Nonlinear left $*$-Lie triple mappings of standard operator algebras

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Abstract. Let $\mathcal{H}$ be an infinite dimensional complex Hilbert space and $\mathcal{A}$ be a standard operator algebra on $\mathcal{H}$ which is closed under the adjoint operation. For $A, B \in \mathcal{A}$, define by $*[A, B] = AB - B^*A$ the left $*$-Lie product of $A$ and $B$. In this paper, we prove that a mapping $\phi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ satisfies $\phi([A, [B, C]]) = \phi([A, B, C])$, for all $A, B, C \in \mathcal{A}$ is automatically linear. Moreover, $\phi$ is an inner $*$-derivation.

Keywords: left $*$-Lie triple product, derivation, standard operator algebras.

1. Introduction

Let $\mathcal{A}$ be an algebra. A mapping $\phi : \mathcal{A} \to \mathcal{A}$ is called a nonlinear Lie derivation if $\phi([A, B]) = [\phi(A), B] + [A, \phi(B)]$ holds true for all $A, B \in \mathcal{A}$, where $[A, B] = AB - BA$ is the usual Lie product. Furthermore, if $\mathcal{A}$ is an algebra with involution, a mapping $\phi : \mathcal{A} \to \mathcal{A}$ is called a nonlinear $*$-Lie derivation if for any $A, B \in \mathcal{A}$, $\phi([A, B]) = [\phi(A), B] + [A, \phi(B)]$, where $[A, B] = AB - BA^*$ is the skew Lie product of $A$ and $B$. Note that for both cases no additivity is assumed on $\phi$. A linear mapping $\phi : \mathcal{A} \to \mathcal{A}$ is called a derivation if $\phi(AB) = \phi(A)B + A\phi(B)$, for all $A, B \in \mathcal{A}$. $\phi$ is a $*$-derivation provided that $\phi(A^*) = \phi(A)^*$. Corresponding author
\[ \phi(A^*) = \phi(A)^* \text{ for all } A \in \mathcal{A}. \]

A derivation on \( \mathcal{A} \) is inner if there exists \( T \in \mathcal{A} \) such that \( \phi(A) = AT - TA \). A linear mapping \( \phi : \mathcal{A} \rightarrow \mathcal{A} \) is called a Jordan derivation if \( \phi(A^2) = \phi(A)A + A\phi(A) \), for all \( A \in \mathcal{A} \). A linear mapping \( \phi : \mathcal{A} \rightarrow \mathcal{A} \) is called a Jordan left \(*\)-derivation if \( \phi(A^2) = \phi(A)A + A^*\phi(A) \) holds true for any \( A \in \mathcal{A} \).

Concerning Lie product, Lu and Liu [6] proved that every Lie derivation on \( \mathcal{B}(\mathcal{X}) \) can be expressed as the sum of an additive derivation of \( \mathcal{B}(\mathcal{X}) \) into itself and a central mapping on \( \mathcal{B}(\mathcal{X}) \) vanishing on each commutator. This result was generalized to the case of Lie derivation on prime rings in [3]. The skew Lie product is found playing an important role in the problem of representing quadratic functionals with sesquilinear functionals (see, for example, [8, 9, 10]) and in the problem of characterizing ideals (see, for example, [1, 7]). In [13] Yu and Zhang showed that every nonlinear \(*\)-Lie derivation from a factor von Neumann algebra on an infinite dimensional complex Hilbert space into itself is an additive \(*\)-derivation. In [5], Li, Lu and Fang arrived the same conclusion on von Neumann algebra without central abelian projections. Recently, Jing [4] proved that every nonlinear \(*\)-Lie derivation of standard operator algebra on complex Hilbert space is an inner \(*\)-derivation.

In this paper, we define left \(*\)-Lie product by \( [A, B] = AB - B^*A \), for all \( A, B \in \mathcal{A} \), in fact, it has a close relationship to Jordan left \(*\)-derivation [11]. And we call a nonlinear mapping \( \phi \) a nonlinear left \(*\)-Lie triple mapping if it satisfies \( \phi([A, [B, C]]) = [\phi(A), [B, C]] + [A, [\phi(B), C]] + [A, [B, \phi(C)]] \) for all \( A, B, C \in \mathcal{A} \). We shall show every nonlinear left \(*\)-Lie triple mapping of standard operator algebras which are closed under adjoint operation on infinite dimensional complex Hilbert space is automatically linear. Moreover it is an inner \(*\)-derivation.

