A new view of closed-CS-module

Majid Mohammed Abed
Department of Mathematics
Faculty of Education For Pure Sciences
Universiti of Anbar
Al-anbar, Iraq
m_m_ukm@gmail.com

Abstract. This paper give a new fact about the extending module. A module M is called extending if every closed submodule N of M is a direct summand. Study of the concepts complement closed submodule (Closed-N)c is achieved. Also we expose to a new way to obtain generalization of extending module by complement closed submodule.

Keywords: extending module, essential submodule, closed submodule, exact sequence.

1. Introduction

In (1976), Goodearl introduced the definition of complement closed submodule and Dungh, Huynh, Smith and Wisbauer [1], studied the extending modules. Wang [5] studied closed-CS-module. A submodule A of M is called essential submodule if A∩K≠0 for every non-zero submodule K of M, equivalently A is a essential in M if and only if every non-zero element of M has a non-zero multiple in A. Therefore if every submodule is essential in a direct summand of M, then M is called extending module. A module M is called extending if every closed submodule N of M is a direct summand of M. Extending modules has been studied in [1] and [2]. Let Z(M)= {IₓM:Iₓ=0, for some ideal I₆ess R}. If Z(M)=M, then M is a singular. Thus we can define another set: Let M/N be a quotient module and let Z(M/N)= {a+IₓM/N:Iₓ=0, for some ideal I₆ess R}. If Z(M/N)=M/N, then M/N is singular. Therefore if Z(M/N)≠M/N, this means the quotient module M/N is non singular.

Remark 1.1. (a) We denote (Closed-N)c to complement closed submodule N of M.
(b) Every semisimple R-module is an extending module. For example Z6 as Z-module.
(c) Not every module M has closed submodule is extending; for example; the module M=Z8⊕Z2 as a Z-module. Let A=(2,1) be the submodule generated by (2,1). Clear that A is closed in M but not a summand. Hence M is not extending.
(d) Let us take (Closed-B)c belong to A; where A and B are submodules in an R-Module M. Then (2,1) is essential in M.
(e) Every (Closed-N)c is closed.
Theorem 1.2. Any module $K$ is singular if and only if there exists a short exact sequence

$$0 \rightarrow N \rightarrow L \rightarrow K \rightarrow 0$$

such that $f$ is an essential monomorphism between $N$ and $L$.

Definition 1.3. (see [4]) Let $M$ be a module. Then $M$ is called closed-CS-module (generalization of extending module) if for every submodule $N$ of $M$; the quotient module $\frac{M}{N}$ is non singular and is direct summand of $M$. (i.e. $M$ has $(\text{Closed-}N)^c$ and direct summand of $M$).

This paper, contain two main sections. In the first section we give some properties of $(\text{Closed-}N)^c$ and in the second section the closed-CS-module is investigated. We prove if $K$ is maximal $(\text{Closed-}K)^c$ of $M$, then $\frac{M}{K}$ is a projective and $K$ is a direct summand of $M$. (see Proposition 2.13). On the other hand, we prove that an $R$-module $M$ is closed-CS-module iff for every $(\text{Closed-}N)^c$ of $M$, there is a decomposition $M=M_1 \oplus M_2$ such that $A$ is a subset of $M_1$ and $A^c = M_2 \subseteq M$. (see Theorem 3.5).

2. Complement closed submodule

Let $N$ be a submodule of an $R$-module $M (N \subseteq M)$. Then we can denote $(\text{Closed-}N)^c$ of $M$ to the complement closed submodule $N$ and (closed-CS-module) means $M$ has $(\text{Closed-}N)^c$. If every $(\text{Closed-}N)^c$ of $M$ is a direct summand, then we obtain a generalization of extending module $M$ (closed-CS-module).

Remark 2.1. If the quotient module $\frac{M}{N}$ is non singular, then $N$ is a $(\text{Closed-}N)^c$.

Definition 2.2. For $N \subseteq M$ and $L \subseteq N$ such that $L \lhd N$, then $M = \frac{N}{L}$. So, if we have $N$ as a module, then $N$ is called generalization of extending module if the quotient module $\frac{N}{L}$ is non singular and is a direct summand in $M$.

