On coefficient inequalities for certain subclasses of meromorphic bi-univalent functions

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Abstract. In the present paper, we investigate and define two subclasses of meromorphic bi-univalent function class $\Sigma'$ which are defined on the domain $U^{*} = \{ z \in \mathbb{C} : 1 < |z| < \infty \}$. Further, by using the well-known coefficients estimates of the Carathéodory functions (i.e functions with positive real part) we obtain the estimates on the coefficients $|b_0|$, $|b_1|$ and $|b_2 + b_3|$ for functions in these subclasses.

Keywords: analytic function, meromorphic function, univalent function, bi-univalent function, meromorphic bi-univalent function.

1. Introduction

Let the class $A = \{ f : U \to \mathbb{C} : f \text{ is analytic in } U \text{ and } f(0) = f'(0) - 1 = 0 \}$ and its subclass $S = \{ f : U \to \mathbb{C} : f \in A \text{ and also univalent in } U \}$ where $U = \{ z \in \mathbb{C} : |z| < 1 \}$ be the open unit disk and such functions $f \in A$ have the form:

\[
f(z) = z + \sum_{k=2}^{\infty} a_k z^k.
\]

(1.1)

In 1972, Ozaki and Nunokawa [14] proved the following Lemma (univalence criterion). In fact, this result is appeared in the paper by Aksentev [1] (also see the paper by Aksentëv and Avhadiev [2]).

Lemma 1.1. If for $f(z) \in A$

\[
\frac{z^2 f'(z)}{(f(z))^2} - 1 < 1 \quad (z \in U),
\]

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then \( f(z) \) is univalent in \( U \) and hence \( f(z) \in \mathcal{S} \).

Also, a function \( f(z) \in \mathcal{A} \) is said to be in the class \( \mathcal{T}(\mu) \), \( (0 < \mu \leq 1) \) if

\[
\left| \frac{z^2 f'(z)}{(f(z))^2} - 1 \right| < \mu \quad (z \in U)
\]

and \( \mathcal{T}(1) = \mathcal{T} \). Clearly, \( \mathcal{T}(\mu) \subset \mathcal{T} \subset \mathcal{S} \). Further (see Kuroki et al. [10]), for \( f(z) \in \mathcal{T}(\mu) \) see that:

\[
\Re \left( \frac{z^2 f'(z)}{(f(z))^2} \right) > 1 - \mu \quad (z \in U).
\]

In particular, for initial coefficient estimates of bi-univalent function classes \( \mathcal{T}_\Sigma(\mu) \) and \( \mathcal{T}_E^\Sigma \), see the paper by Naik and Patil [12].

In 1967, Lewin [11] introduced and studied the bi-univalent function class \( \Sigma \). After which some researchers (viz. \[3, 13\]) found the initial coefficient estimates for the functions in \( \Sigma \). Later, Srivastava et al. [17] revived it for the subclasses of \( \Sigma \). Recently, the concept of bi-univalent functions is extend to meromorphic bi-univalent functions.

Let \( \mathcal{S}' \) denote the class of meromorphic univalent functions \( g \) of the form:

\[
g(z) = z + \sum_{n=0}^{\infty} \frac{b_n}{z^n},
\]

defined on the domain \( U^* = \{ z : z \in \mathbb{C}, 1 < |z| < \infty \} \). Clearly, \( g \in \mathcal{S}' \) has an inverse say \( g^{-1} \), defined by:

\[
g^{-1}(g(z)) = z, \quad (z \in U^*)
\]

and

\[
g(g^{-1}(w)) = w, \quad (0 < M < |w| < \infty),
\]

which has a series expansion of the form:

\[
g^{-1}(w) = h(w) = w + \sum_{n=0}^{\infty} \frac{c_n}{w^n}, \quad (0 < M < |w| < \infty).
\]

Some simple computations using equation (1.2) shows that:

\[
(1.3) \quad g^{-1}(w) = h(w) = w - b_0 - \frac{b_1}{w} - \frac{b_2 + b_0b_1}{w^2} - \frac{b_3 + 2b_0b_2 + b_0^2b_1 + b_1^2}{w^3} + \cdots.
\]

Let \( \Sigma' = \{ g \in \mathcal{S}' : \text{both } g \text{ and } g^{-1} \text{ are meromorphic univalent in } U^* \} \) denote the class of all meromorphic bi-univalent functions in \( U^* \). Recently the coefficient estimate on functions of various subclasses of \( \Sigma' \) were obtained by some researchers viz. Halim et al. [6], Hamidi et al. [7, 8], Panigrahi [15], Janani
and Murugusundaramoorthy [9], Bulut [4], etc. In the present investigation, we define two new subclasses of the function class \( \Sigma' \) and obtain the estimate on \( |b_0|, |b_1| \) and \( |b_2 + b_3^2| \) for the functions in these new subclasses.