Throughout this paper, \( \mathbb{R} \) and \( \mathbb{C} \) denote respectively the real field and complex field, \( \mathcal{B}(\mathcal{H}) \) will represent the algebra of all bounded linear operators on a complex Hilbert space \( \mathcal{H} \). We will denote by \( \mathcal{F}(\mathcal{H}) \subseteq \mathcal{B}(\mathcal{H}) \) the subalgebra of all bounded finite rank operators. We call a subalgebra \( \mathcal{A} \) of \( \mathcal{B}(\mathcal{H}) \) a standard operator algebra if it contain \( \mathcal{F}(\mathcal{H}) \). Note that, different from von Neumann algebra which are always weakly closed, a standard operator algebra is not necessarily closed. Recall that an algebra \( \mathcal{A} \) is prime if \( \mathcal{A}AB = \{0\} \) for \( A, B \in \mathcal{A} \) implies either \( A = 0 \) or \( B = 0 \). An operator \( P \in \mathcal{B}(\mathcal{H}) \) is said to be a projection provided \( P^2 = P \) and \( P^2 = P \). It is well known that every standard operator algebra is prime and its commutant is \( \mathbb{C}I \).

2. The main result and its proof

The main result in this paper is as follows.

**Theorem 2.1.** Let \( \mathcal{H} \) be an infinite dimensional complex Hilbert space and \( \mathcal{A} \) be a standard operator algebra on \( \mathcal{H} \) containing the identity operator \( I \). If \( \mathcal{A} \) is
closed under the adjoint operation and \( \phi : \mathcal{A} \to \mathcal{B}(\mathcal{H}) \) satisfies
\[
\phi(\ast [A, \ast [B, C]]) = \ast [\phi(A), \ast [B, C]] + \ast [A, \ast [\phi(B), C]] + \ast [A, \ast [B, \phi(C)]]
\]
for all \( A, B, C \in \mathcal{A} \), then \( \phi \) is a linear *-derivation. Moreover, there exists an operator \( T \in \mathcal{B}(\mathcal{H}) \) satisfying \( T + T^* = 0 \) such that \( \phi(A) = AT - TA \), for all \( A \in \mathcal{A} \), that is, \( \phi \) is inner.

To complete the proof of the main theorem, we begin with the following lemmas.

**Lemma 2.1.** Let \( \mathcal{A} \) be a standard operator algebra containing identity \( I \) on a complex Hilbert space which is closed under adjoint operation. If \( AB = B^*A \) holds true for all \( A \in \mathcal{A} \), then \( B \in \mathbb{R}I \).

**Proof.** In fact, take \( A = I \), then \( B = B^* \). Thus the condition becomes \( AB = BA \). It follows that \( B \in \mathbb{C}I \), the center of \( \mathcal{A} \), and so \( B \in \mathbb{R}I \). \( \square \)

**Lemma 2.2.** \( \phi(0) = 0 \).

**Proof.** It follows from the following:
\[
\phi(0) = \phi(\ast [0, \ast [0, 0]]) = \ast [\phi(0), \ast [0, 0]] + \ast [0, \ast [\phi(0), 0]] + \ast [0, \ast [0, \phi(0)]] = 0.
\]
\( \square \)

**Lemma 2.3.** \( \phi(\mathbb{R}I) \subseteq \mathbb{R}I, \phi(\mathbb{C}I) \subseteq \mathbb{C}I. \) For any \( A \in \mathcal{A} \) with \( A = A^* \), \( \phi(A^*) = \phi(A)^* \).

**Proof.** For any \( \lambda \in \mathbb{R} \), we consider
\[
0 = \phi(\ast [I, \ast [A, \lambda I]])
= \ast [\phi(I), \ast [A, \lambda I]] + \ast [I, \ast [\phi(A), \lambda I]] + \ast [I, \ast [A, \phi(\lambda I)]]
= \ast [I, \ast [A, \phi(\lambda I)]]
= (A + A^*)\phi(\lambda I) - \phi(\lambda I)^*(A + A^*).
\]
This gives us \( (A + A^*)\phi(\lambda I) = \phi(\lambda I)^*(A + A^*) \) holds true for all \( A \in \mathcal{A} \). That is, \( B\phi(\lambda I) = \phi(\lambda I)^*B \) holds true for all \( B = B^* \in \mathcal{A} \). Since every element in \( \mathcal{A} \) is a linear span of two self-adjoint operators, it follows that \( B\phi(\lambda I) = \phi(\lambda I)^*B \) holds true for all \( B \in \mathcal{A} \). By Lemma 2.1, we have \( \phi(\lambda I) \in \mathbb{R}I \). Hence \( \phi(\mathbb{R}I) \subseteq \mathbb{R}I \).