Note that, if $(\text{Closed-}N)^c$ is a subset of $M$, then $N$ subset of $(\text{Closed-}K)^c$ and from the second isomorphism theorem, we have; $N$ subset of $(\text{Closed-}N)^c + K \iff (N \cap K)$ is a subset of $(\text{Closed-}K)^c$. Also, by the third isomorphism theorem we can say: $N$ is a subset of $K$ and $K$ is a subset of $M \implies K$ is a subset of $(\text{Closed-}N)^c$ of $M \iff K$ is a subset of $(\text{Closed-}K)^c$.

Lemma 2.3. Let $M$ be an $R$-module and let $B\alpha$ in $\Lambda$, be an independent family of submodules of $M$ and $A\alpha$ is a subset of $B\alpha$, for all $\alpha$ in $\Lambda$. Then $\prod A\alpha$ is a subset of $(\text{Closed-}N)^c$ of $B\alpha$ if and only if $A\alpha$ is a subset of $(\text{Closed-}N)^c$ of $B\alpha$, for all $\alpha$ in $\Lambda$.

Proof. Suppose that $\prod A\alpha$ is a subset of $\prod B\alpha$. We have, $\frac{B\alpha}{A\alpha} \cong \frac{B\alpha}{A\alpha}$. Then $A\alpha$ subset of $(\text{Closed-}N)^c$ of $B\alpha$, for all $\alpha$ in $\Lambda$. Conversely, $A\alpha$ is a subset of $(\text{Closed-}N)^c$ of $B\alpha$, for all $\alpha$ in $\Lambda$. Then $\frac{B\alpha}{A\alpha}$ is non-singular, for all $\alpha \in \Lambda$ and hence $\frac{B\alpha}{A\alpha}$ is non-singular. But $\frac{B\alpha}{A\alpha} \cong \frac{B\alpha}{A\alpha}$. So $A\alpha$ is a subset of $(\text{Closed-}N)^c$ of $\prod B\alpha$. □
Theorem 2.4. Let $M$ be an $R$-module and let $N$ and $K$ are submodules of $M$. Then $(N \cap K)$ is a subset of $(\text{Closed}(N))^c$ in $M$.

**Proof.** Let $N$ be a subset of $(\text{closed-CS-module})$ and let $K$ be a subset of $(\text{closed-CS-M})$. We must prove that $(N \cap K)$ is a subset of $(\text{Closed}(N))^c$ in $M$. Let us take an element $m \in M$ such that $m+N \cap K$ belong to $Z(M/N \cap K)$. Thus Annihilator of $(m+N \cap K)$ is a subset of $(eR)$. Since Annihilator of $(m+N \cap K)$ is a subset of Annihilator of $(m+N)$, then Annihilator of $(m+N)$ is a subset of $(eR)$. We have $Z(M/N)=0$, therefore $m+N=N$. Similar, we get $m+K=K$. Thus $m$ belong to $N \cap K$ and then $Z(M/N \cap K)=0$.

Lemma 2.5. Let $L$ and $K$ be a submodules of an $R$-module $M$. If $L$ is a subset of $(\text{Closed}(K))^c$ and $K$ is a subset of $(\text{closed-CS-module})$, then $L$ is a subset of $(\text{closed-CS-module})$.

**Proof.** Let $L$ be a subset of $(\text{Closed}(K))^c$ and let $K$ be a subset of $(\text{closed-CS-module})$. Let us take short exact sequence:

$$0 \rightarrow \left( \frac{K}{L} \right) \rightarrow \left( \frac{M}{L} \right) \rightarrow \left( \frac{M}{L}/\left( \frac{K}{L} \right) \right) \rightarrow 0.$$ 

Such that $i$ is the inclusion map from $(\frac{K}{L})$ into $(\frac{M}{L})$ and $\pi$ is the natural epimorphism from $(\frac{M}{L})$ into $(\frac{M}{L}/\left( \frac{K}{L} \right))$. Since $L$ is a subset of $K$ and $K$ is a subset of $(\text{closed-CS-module})$, then $(\frac{K}{L})$ is a subset of $(\text{Closed}(N))^c$ of $(\frac{M}{L})$, (see Theorem 2.4). Since $(\frac{K}{L})$ and $(\frac{M}{L}/\left( \frac{K}{L} \right))$ are non-singular, then $(\frac{M}{L})$ is non-singular.