We need to recall the Carathéodory lemma in the following form to prove our main results (see [5], [16]).

**Lemma 1.2.** If \( p(z) \in \mathcal{P} \), the class of all functions analytic in \( \mathbb{U}^* \), for which

\[
\Re(p(z)) > 0,
\]

then \( |p_n| \leq 2 \) for each \( n \in \mathbb{N} := \{1, 2, 3, \ldots\} \), where

\[
p(z) = 1 + \frac{p_1}{z} + \frac{p_2}{z^2} + \frac{p_3}{z^3} + \cdots, \quad (z \in \mathbb{U}^*).
\]

### 2. Coefficient estimates

**Definition 2.1.** A function \( g(z) \in \Sigma' \) given by (1.2) is said to be in the class \( \mathcal{T}_{\Sigma'}(\mu) \) if the following conditions are satisfied:

\[
\Re \left( \frac{z^2 g'(z)}{(g(z))^2} \right) > 1 - \mu, \quad (z \in \mathbb{U}^*; \ 0 < \mu \leq 1)
\]

and

\[
\Re \left( \frac{w^2 h'(w)}{(h(w))^2} \right) > 1 - \mu, \quad (w \in \mathbb{U}^*; \ 0 < \mu \leq 1),
\]

where the function \( h \) is an inverse of \( g \) given by (1.3).

**Theorem 2.2.** Let the function \( g(z) \in \Sigma' \) given by (1.2) be in the class \( \mathcal{T}_{\Sigma'}(\mu) \), where \( 0 < \mu \leq 1 \). Then,

\[
|b_0| \leq \begin{cases} 
\mu; & (0 < \mu \leq \frac{3}{2}) \\
\frac{\mu}{3}; & (\frac{3}{2} \leq \mu \leq 1)
\end{cases}
\]

(2.1)

\[
|b_1| \leq \frac{2\mu}{3},
\]

(2.2)

\[
|b_2 + b_3^2| \leq \frac{\mu}{2}.
\]

(2.3)

**Proof.** Let the function \( g(z) \in \mathcal{T}_{\Sigma'}(\mu) \). See that clearly, the conditions given in the definition of meromorphic bi-univalent function class \( \mathcal{T}_{\Sigma'}(\mu) \) can be written as:

\[
\frac{z^2 g'(z)}{(g(z))^2} = (1 - \mu) + \mu s(z)
\]

(2.4)
\[ w^2 \frac{h'(w)}{(h(w))^2} = (1 - \mu) + \mu t(w), \]

where \( s(z), t(w) \in \mathbb{P} \) have the form:

\[ s(z) = 1 + \frac{s_1}{z} + \frac{s_2}{z^2} + \frac{s_3}{z^3} + \cdots, \quad (z \in \mathbb{U}^{*}) \]

and

\[ t(w) = 1 + \frac{t_1}{w} + \frac{t_2}{w^2} + \frac{t_3}{w^3} + \cdots, \quad (w \in \mathbb{U}^{*}). \]

Hence we have:

\[ (1 - \mu) + \mu s(z) = 1 + \frac{\mu s_1}{z} + \frac{\mu s_2}{z^2} + \frac{\mu s_3}{z^3} + \cdots \]

and

\[ (1 - \mu) + \mu t(w) = 1 + \frac{\mu t_1}{w} + \frac{\mu t_2}{w^2} + \frac{\mu t_3}{w^3} + \cdots. \]

Also, using (1.2) and (1.3) we obtain:

\[ \frac{z^2 g'(z)}{(g(z))^2} = 1 - \frac{2b_0}{z} + \frac{3 \left(b_0^2 - b_1\right)}{z^2} + \frac{8b_0b_1 - 4b_2 - 4b_0^3}{z^3} + \cdots \]

and

\[ \frac{w^2 h'(w)}{(h(w))^2} = 1 + \frac{2b_0}{w} + \frac{3 \left(b_0^2 + b_1\right)}{w^2} + \frac{12b_0b_1 + 4b_2 + 4b_0^3}{w^3} + \cdots. \]

Now, equating the coefficients in (2.4) and (2.5) we get:

\[ -2b_0 = \mu s_1, \]

\[ 3 \left(b_0^2 - b_1\right) = \mu s_2, \]

\[ 8b_0b_1 - 4b_2 - 4b_0^3 = \mu s_3, \]

\[ 2b_0 = \mu t_1, \]

\[ 3 \left(b_0^2 + b_1\right) = \mu t_2, \]

\[ 12b_0b_1 + 4b_2 + 4b_0^3 = \mu t_3. \]