Let \( A = A^* \in \mathcal{A} \). Since \( \phi(I) \in \mathbb{R}I \), we have that
\[
0 = \phi(\ast [I, \ast [I, A]])
= \ast [\phi(I), \ast [I, A]] + \ast [I, \ast [\phi(I), A]] + \ast [I, \ast [I, \phi(A)]]
= \ast [I, \ast [I, \phi(A)]]
= 2\phi(A) - 2\phi(A)^*.
\]
Hence $\phi(A) = \phi(A)^*$. For any $\lambda \in \mathbb{C}$ and $A \in \mathcal{A}$ with $A = A^* \in \mathcal{A}$, applying above results, we see that
\[
0 = \phi(\star[C, \star[\lambda I, A]])
= \star[\phi(C), \star[\lambda I, A]] + \star[C, \star[\phi(\lambda I), A]] + \star[C, \star[\lambda I, \phi(A)]]
= \star[C, \star[\phi(\lambda I), A]]
\]
holds true for all $C \in \mathcal{A}$. It follows from Lemma 2.1 that $\star[\phi(\lambda I), A] \in \mathbb{R}I$. This yields that $[\phi(\lambda I), A] \in \mathbb{R}I$, for all $A \in \mathcal{A}$ with $A = A^*$. By the Kleinecke-Shirokov theorem (cf. [2, Problem 230]) , we get $[\phi(\lambda I), A] = 0$, that is, $\phi(\lambda I)A = A\phi(\lambda I)$, for all $A \in \mathcal{A}$ with $A = A^*$. It follows that $\phi(\lambda I)A = A\phi(\lambda I)$ for any $A \in \mathcal{A}$, and so $\phi(\lambda I) \in \mathbb{C}I$. Therefore, $\phi(\mathbb{C}I) \subseteq \mathbb{C}I$.

**Lemma 2.4.** $\phi(\frac{1}{2}I) = \phi(\frac{1}{2}iI) = 0$ and $\phi(iA) = i\phi(A)$, for all $A \in \mathcal{A}$, where $i$ is the imaginary unit.

**Proof.** We compute
\[
0 = \phi(\star[-\frac{1}{2}I, \star[-\frac{1}{2}iI, -\frac{1}{2}iI]])
= \star[\phi(-\frac{1}{2}I), \star[-\frac{1}{2}I, -\frac{1}{2}iI]] + \star[-\frac{1}{2}I, \star[\phi(-\frac{1}{2}iI), -\frac{1}{2}iI]]
+ \star[-\frac{1}{2}I, \star[-\frac{1}{2}iI, \phi(-\frac{1}{2}iI)]]
= \star[\phi(-\frac{1}{2}I), -\frac{1}{2}I] + \star[-\frac{1}{2}I, -i\phi(-\frac{1}{2}iI)] + \star[-\frac{1}{2}I, -\frac{1}{2}i(\phi(-\frac{1}{2}iI) - \phi(-\frac{1}{2}iI)^*)]
= i\phi(-\frac{1}{2}I) - i\phi(-\frac{1}{2}iI)^*.
\]

It follows that $\phi(-\frac{1}{2}iI) = -\phi(-\frac{1}{2}iI)^*$. Similarly, by the equality $0 = \star[\frac{1}{2}I, \star[\frac{1}{2}iI, \frac{1}{2}iI]]$, we can get $\phi(\frac{1}{2}iI) = -\phi(\frac{1}{2}iI)^*$. We may also compute
\[
\phi(-\frac{1}{2}iI) = \phi(\star[-\frac{1}{2}I, \star[-\frac{1}{2}I, -\frac{1}{2}iI]])
= \star[\phi(-\frac{1}{2}I), -\frac{1}{2}iI] + \star[-\frac{1}{2}I, -i\phi(\frac{1}{2}iI)] + \star[-\frac{1}{2}I, -\phi(-\frac{1}{2}iI)]
= 2i\phi(-\frac{1}{2}I) + \phi(-\frac{1}{2}iI).
\]

It follows that $\phi(-\frac{1}{2}I) = 0$. The equality $-\frac{1}{2}I = \star[\frac{1}{2}iI, \star[-\frac{1}{2}I, -\frac{1}{2}iI]]$ implies
\[
0 = \phi(-\frac{1}{2}I) = \phi(\star[\frac{1}{2}iI, \star[-\frac{1}{2}I, -\frac{1}{2}iI]])
= \star[\phi(\frac{1}{2}iI), \star[-\frac{1}{2}I, -\frac{1}{2}iI]] + 0 + \star[\frac{1}{2}iI, \star[-\frac{1}{2}I, \phi(-\frac{1}{2}iI)]]
= i\phi(\frac{1}{2}iI) - i\phi(-\frac{1}{2}iI).
\]
Hence

\[
(1) \quad \phi(\frac{1}{2}i) = \phi(-\frac{1}{2}i).
\]