Let $M$ be an $R$-module such that $L$ subset of $K$ and $K$ subset of $M$. If $K$ subset of $(\text{Closed}(N))^c$ of $M$, then $L$ need not be $(\text{Closed}(N))^c$. See the following example:

**Example 2.6.** Consider $Z$ as $Z$-module, it is clear that $Z$ subset of $(\text{Closed}(N))^c$ of $Z$. But $Z(2Z \leq Z)=Z(Z_2)=Z_2$ is singular. On the other hand, if $L$ subset of $(\text{Closed}(N))^c$ of $M$, then $K$ need not be $(\text{Closed}(K))^c$.

**Example 2.7.** Let $0$ subset of $2Z$ and $2Z$ subset of $Z$. Clearly $0$ subset of (closed-CS-$Z$). But $Z(\frac{Z}{2Z})=Z(Z_2)=Z_2$ is singular. Also, an epimorphic image of an (Closed-N)$^c$ need not be (closed-CS-module). We have the natural epimorphism $\pi:Z \rightarrow \frac{Z}{2Z}$. That is means $0$ subset of (Closed-N)$^c$ of $Z$. On the other hand, since $\frac{Z}{2Z} \cong Z_4$ is a singular imply the image of zero always equal zero and moreover it is not (closed-CS-$\frac{Z}{2Z}$).

**Proposition 2.8.** Let $\lambda: M \rightarrow N$ be an epimorphism and $L$ subset of (closed-CS-module). If ker($f$) subset of $L$, then $f(L)$ subset of $(\text{Closed}(N))^c$.

**Proof.** Assume that $L$ subset of (closed-CS-module). To show that $f(L)$ subset of (Closed-N)$^c$. Let $n$ belong to $N$ such that Annihilator($n+f(L)$) subset of $eR$. Since $f$ is an epimorphism, then $n=f(m)$, for some $m \in M$. Since ker($f$) subset of $L$, then Annihilator($n+f(L)$) subset of Annihilator($m+L$) and hence Annihilator($n+f(L)$) subset of $eR$. But $L$ subset of (Closed-N)$^c$ of $M$, so $m \in L$. Thus $n=f(m) \in f(L)$.
Theorem 2.9. Let \( \lambda : M \rightarrow N \) be an \( R \)-homomorphism and \( K \) \((\text{Closed-N})^c\), then for every singular submodule \( L \) of \( M \), \( f(L) \) subset of \( K \).

Proof. Let \( \mu : N \rightarrow \frac{N}{K} \) be the natural epimorphism. Let \( \mu \circ \lambda : M \rightarrow \frac{N}{K} \). Now \( \mu \circ \lambda|_{L} : L \rightarrow \frac{N}{K} \). But \( N \) is a singular and \( \frac{N}{K} \) is non-singular. Thus \( \mu \circ \lambda|_{L} = 0 \). So \( \mu(\lambda(L)) = 0 \) and hence \( \lambda(L) \) subset of \( \ker(\mu) = K \).

As a result from Theorem 2.9, we introduce the following good corollary.

Corollary 2.10. If \( N \) is a module and \( K \) subset of \((\text{Closed-N})^c\). Then \( \frac{\text{Hom}(M,N)}{M} \) subset of \( K \), such that \( Z(M) = M \).

Example 2.11. Suppose that \( M \) is an \( R \)-module. Let \( L \) subset of \((\text{closed-CS-module})\). Then \( Z(M) = Z(L) \).

