P_2)),$$

we have $Z(P_2) = C_2$. Then $Z(G) = Z(P_2) = C_2$, a contradiction. Hence G is a 2-group.

Step 5. If $G' = C_2 = Z(G)$, then $\Phi(G) = C_2$ and G is a extra-special 2-group.

For any $a, b \in G$, $[a^2, b] = [a, b]^2 = 1$ by $Z(G) = C_2 = G'$. Then $a^2 \in Z(G)$, and so $\mho_1(G) \leq Z(G)$. Hence $\Phi(G) = G'\mho_1(G) = Z(G) = C_2$, and so G is a extra-special 2-group.

Step 6. If G is a extra-special 2-group, then $G = Q_8$.

It is well-known that G is a central product of D_8's or a central product of D_8's and a single Q_8 if G is a extra-special 2-group. It is easy to see that $Inn(D_8) \leq Aut(D_8) = D_8$ and $Inn(D_8)$ is elementary abelian 2-group of order 4.

If G contains D_8, then G contains an elementary abelian 2-group of order 4. Let H be an elementary abelian 2-group of order 4. Then

$$N_G(H)/C_G(H) = C_{Aut(H)}(Inn(H)) = Aut(H) = S_3.$$

It is obviously that $N_G(H)/C_G(H)$ is a 2-group, a contradiction. That is, G does not contain D_8, so $G = Q_8$, as required.

Corollary 3.4. Let G be a non-abelian group. Then $Aut_G(H) = Aut_c(H)$ for every subgroup H of G if and only if $G = Q_8$.

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