Since the equality \( \frac{1}{2}i = \star[\frac{1}{2}i, \star[-\frac{1}{2}I, -\frac{1}{2}i]] \) hold true, we have

\[
\phi(\frac{1}{2}i) = \phi(\star[\frac{1}{2}i, \star[-\frac{1}{2}I, -\frac{1}{2}i]])
\]
\[
= \star[\phi(\frac{1}{2}i), \star[-\frac{1}{2}I, -\frac{1}{2}i]] + 0 + \star[\frac{1}{2}I, \star[-\frac{1}{2}I, \phi(-\frac{1}{2}i)]]
\]
\[
= \star[\phi(\frac{1}{2}i), \frac{1}{2}iI] + \star[\frac{1}{2}I, -\phi(-\frac{1}{2}i)]
\]
\[
= i\phi(\frac{1}{2}I) - \phi(-\frac{1}{2}i).
\]

It follows that

\[
(2) \quad \phi(\frac{1}{2}i) + \phi(-\frac{1}{2}i) = i\phi(\frac{1}{2}i).
\]

Finally, by the equality \( \frac{1}{2}I = \star[-\frac{1}{2}I, \star[-\frac{1}{2}I, -\frac{1}{2}i]] \), we can get

\[
\phi(\frac{1}{2}I) = \phi(\star[-\frac{1}{2}I, \star[-\frac{1}{2}I, \frac{1}{2}iI]])
\]
\[
= \star[\phi(-\frac{1}{2}I), \star[-\frac{1}{2}I, \frac{1}{2}iI]] + 0 + \star[-\frac{1}{2}I, \star[-\frac{1}{2}I, \phi(\frac{1}{2}iI)]]
\]
\[
= \star[\phi(-\frac{1}{2}I), \frac{1}{2}I] + \star[-\frac{1}{2}I, -\phi(-\frac{1}{2}iI)]
\]
\[
= i\phi(-\frac{1}{2}I) + i\phi(-\frac{1}{2}i) = 2i\phi(-\frac{1}{2}i).
\]

It follows that

\[
(3) \quad 2\phi(-\frac{1}{2}i) = -i\phi(\frac{1}{2}I).
\]

Hence by Eq. (1), Eq. (2) and Eq. (3), we have \( \phi(\frac{1}{2}iI) = \phi(-\frac{1}{2}iI) = 0 \). For every \( A \in \mathcal{A} \), it follows from \( iA = \star[A, \star[\frac{1}{2}I, \frac{1}{2}iI]] \) that

\[
\phi(iA) = \phi(\star[A, \star[\frac{1}{2}I, \frac{1}{2}iI]]) = \star[\phi(A), \star[\frac{1}{2}I, \frac{1}{2}iI]] = i\phi(A). \quad \Box
\]

We now choose a nontrivial projection \( P_1 \in \mathcal{A} \) and let \( P_2 = I - P_1 \). Denote \( A_{ij} = P_iAP_j, i, j = 1, 2, \). Then we have the Peirce decomposition of \( A \) as \( A = \sum_{i,j=1}^2 A_{ij} \). Note that any operator \( A \in \mathcal{A} \) can be expressed as \( A = A_{11} + A_{12} + A_{21} + A_{22} \), and \( A_{ij}^* \in \mathcal{A}_{ji} \) for any \( A_{ij} \in \mathcal{A}_{ij} \).

**Lemma 2.5.** For any \( A \in \mathcal{A} \),

1. \( \star[A, \star[\frac{1}{2}iI, \frac{1}{2}iI]] = 0 \) implies \( A_{11} = A_{22} = 0 \),
2. \( \star[I, \star[P_1, A]] = 0 \) implies \( A_{12} = 0 \),
3. \( \star[I, \star[P_2, A]] = 0 \) implies \( A_{21} = 0 \),
4. \( \star[A, \star[I, P_1]] = 0 \) implies \( A_{11} = A_{12} = A_{21} = 0 \),
5. \( \star[A, \star[I, P_2]] = 0 \) implies \( A_{22} = A_{12} = A_{21} = 0 \).
We only show (1). The proofs of (2), (3), (4) and (5) go similarly. We compute

\[
0 = \ast[A, \ast[I, i(P_2 - P_1)]] = \ast[A, 2i(P_2 - P_1)] \\
= 2i(A(P_2 - P_1) + (P_2 - P_1)A) \\
= 4i(A_{22} - A_{11}),
\]

which leads to \( A_{22} = A_{11} = 0 \). \qed

**Lemma 2.6.** For any \( A_{12} \in A_{12} \) and \( B_{21} \in A_{21} \), we have

\[
\phi(A_{12} + B_{21}) = \phi(A_{12}) + \Phi(B_{21}).
\]