Proof. We must prove that \( Z(M) \) is a subset of \( Z(L) \). Let \( i: Z(M) \rightarrow M \) be the inclusion map and \( \mu : M \rightarrow \frac{M}{L} \) be the natural epimorphism from \( M \) into \( \frac{M}{L} \). We take the map \( \mu \circ i : Z(M) \rightarrow \frac{M}{L} \). Since \( Z(M) \) is a singular and \( \frac{M}{L} \) is non-singular, then \( \mu \circ i = 0 \). So \( \mu \circ i (Z(M)) = 0 \). Thus \( Z(M) \) is a subset of \( \ker(\mu) = L \).

We know that \( Z(L) = Z(M) \backslash A \). So \( Z(L) = Z(M) \).

Theorem 2.12. Let \( M \) be an \( R \)-module and let \( L \subseteq K \subseteq M \) and \( N \subseteq (\text{closed-CS-module}) \), then \( \frac{M}{K} \) is a singular if and only \( K \) subset of \((\text{closed-CS-module})\).

Proof. Let \( L \) subset of \((\text{Closed-N})^c \) of \( M \) and \( \frac{M}{K} \) is singular. By the third isomorphism theorem \( \frac{M}{K} \cong \frac{\frac{M}{L}}{\frac{K}{L}} \). Since \( \frac{M}{K} \) is non-singular, then \( \frac{\frac{M}{L}}{\frac{K}{L}} \) subset of \((\text{closed-CS-module}) \). Let \( \mu : M \rightarrow \frac{M}{K} \) be the natural epimorphism. We have \( K = \mu^{-1}(\frac{K}{L}) \) is a subset of \( \mu^{-1}(\frac{M}{L}) = M \). The converse is clear by [3].

Proposition 2.13. Let \( M \) be an \( R \)-module and \( K \) is maximal \((\text{Closed-K})^c \) of \( M \). Then \( \frac{M}{K} \) is projective and \( K \) is a direct summand of \( M \).

Proof. Since \( K \) is maximal submodule of \( M \), then \( \frac{M}{K} \) is simple and hence semisimple. But \( \frac{M}{K} \) is non-singular, therefore \( \frac{M}{K} \) is projective. Now consider the following short exact sequence \( 0 \rightarrow K \rightarrow M \rightarrow \frac{M}{K} \rightarrow 0 \): where \( i \) is the inclusion map and \( \pi \) is the natural epimorphism from \( M \) into \( \frac{M}{K} \). Since \( \frac{M}{K} \) is projective, then the sequence is splits, (see [6]). Thus \( K \) is a direct summand of \( M \). Let \( M \) be an \( R \)-module and \( N \) subset of \( M \). Recall that the residual of \( M \) in \( N \) (denoted by \([N:M]\)) is defined as follows: \([N:M] = r \in R, rM \subseteq N\), (see [7]).

3. Closed-CS-module

In this section, we introduce main theorems which explain the new ways to obtain a generalization of extending module.

Proposition 3.1. Let \( M \) be a \((\text{Closed-N})^c \) and \( N \leq M \), then the quotient module is a \((\text{Closed-N})^c \) of \( M \).
Proof. Let $\frac{K}{N}$ subset of $(\text{Closed-N})^c$ of $\frac{M}{N}$. Then by Theorem 2.4 and Lemma 2.5, $K$ is a subset of $(\text{Closed-N})^c$ in $M$. But $M$ is a closed-CS-module. (i.e. has $(\text{Closed-N})^c$) of $M$, therefore $M=N \oplus K$, $K$ is a subset of $M$. Since $N$ is a subset of $K$, then one can easily show that $\frac{M}{N}=(\frac{K}{N}) \oplus (\frac{K+N}{N})$. Thus $\frac{M}{N}$ is a closed-CS-module.

Recall that a module $M$ is called closed-CS-module if for any submodule $N$ of $M$, there is a direct summand $K$ of $M$ such that $N$ is a subset of $K$ and $\frac{K}{N}$ is singular.

Let $N$ subset of $(\text{Closed-N})^c$. Since $M$ is $(\text{Closed-N})^c$, then there exists a direct summand $K$ of $M$ such that $N$ is a subset of $K$ and $Z(\frac{K}{N})=(\frac{K}{N})$; $(\frac{K}{N}$ is a singular). But $\frac{K}{N}$ is a subset of $\frac{M}{N}$, so is non-singular. Thus $K=N$. So any $(\text{Closed-M})^c$ is closed-CS-module.