Clearly, equation (2.8) and (2.11) in light of Lemma 1.2 gives:

\[ |b_0| \leq \mu. \]
Also by adding (2.9) in (2.12), we obtain:

\[ 6b_0^2 = \mu (s_2 + t_2) \]

which, by using Lemma 1.2 gives:

\[ |b_0^2| \leq \frac{2\mu}{3}. \]

Equation (2.14) and (2.15) together yields:

\[ |b_0| \leq \min \left\{ \mu, \sqrt{\frac{2\mu}{3}} \right\}, \]

which, for \( 0 < \mu \leq 1 \) gives the desired result (2.1).

Now, by subtracting (2.9) from (2.12), we get:

\[ 6b_1 = \mu (t_2 - s_2) \]

which, by using Lemma 1.2 gives:

\[ |b_1| \leq \frac{2\mu}{3}. \]

This is the desired result (2.2).

Finally, for the last inequality subtracting (2.10) from (2.13), we get:

\[ 4b_0b_1 + 8b_2 + 8b_0^3 = \mu (t_3 - s_3). \]

Also, by adding (2.10) in (2.13), we get:

\[ 20b_0b_1 = \mu (s_3 + t_3). \]

Eliminating \( b_0b_1 \) from (2.17) and (2.18), we obtain:

\[ 40 \left( b_2 + b_0^3 \right) = \mu (4t_3 - 6s_3) \]

which, in light of Lemma 1.2, yields the desired inequality (2.3).

This completes the proof of Theorem 2.2.

**Definition 2.3.** A function \( g(z) \in \Sigma' \) given by (1.2) is said to be in the class \( \mathcal{T}_g' \) if the following conditions are satisfied:

\[ \left| \arg \left( \frac{z^2g'(z)}{(g(z))^2} \right) \right| < \frac{\alpha \pi}{2} \quad (z \in \mathbb{U}^*; 0 < \alpha \leq 1) \]

and

\[ \left| \arg \left( \frac{w^2h'(w)}{(h(w))^2} \right) \right| < \frac{\alpha \pi}{2} \quad (w \in \mathbb{U}^*; 0 < \alpha \leq 1), \]

where the function \( h \) is an inverse of \( g \) given by (1.3).
Theorem 2.4. Let the function \( g(z) \in \Sigma' \) given by (1.2) be in the class \( T_{\Sigma'}^0 \), where \( 0 < \alpha \leq 1 \). Then,

\[
|b_0| \leq \sqrt{\frac{2}{3}} \alpha, \\
|b_1| \leq \frac{2}{3} \alpha^2, \\
|b_2 + b_0^3| \leq \frac{\alpha (2\alpha^2 + 1)}{6}.
\]

Proof. Since \( g(z) \in T_{\Sigma'}^0 \); for \( s(z), t(w) \in \mathcal{P} \) the conditions given in the definition of the function class \( T_{\Sigma'}^0 \) can be written as:

\[
\frac{z^2 g'(z)}{(g(z))^2} = [s(z)]^\alpha
\]
and
\[
\frac{w^2 h'(w)}{(h(w))^2} = [t(w)]^\alpha,
\]
where \( s(z) \) and \( t(w) \) have the form as given in (2.6) and (2.7), respectively.

Clearly, we have:

\[
[s(z)]^\alpha = 1 + \frac{\alpha s_1}{z} + \frac{\frac{1}{2} \alpha (\alpha - 1) s_1^2 + \alpha s_2}{z^2} + \frac{\frac{1}{6} \alpha (\alpha - 1)(\alpha - 2) s_1^3 + \alpha (\alpha - 1) s_1 s_2 + \alpha s_3}{z^3} + \ldots
\]
and

\[
[t(w)]^\alpha = 1 + \frac{\alpha t_1}{w} + \frac{\frac{1}{2} \alpha (\alpha - 1) t_1^2 + \alpha t_2}{w^2} + \frac{\frac{1}{6} \alpha (\alpha - 1)(\alpha - 2) t_1^3 + \alpha (\alpha - 1) t_1 t_2 + \alpha t_3}{w^3} + \ldots.
\]

Also, just as in proof of Theorem 2.2 we have:

\[
\frac{z^2 g'(z)}{(g(z))^2} = 1 - \frac{2b_0}{z} + \frac{3 \left( b_0^2 - b_1 \right)}{z^2} + \frac{8b_0b_1 - 4b_2 - 4b_0^3}{z^3} + \ldots
\]
and
\[
\frac{w^2 h'(w)}{(h(w))^2} = 1 + \frac{2b_0}{w} + \frac{3 \left( b_0^2 + b_1 \right)}{w^2} + \frac{12b_0b_1 + 4b_2 + 4b_0^3}{w^3} + \ldots.
\]