**Proof.** Let \( M = \phi(A_{12} + B_{21}) - \phi(A_{12}) - \phi(B_{21}) \). We now show that \( M = 0 \).

On one hand, since \( \ast[A_{12}, \ast[I, i(P_2 - P_1)]] = \ast[B_{21}, \ast[I, i(P_2 - P_1)]] = 0 \), we have

\[
0 = \phi(\ast[A_{12} + B_{21}, \ast[I, i(P_2 - P_1)]] \\
= \ast[\phi(A_{12} + B_{21}), \ast[I, i(P_2 - P_1)] + \ast[A_{12} + B_{21}, \ast[\phi(I), i(P_2 - P_1)]] \\
+ \ast[A_{12} + B_{21}, \ast[I, \phi(i(P_2 - P_1))]].
\]

On the other hand,

\[
0 = \phi(\ast[A_{12}, \ast[I, i(P_2 - P_1)]] + \phi(\ast[B_{21}, \ast[I, i(P_2 - P_1)]])) \\
= \ast[\phi(A_{12}) + \phi(B_{21}), \ast[I, i(P_2 - P_1)] + \ast[A_{12} + B_{21}, \ast[\phi(I), i(P_2 - P_1)]] \\
+ \ast[A_{12} + B_{21}, \ast[I, \phi(i(P_2 - P_1))]].
\]

Comparing the above two equalities, we arrive at \( \ast[M, \ast[I, i(P_2 - P_1)]] = 0 \). It follows from Lemma 2.5 (1), that \( M_{11} = M_{22} = 0 \).

Since \( \ast[I, \ast[P_1, B_{21}]] = 0 \), we have that

\[
\ast[\phi(I), \ast[P_1, A_{12} + B_{21}]] = \ast[I, \ast[\phi(P_1), A_{12} + B_{21}]] + \ast[I, \ast[P_1, \phi(A_{12} + B_{21})]] \\
= \phi(\ast[I, \ast[P_1, A_{12} + B_{21}]]) \\
= \phi(\ast[I, \ast[P_1, A_{12}]]) + \phi(\ast[I, \ast[P_1, B_{21}]]]) \\
= \ast[\phi(I), \ast[P_1, A_{12} + B_{21}]] + \ast[I, \ast[\phi(P_1), A_{12} + B_{21}]] + \ast[I, \ast[P_1, \phi(A_{12} + B_{21})]]].
\]

Hence \( \ast[I, \ast[P_1, M]] = 0 \). By Lemma 2.5 (2), we get that \( M_{12} = 0 \). Similarly, by using the fact \( \ast[I, \ast[P_2, A_{12}]] = 0 \), one can show \( M_{21} = 0 \). \qed

**Lemma 2.7.** For any \( A_{11} \in A_{11}, B_{12} \in A_{12}, C_{21} \in A_{21}, \) and \( D_{22} \in A_{22}, \)

1. \( \phi(A_{11} + B_{12} + C_{21}) = \phi(A_{11}) + \phi(B_{12}) + \phi(C_{21}) \).
2. \( \phi(B_{12} + C_{21} + D_{22}) = \phi(B_{12}) + \phi(C_{21}) + \phi(D_{22}) \).
Proof. (1) Since \( [A_{11}, [I, iP_2]] = 0 \), by Lemma 2.6, we obtain
\[
\begin{align*}
&\star [\phi(A_{11} + B_{12} + C_{21}), [I, iP_2]] + \star [A_{11} + B_{12} + C_{21}, \star \phi(I), iP_2] \\
+ &\star [A_{11} + B_{12} + C_{21}, \star I, i\phi(iP_2)] \\
= &\phi(\star [A_{11} + B_{12} + C_{21}, [I, iP_2]]) \\
= &\phi(\star [A_{11}, [I, iP_2]] + \phi(\star [B_{12} + C_{21}, [I, P_2]]) \\
= &\star \phi(A_{11}) + \phi(B_{12}) + \phi(C_{21}), [I, iP_2]) + \star [A_{11} + B_{12} + C_{21}, \star \phi(I), iP_2] \\
+ &\star [A_{11} + B_{12} + C_{21}, [I, i\phi(iP_2)].
\end{align*}
\]
Letting \( M = \phi(A_{11} + B_{12} + C_{21}) - \phi(A_{11}) - \phi(B_{12}) - \phi(C_{21}) \), we get \( \star [M, [I, iP_2]] = 0 \).