Theorem 3.2. An $R$-module $M$ is a closed-CS-module if and only if for every $N$ submodule of $M$, $(\text{Closed-N})^c$, there is a decomposition $M=M_1 \oplus M_2$ such that $N$ is a subset of $M_1$ and $M_2$ is a complement of $N$ in $M$.

Proof. $\implies$ Clear.

$\leftarrow$ Let $N$ be a subset of $(\text{Closed-N})^c$, then by our assumption, there exists decomposition $M=M_1 \oplus M_2$ such that $N$ is a subset of $M_1$ and $M_2$ is a complement of $N$ in $M$. So $N \oplus M_2$ is a subset of $(\text{Closed-N})^c$ of $M$. Thus $N$ is a subset of $(\text{Closed-N})^c$ of $M_1$ and hence $Z(\frac{N}{M_1})=(\frac{N}{M_1})$; $(\frac{N}{M_1}$ is singular). But $N$ is a subset of $M_1$ and $N$ is a subset of $(\text{Closed-N})^c$ of $M$, therefore $N$ is a subset of $(\text{Closed-N})^c$ of $M_1$, (see Theorem 2.4). Thus $N=M_1$. 

Corollary 3.3. Every $(\text{Closed-L})^c$ of closed-CS-module $M$ is closed-CS-module.

Proof. Let $M$ be a closed-CS-module and let $N$ be a subset of $M$. We must prove that $N$ is a closed-CS-module. Let $K$ subset of $(\text{Closed-N})^c$, then by Theorem 2.4, $L$ is a subset of $(\text{Closed-N})^c$ of $M$. But $M$ is a closed-CS-module, therefore $L$ is a direct summand of $M$ and hence $K$ is a direct summand of $A$.

Lemma 3.4. An $R$-module $M$ is closed-CS-module if and only if every $(\text{Closed-N})^c$ of $M$ is essential in a direct summand.

Proof. $\implies$ Clear.

$\leftarrow$ let $N$ subset of $(\text{Closed-N})^c$, we need to show that $N$ is a direct summand of $M$. Since $N$ subset of $(\text{Closed-N})^c$ of $M$, then by our assumption $N$ is a subset of $(\text{Closed-N})^c$ of $M$, where $D$ is a direct summand of $M$. Thus $Z(\frac{N}{M})=(\frac{N}{M})$; $(\frac{N}{M}$ is singular). But $\frac{N}{M}$ subset of $\frac{M}{N}$, therefore $\frac{N}{M}$ is non-singular. Thus $N=D$ and hence $M$ is closed-CS-module.

Theorem 3.5. An $R$-module $M$ is closed-CS-module if and only if for every $(\text{Closed-N})^c$ of $M$, there exists a decomposition $M=M_1 \oplus M_2$ such that $N$ is a subset of $M_1$ and $N \oplus M_2$ is a subset of $(\text{Closed-N})^c$ of $M$. 

Proof. \(\implies\) Clear.

\(\Leftarrow\) Let \(N\) be a subset of \((\text{Closed-}N)^c\) of \(M\), we need to show that \(N\) is a direct summand of \(M\). Since \(N\) is a subset of \((\text{Closed-}N)^c\) of \(M\), then by assumption there exists a decomposition \(M = M_1 \oplus M_2\) such that \(N \subseteq M_1\) and \((N \subseteq M_2)\) is a subset of \((\text{Closed-}N)^c\) of \(M\). So \(\frac{M}{(N \oplus M_2)}\) is a singular. But \(N \oplus M_1\) and \(A\) are subset of \((\text{Closed-}N)^c\) of \(M\), therefore by Theorem 2.4, \(N\) is a subset of \((\text{Closed-}N)^c\) of \(M_1\). Since \(M_2\) is a subset of \((\text{Closed-}N)^c\) of \(M_2\), then by Lemma 2.3, \((N \oplus M_2)\) is a subset of \((\text{Closed-}N)^c\) of \(M_1 \oplus M_2 = M\). So \(\frac{M}{(N \oplus M_2)}\) is non-singular. Thus \(M = N \oplus M_2\).