Now, equating the coefficients in (2.22) and (2.23) we get:

\[
-2b_0 = \alpha s_1,
\]
\begin{align*}
(2.25) \quad 3 \left( b_0^2 - b_1 \right) &= \frac{1}{2} \alpha (\alpha - 1) s_1^2 + \alpha s_2, \\
(2.26) \quad 8b_0b_1 - 4b_2 - 4b_0^3 &= \frac{1}{6} \alpha (\alpha - 1) (\alpha - 2) s_1^3 + \alpha (\alpha - 1) s_1 s_2 + \alpha s_3, \\
(2.27) \quad 2b_0 &= \alpha t_1, \\
(2.28) \quad 3 \left( b_0^2 + b_1 \right) &= \frac{1}{2} \alpha (\alpha - 1) t_1^2 + \alpha t_2, \\
(2.29) \quad 12b_0b_1 + 4b_2 + 4b_0^3 &= \frac{1}{6} \alpha (\alpha - 1) (\alpha - 2) t_1^3 + \alpha (\alpha - 1) t_1 t_2 + \alpha t_3.
\end{align*}

Clearly, equation (2.24) and (2.27) in light of Lemma 1.2 gives:
\begin{equation}
(2.30) \quad |b_0| \leq \alpha.
\end{equation}

Also by adding (2.25) in (2.28), we obtain:
\begin{equation*}
6b_0^2 = \frac{1}{2} \alpha (\alpha - 1) (s_1^2 + t_1^2) + \alpha (s_2 + t_2)
\end{equation*}
which, by using Lemma 1.2 gives:
\begin{equation}
(2.31) \quad |b_0^2| \leq \frac{2}{3} \alpha^2.
\end{equation}

Obviously, from (2.30) and (2.31) we can write:
\begin{equation*}
|b_0| \leq \sqrt{\frac{2}{3}} \alpha \leq \alpha; \quad (0 < \alpha \leq 1).
\end{equation*}

This gives the desired result (2.19).

Now, by subtracting (2.25) from (2.28), we get:
\begin{equation*}
6b_1 = \frac{1}{2} \alpha (\alpha - 1) (t_1^2 - s_1^2) + \alpha (t_2 - s_2)
\end{equation*}
which, by using Lemma 1.2 gives:
\begin{equation*}
|b_1| \leq \frac{2}{3} \alpha^2.
\end{equation*}

This is the desired result (2.20).

Finally, subtracting (2.26) from (2.29), we get:
\begin{equation}
(2.32) \quad 24 \left( b_0b_1 + 2b_2 + 2b_0^3 \right) = \alpha (\alpha - 1) (\alpha - 2) (t_1^3 - s_1^3) + 6\alpha (\alpha - 1) (t_1 t_2 - s_1 s_2) + 6\alpha (t_3 - s_3).
\end{equation}
Also, by adding (2.26) in (2.29), we get:

\[(2.33) \quad 120 b_0 b_1 = \alpha (\alpha - 1)(\alpha - 2)(s_1^3 + t_1^3) + 6\alpha (\alpha - 1)(s_1 s_2 + t_1 t_2) + 6\alpha (s_3 + t_3).\]

Eliminating \(b_0 b_1\) from (2.32) and (2.33), we obtain:

\[
240 \left( b_2 + b_0^3 \right) = \alpha (\alpha - 1)(\alpha - 2)(4t_1^3 - 6s_1^3) + 6\alpha (\alpha - 1)(4t_1 t_2 - 6s_1 s_2) + 6\alpha (4t_3 - 6s_3)
\]

which, in light of Lemma 1.2, yields the desired inequality (2.21).

This completes the proof of Theorem 2.4.

\[\square\]

3. Conclusion

It is interesting that, for functions in both the subclasses \(T_{\Sigma}^{\prime}(\mu)\) and \(T_{\Sigma}^{\prime}\), \((0 < \mu, \alpha \leq 1)\); all the coefficient inequalities are similar in the following sense:

\[
\max_{g \in \Sigma} \left| b_0 \right| \leq \sqrt{\frac{2}{3}},
\]

\[
\max_{g \in \Sigma} \left| b_1 \right| \leq \frac{2}{3},
\]

\[
\max_{g \in \Sigma} \left| b_2 + b_0^3 \right| \leq \frac{1}{2}.
\]

References


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