We now show that \( M_{11} = 0 \). By noting \( [B_{12}, [I, i(P_2 - P_1)] = [C_{21}, [I, i(P_2 - P_1)] = 0 \), we have
\[
\begin{align*}
&\phi(\star [A_{11} + B_{12} + C_{21}, [I, i(P_2 - P_1)]]) \\
= &\phi(\star [A_{11}, [I, i(P_2 - P_1), I]) + \phi(\star [B_{12}, [I, i(P_2 - P_1)]) \\
+ &\phi(\star [C_{21}, [I, i(P_2 - P_1)])].
\end{align*}
\]
By using the similar argument, we can get \( \star [M, [I, i(P_2 - P_1)] = 0 \). Therefore, \( M_{11} = 0 \) by Lemma 2.5 (3).

(2) Considering \( [\phi(A_{11}) + \phi(B_{12}) + \phi(C_{21}), [I, iP_1]] \) and \( \phi(\star [A_{11} + B_{12} + C_{21}, [I, i(P_2 - P_1)]) \), with the same argument as in (1), one can get \( \phi(B_{12} + C_{21} + D_{22}) = \phi(B_{12}) + \phi(C_{21}) + \phi(D_{22}). \)

Lemma 2.8. For any \( A_{11} \in A_{11}, B_{12} \in A_{12}, C_{21} \in A_{21}, \) and \( D_{22} \in A_{22}, \)
\[
\phi(A_{11} + B_{12} + C_{21} + D_{22}) = \phi(A_{11}) + \phi(B_{12}) + \phi(C_{21}) + \phi(D_{22}).
\]
Proof. Let \( M = \phi(A_{11} + B_{12} + C_{21} + D_{22}) - \phi(A_{11}) - \phi(B_{12}) - \phi(C_{21}) - \phi(D_{22}). \)
Noticing that \( [D_{22}, [I, iP_1]] = 0 \) and applying (1) in Lemma 2.7, we have
\[
\begin{align*}
&\star [\phi(A_{11} + B_{12} + C_{21} + D_{22}), [I, iP_1]] + \star [A_{11} + B_{12} + C_{21} + D_{22}, [\phi(I), iP_1]] \\
+ &\star [A_{11} + B_{12} + C_{21} + D_{22}, [I, \phi(iP_1)] \\
= &\phi(\star [A_{11} + B_{12} + C_{21} + D_{22}, [I, iP_1]) \\
= &\phi(\star [A_{11} + B_{12} + C_{21}, [I, iP_1])] + \phi(\star [D_{22}, [I, iP_1]]) \\
= &\star \phi(A_{11}) + \phi(B_{12}) + \phi(C_{21}) + \phi(D_{22}), [I, iP_1]) \\
+ &\star [A_{11} + B_{12} + C_{21} + D_{22}, [I, \phi(iP_1)] \\
+ &\star [A_{11} + B_{12} + C_{21} + D_{22}, [I, i\phi(iP_1)].
\end{align*}
\]
It follows that \( \star [M, [I, iP_1]] = 0 \), so \( M_{11} = M_{12} = M_{21} = 0 \) by Lemma 2.5.

Using the fact that \( \star [A_{11}, [I, iP_2]] = 0 \) and the similar argument above, we can get \( \star [M, [I, iP_2]] = 0 \) which leads \( M_{22} = 0 \), completing the proof. \( \square \)
** Lemma 2.9.** For any \( A_{jk}, B_{jk} \in A_{jk} \), where \( 1 \leq j \neq k \leq 2 \), we have

\[
\phi(A_{jk} + B_{jk}) = \phi(A_{jk}) + \phi(B_{jk}).
\]

**Proof.** On one hand, by Lemma 2.7,

\[
\phi(iA_{jk} + iB_{jk} + iA_{jk}^* + iA_{jk}^* B_{jk}) = \phi(iA_{jk} + iB_{jk}) + \phi(iA_{jk}^*) + \phi(iA_{jk}^* B_{jk}).
\]