Proposition 3.6. An \(R\)-module \(M\) is a closed-CS-module if and only if for every direct summand \(A\) of the injective hull \(E(M)\) of \(M\) such that \((A \cap M)^c\) is a subset of \((\text{closed-CS-module})\), then \((A \cap M)\) is a direct summand of \(M\).

Proof. \(\implies\) Clear.

\(\Leftarrow\) Let \(N\) be a subset of \((\text{Closed-}N)^c\) of \(M\) and let \(K\) be a relative complement of \(N\), then \((N \setminus K)\) is a subset of \((\text{Closed-}N)^c\) of \(M\). Since \(M\) is a subset of \((\text{Closed-}N)^c\) of \(E(M)\), then \((N \setminus K)\) is a subset of \((\text{Closed-}N)^c\) of \(E(M)\). Thus \(E(N) \oplus E(K) = E(N \setminus K) = E(M)\). Since \(E(N)\) is a summand of \(E(M)\), then by our assumption \(E(N) \setminus M\) is a summand of \(M\). Now \(N\) is a subset of \((\text{Closed-}N)^c\) of \(E(N)\) and \(M\) is a subset of \((\text{Closed-}N)^c\) of \(M\), thus \(N = (N \setminus M)\) is a subset of \((\text{Closed-}N)^c\) of \(E(M) \setminus M\). Hence by Lemma 3.5, \(M\) is closed-CS-module.

Theorem 3.7. Let \(R\) be a ring, then \(R\) is a closed-CS-module if and only if every cyclic non-singular \(R\)-module is projective.

Proof. Let \(R\) be a closed-CS-ring and \(M = Ra, a \in M\) be a nonsingular \(R\)-module. Let the following be a short exact sequence.

\[
0 \to \text{Annihilator}(a) \to R \to Ra \to 0,
\]

where \(i\) is the inclusion homomorphism and \(f\) is a map defined by \(f(r) = ra, r \in R\). So \(f\) is an epimorphism and \(\ker(f)\) equal Annihilator of \((a)\). Hence from the first isomorphism theorem, Annihilator of \((a)R \cong Ra\). But \(Ra\) is non-singular, therefore Annihilator of \((a)\) subset of \((\text{Closed-}N)^c\) of \(R\). Since \(R\) is closed-CS-ring, then Annihilator of \((a)\) is a direct summand of \(R\), so the sequence is split. Thus \(R\) is equivalent to Annihilator of \((a)\) \(Ra\). Since \(R\) is projective, then \(Ra\) is projective. Conversely, let \(A\) be a \((\text{Closed-}N)^c\) of \(I\), \(I\) an ideal in \(R\), then \(\frac{R}{A}\) is non-singular. Since \(R\) is cyclic, then \(\frac{R}{A}\) is cyclic. By our assumption \(\frac{R}{A}\) is a projective. Now consider the following short exact sequence:

\[
0 \to A \to R \to AR \to 0,
\]

where \(i\) is the inclusion homomorphism and \(\pi\) is the natural epimorphism from \(R\) into \(Ra\). Since \(\frac{R}{A}\) is projective, then the sequence is split. Thus \(A\) is a summand of \(R\). Also a direct sum of closed-CS-module need not to be closed-CS-modules (see [4]).
Proposition 3.8. Let $M$ and $N$ be closed-CS-modules such that Annihilator of $M+\text{Annihilator of } N$ equal $R$. Then $M \oplus N$ is closed-CS-module.

Proof. Let $A$ be a $(\text{Closed-N})_c$ submodule of $M \oplus N$.

Since Annihilator of $M+\text{Annihilator of } N=R$, then by the same way of the prove [9, Proposition 4.2, CH.1], $A=C \oplus D$, where $C$ is a submodule of $M$ and $D$ is a submodule of $N$. Since $A=(C \oplus D)$ is a subset of $(\text{Closed-N})_c$ of $M \oplus N$, then $C$ and $D$ are $(\text{Closed-N})_c$ of $M$ and $N$ respectively by Lemma 2.3. But $M$ and $N$ are closed-CS-modules, therefore $C$ is a summand of $M$ and $D$ is a summand of $N$. So $A=C \oplus D$ is a summand of $M \oplus N$. Thus $M \oplus N$ is a closed-CS-module. Recall that a submodule $N$ of $R$-module $M$ is called a fully invariant submodule of $M$, if for every endomorphism $f: M \rightarrow M$, $f(N)$ subset of $N$, $(N$ is fully invariant) (see [8]).