On the other hand, since

\[
*[P_j + B_{jk}, \ [P_k + A_{jk}, \frac{i}{2} I]] = i(A_{jk} + B_{jk}) + i(A_{jk}^*) + i(A_{jk}^* B_{jk}),
\]

using Lemma 2.8 again,

\[
\phi(iA_{jk} + iB_{jk} + iA_{jk}^* + iA_{jk}^* B_{jk}) = \phi(*[P_j + B_{jk}, \ [P_k + A_{jk}, \frac{i}{2} I]])
\]

\[
= *[\phi(P_j + B_{jk}), \ [P_k + A_{jk}, \frac{i}{2} I]] + *[P_j + B_{jk}, \ *[P_k + A_{jk}, \frac{i}{2} I]]
\]

\[
+ *[P_j + B_{jk}, \ *[P_k + A_{jk}, \phi(\frac{i}{2} I)]]
\]

\[
= *[\phi(P_j) + \phi(B_{jk}), \ *[P_k + A_{jk}, \frac{i}{2} I]] + *[P_j + B_{jk}, \ *[\phi(P_k) + \phi(A_{jk}), \frac{i}{2} I]]
\]

\[
+ *[P_j + B_{jk}, \ *[P_k + A_{jk}, \phi(\frac{i}{2} I)]]
\]

\[
= \phi(*[P_j, \ *[P_k, \frac{i}{2} I]]) + \phi(*[B_{jk}, \ *[P_j, \frac{i}{2} I]]) + \phi(*[P_j, \ *[A_{jk}, \frac{i}{2} I]])
\]

\[
+ \phi(*[B_{jk}, \ *[A_{jk}, \frac{i}{2} I]])
\]

\[
= \phi(iB_{jk}) + \phi(iA_{jk}) + \phi(iA_{jk}^* B_{jk})
\]

\[
= \phi(iB_{jk}) + \phi(iA_{jk} + iA_{jk}^*) + \phi(iA_{jk}^* B_{jk})
\]

\[
= \phi(A_{jk} + B_{jk}) = \phi(A_{jk}) + \phi(B_{jk})
\]

Note that in the last identity above, we are using Lemma 2.6. We now can conclude that \( \phi(A_{jk} + B_{jk}) = \phi(A_{jk}) + \phi(B_{jk}) \) by Lemma 2.4.

** Lemma 2.10.** For any \( A_{jj}, B_{jj} \in A_{jj} \), where \( 1 \leq j \leq 2 \), we have

\[
\phi(A_{jj} + B_{jj}) = \phi(A_{jj}) + \phi(B_{jj}).
\]

**Proof.** Let \( k \in \{1, 2\} \), with \( k \neq j \). We compute

\[
*[\phi(A_{jj} + B_{jj}), \ [I, iP_k]] + *[A_{jj} + B_{jj}, \ *[I, \phi(iP_k)]] + *[A_{jj} + B_{jj}, \ [I, \phi(iP_k)]]
\]

\[
= \phi(*[A_{jj} + B_{jj}, \ [I, iP_k]]) + \phi(*[A_{jj} + B_{jj}, \ [I, \phi(iP_k)]])
\]

\[
= \phi(*[A_{jj} + B_{jj}, \ [I, iP_k]]) + \phi(*[A_{jj} + B_{jj}, \ [I, \phi(iP_k)]])
\]

\[
= \phi(*[A_{jj} + B_{jj}, \ [I, iP_k]]) + \phi(*[A_{jj} + B_{jj}, \ [I, \phi(iP_k)]])
\]

\[
+ *[A_{jj} + B_{jj}, \ [I, \phi(iP_k)]]
\]

...
Write $M = \phi(A_{jj} + B_{jj}) - \phi(A_{jj}) - \phi(B_{jj})$. The above computation yields that $s[M, s[I, iP_k]] = 0$. By Lemma 2.4, we have $M_{kj} = M_{jk} = M_{kk} = 0$. We now show that $M_{jj} = 0$. For any $C_{jk} \in A_{jk}$, by Lemma 2.7, \[
s[\phi(C_{jk}), s[A_{jj} + B_{jj}, \frac{1}{2} iP_j]] + s[C_{jk}, s[\phi(A_{jj} + B_{jj}), \frac{1}{2} iP_j]]
+ s[C_{jk}, s[A_{jj} + B_{jj}, \phi(\frac{1}{2} iP_j)]] = \phi(s[C_{jk}, s[A_{jj} + B_{jj}, \frac{1}{2} iP_j]]),
= \phi(s[C_{jk}, s[A_{jj}, \frac{1}{2} iP_j]]) + \phi(s[C_{jk}, s[B_{jj}, \frac{1}{2} iP_j]])
+ \phi(s[C_{jk}, s[A_{jj} + B_{jj}, \phi(\frac{1}{2} iP_j)])] = 0.
\]
Therefore, $s[C_{jk}, s[M, \frac{1}{2} iP_j]] = 0$ which leads to $M_{jj}^* C_{jk} = 0$, for all $C_{jk} \in A_{jk}$. Since $\mathcal{A}$ is prime, we see that $M_{jj} = 0$. \hfill \qed

**Lemma 2.11.** $\phi$ is an additive derivation with $\phi(A^*) = \phi(A)^*$, for all $A \in \mathcal{A}$.

**Proof.** We first show that $\phi$ is additive. For arbitrary $A, B \in \mathcal{A}$, we write $A = \sum_{i,j=1}^2 A_{ij}$ and $B = \sum_{i,j=1}^2 B_{ij}$. By Lemma 2.8, Lemma 2.9 and Lemma 2.10, we obtain \[
\phi(A + B) = \phi(\sum_{i,j=1}^2 A_{ij} + \sum_{i,j=1}^2 B_{ij}) = \sum_{i,j=1}^2 \Phi(A_{ij} + B_{ij})
= \sum_{i,j=1}^2 \phi(A_{ij}) + \sum_{i,j=1}^2 \phi(B_{ij}) = \phi(\sum_{i,j=1}^2 A_{ij}) + \phi(\sum_{i,j=1}^2 B_{ij})
= \phi(A) + \phi(B).
\]
We now show $\phi(A^*) = \phi(A)^*$. For every $A \in \mathcal{A}$, we write $A = A_1 + iA_2$, where $A_1 = \frac{A + A^*}{2}$ and $A_2 = \frac{A - A^*}{2i}$ are self-adjoint elements. By Lemma 2.3 and Lemma 2.4, we have \[
\phi(A^*) = \phi(A_1 - iA_2) = \phi(A_1) - i\phi(A_2)
= \phi(A_1) - i\phi(A_2) = \phi(A_1)^* - i\phi(A_2)^*
= \phi(A_1)^* + (i\phi(A_2))^* = \phi(A_1 + iA_2)^* = \phi(A)^*.
\]
To complete the proof, we need to show that $\phi$ is a derivation. By the additivity of $\phi$ and Lemma 2.5, we have $\phi(iI) = 2\phi(\frac{1}{2} iI) = 0$. Note that $s[A, s[B, iI]] = 2i(AB + B^* A)$. We compute \[
2i\phi(AB + B^* A) = \phi(2i(AB + B^* A))
= \phi(s[A, s[B, iI]])
= s[\phi(A), s[B, iI]] + s[A, s[\phi(B), iI]] + s[A, s[B, \phi(iI)]
= 2i(\phi(A)B + B^* \phi(A)) + A\phi(B) + \phi(B)^* A).
\]
It follows that
\[ \phi(AB + B^*A) = \phi(A)B + B^*\phi(A) + A\phi(B) + \phi(B)^*A. \]
Replacing \( B \) by \( iB \) in the above equality, we get
\[ \phi(AB - B^*A) = \phi(A)B - B^*\phi(A) + A\phi(B) - \phi(B)^*A. \]
Thus \( \phi(AB) = \phi(A)B + A\phi(B) \), it is a derivation. □

**The proof of the main theorem.** By Lemma 2.11, we see that \( \phi \) is an additive derivation with \( \phi(A^*) = \phi(A)^* \). It follows from [12, Theorem 2.3] that \( \phi \) is a linear inner derivation, that is, there exists an operator \( A \in \mathcal{B}(\mathcal{H}) \) such that \( \phi(A) = AS - SA \), for all \( A \in \mathcal{A} \). Since \( \phi(A^*) = \phi(A)^* \), we have
\[ A^*S - SA^* = \phi(A^*) = \phi(A)^* = S^*A^* - A^*S^* \]
for any \( A \in \mathcal{A} \). This leads to \( A^*(S + S^*) = (S + S^*)A^* \). Hence, \( S + S^* = \lambda I \) for some \( \lambda \in \mathbb{R} \). Letting \( T = S - \frac{1}{2}\lambda I \), one can check that \( T + T^* = 0 \) and \( \phi(A) = AT - TA \), for all \( A \in \mathcal{A} \).

**Corollary 2.1.** Let \( \mathcal{H} \) be an infinite dimensional complex Hilbert space and \( \phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}) \) is nonlinear left \( \ast \)-Lie triple mapping, then \( \phi \) is an inner \( \ast \)-derivation, that is, there exists an operator \( T \in \mathcal{B}(\mathcal{H}) \) satisfying \( T + T^* = 0 \) such that \( \phi(A) = AT - TA \), for all \( A \in \mathcal{A} \).

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**References**


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