Corollary 3.9. Let $M=\bigoplus_{i \in I} M_i$ be an $R$-module, such that every $(\text{Closed-N})_c$ of $M$ is fully invariant, then $M$ is closed-CS-module if and only if $M_i$ is closed-CS-module; $i \in I$.

Proof. $\implies$ Clear.

$\Leftarrow$ let $S$ be a $(\text{Closed-N})_c$ of $M$. For each $i \in I$, let $\pi_i: M \rightarrow M_i$ be the projection map. Let $x \in S$, then $x=\sum m_i$, $m_i \in M_i$ and $m_i=0$ for all but finite many element of $i \in I$, $\pi_i(x)=m_i$. Since we have $(\text{Closed-S})_c$, then by our assumption, $S$ is fully invariant and hence $\pi_i(x)=m_i \cap M_i$. So $x \in \bigotimes_i (S \cap M_i)$. Thus $S$ subset of $\bigoplus (S \cap M_i)$. But $\bigoplus (S \cap M_i)$ subset of $S$, therefore $S=\bigoplus (S \cap M_i)$. Since $S$ is a subset of $(\text{Closed-M})_c$, then by Theorem 2.4, $(S \cap M_i)$ is a subset of $(\text{Closed-N})_c$ of $M_i \forall i \in I$. But $M_i$ closed-CS-modules for all $i \in I$, therefore $(S \cap M_i)$ is a direct summand of $M_i$. Thus $S$ is a direct summand on $M$.

An $R$-module $M$ is called a distributive module if $A \cap (B+C)=(A \cap B)+(A \cap C)$, for all submodules $A$, $B$ and $C$ of $M$, (see [9]).

Corollary 3.10. Let $M=M_1 \oplus M_2$ be distributive $R$-module. Then $M$ is closed-CS-module if and only if $M_1$ and $M_2$ are closed-CS-module.

Proof. $\implies$ Clear.

$\iff$ Let $K$ be a subset of $(\text{closed-N})_c$ in $M$. Since $M=M_1 \oplus M_2$, then $K=K \cap (M_1 \oplus M_2)$. But $M$ is a distributive, therefore $K=(K \cap M_1) \oplus (K \cap M_2)$. By Lemma 2.3, $(K \cap M_1)$ is a subset of $(\text{Closed-N})_c$ of $M_1$ and $(K \cap M_2)$ is a subset of $(\text{Closed-N})_c$. Since $M_1$ and $M_2$ are closed-CS-modules, then $(K \cap M_1)$ is a direct summand of $M_1$ and $(K \cap M_2)$ is a direct summand of $M_2$. Clearly that $K=(K \cap M_1) \oplus (K \cap M_2)$ is a direct summand of $M$.

Corollary 3.11. Let $M$ be an $R$-module and let $N$ be a subset of $(\text{closed-CS}-M$. Then $[N:M]$ is a subset of $(\text{closed-CS}-R)$.
Proof. Let $N$ be a subset of closed-CS-module. Assume that $[N:M]$ is not closed-CS-module in $R$. So there exists $r \in R$ such that $[N:M] \neq r+[N:M] \in Z(N \uparrow M)$. Thus $rM$ not subset of $N$ and hence there exists $m_0 \in M$ such that $rm_0 \notin N$. One can easily show that Annihilator of $(r+[N:M])$ is a subset of Annihilator of $(rm_0+N)$. Since Annihilator of $(r+[N:M])$ is a subset of $eR$, then Annihilator of $(rm_0+N)$ is a subset of $eR$. But $\frac{M}{N}$ is non-singular, therefore $rm_0+N=N$ which is contradiction. 

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References


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