On almost generalized pseudo-Ricci symmetric spacetime

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Abstract. The notion of an almost generalized pseudo-Ricci symmetric spacetime has been introduced and studied. The beauty of such spacetime is that it has the flavour of Ricci symmetric, Ricci recurrent, generalized Ricci recurrent, pseudo-Ricci symmetric, generalized pseudo-Ricci symmetric and almost pseudo-Ricci symmetric space. Having found, faulty example in [8] the present paper attempts to construct a nontrivial example of an almost pseudo Ricci symmetric spacetime.

Keywords: almost pseudo Ricci symmetric spacetime, quasi-Einstein.

1. Introduction

In the example given in ([8], page 2884-2885) authors have calculated or assumed the value of the covariant derivatives corresponding to the vanishing component of the Ricci tensor \( R_{13} \& R_{14} \) (namely, \( R_{13,3} \& R_{14,4} \)) to be zero. But, those value are found to be \( R_{13,3} = \frac{2q^2(1-q)}{(1+2q)^3} = -R_{14,4} \) which are non-zero as \( q \neq 0,1 \). Consequently for their [8] choice of the 1-forms

\[
A_i(x) = \begin{cases} \frac{q}{1+2q} & \text{for } i=1, \\ 0 & \text{otherwise,} \end{cases}
\]

\[
B_i(x) = \begin{cases} \frac{1+q}{1+2q} & \text{for } i=1, \\ 0 & \text{otherwise,} \end{cases}
\]

the relations

\[
R_{13,3} = (A_3 + B_3)R_{13} + A_1R_{33} + A_3R_{13},
\]

\[
R_{14,4} = (A_4 + B_4)R_{14} + A_1R_{44} + A_4R_{14},
\]

do not stand. Hence, \((\mathbb{R}^4, g)\) under-considered metric ([8], equation 6.2, page 2884) can not be an almost pseudo-Ricci symmetric spacetime. Coming back to our present paper, we structured it as follows: Keeping in tune with Dubey[11], a new type of spacetime called an almost generalized pseudo-Ricci symmetric

* The author dedicates this work to the memory of Late Professor M. C. Chaki
spacetime which is abbreviated by $A(GPRS)_n$-spacetime is introduced in section 2. Some interesting results of a conformally flat almost generalized pseudo-Ricci symmetric spacetime are obtained. A non-trivial example of an almost pseudo-Ricci symmetric spacetime is constructed in section 3. Finally, we ensured that there exists a spacetime $(\mathbb{R}^4, g)$ which is an almost generalized pseudo-Ricci symmetric for some choice of the 1-forms.

2. $A(GPRS)_n$-spacetime

In the sense of Chaki and Kawaguchi, a non-flat $n$-dimensional semi-Riemann manifold $(M^n, g)(n > 3)$ is said to be an almost pseudo-Ricci symmetric manifold, [7] if its Ricci tensor $S$ of type $(0, 2)$ is not identically zero and satisfies the equation

$$\nabla_X S(Y, U) = [A(X) + B(X)] S(Y, U) + A(Y) S(X, U) + A(U) S(Y, X)$$

where $A(X)$ and $B(X)$ are two non-zero 1-forms defined by $A(X) = g(X, \theta)$ and $B(X) = g(X, \varrho)$, $\nabla$ being the operator of the covariant differentiation. The local expression of the above equation is

$$R_{ik;tl} = (A_l + B_l) R_{ik} + A_i R_{kl} + A_k R_{il},$$

where $A_l$ and $B_l$ are two non-zero co-vectors and comma followed by indices denotes the covariant differentiation with respect to the metric tensor $g$. An $n$-dimensional manifold of this kind is abbreviated by $APRS_n$.

Generalizing the sense of Chaki and Kawaguchi, in the present paper, we attempt to introduce a new type of spacetime called almost generalized pseudo-Ricci symmetric spacetime which is abbreviated by $A(GPRS)_n$-spacetime and defined as follows:

A non-flat $n$-dimensional semi-Riemann manifold $(M^n, g)(n > 3)$, is termed as almost generalized pseudo-Ricci symmetric manifold, if its Ricci tensor $S$ of type $(0, 2)$ is not identically zero and admits the identity([2], [3])

$$\nabla_X S(Y, U) = [A(X) + B(X)] S(Y, U) + A(Y) S(X, U) + A(U) S(Y, X)$$

$$+ [C(X) + D(X)] g(Y, U) + C(Y) g(X, U) + C(U) g(X, Y)$$

where $A(X)$, $B(X)$, $C(X)$ and $D(X)$ are non-zero 1-forms defined by $A(X) = g(X, \theta)$, $B(X) = g(X, \varrho)$, $C(X) = g(X, \pi)$ and $D(X) = g(X, \delta)$. The beauty of such $A(GPRS)_n$-spacetime is that it has the flavour of

(a) Ricci symmetric space in the sense of Cartan (for $A = B = C = D = 0$),
(b) Ricci recurrent space by E. M. Patterson [14] (for $B \neq 0$ and $A = C = D = 0$),
(c) generalized Ricci recurrent space by De, Guha and Kamilya [9] (for $B \neq 0$, $D \neq 0$ and $A = C = 0$),
(d) pseudo-Ricci symmetric space by Chaki [6] (for $A = B \neq 0$ and $C = D = 0$),
(e) generalized pseudo-Ricci symmetric space, by Baishya [1] (for $A = B \neq 0$ and $C = D \neq 0$) and
(f) almost pseudo-Ricci symmetric space by Chaki and Kawaguchi [7] (for $A = B \neq 0$ and $C = D = 0$).

Next, if the vector fields associated to the 1-forms $A$ & $B$ are co-directional with that of $C$ & $D$ respectively, that is $C = \phi A$ & $D = \phi B$ where $\phi$ being constant, then the relation (2.3) turns into

$$ (\nabla_X Z)(Y, U) = [A(X) + B(X)]Z(Y, U) + A(Y) Z(X, U) + A(U)Z(X, U) $$

where $Z(X, Y) = S(X, Y) + \phi \, g(X, Y)$ is a well known Z-tensor introduced in ([12], [13]). This leads to the following:

**Theorem 2.1 ([13]).** Every $A(GPRS)_n$-spacetime is an almost pseudo Z-symmetric spacetime provided that the vector fields associated to the 1-forms $A$ & $B$ are co-directional with that of $C$ & $D$ respectively.

It is to be noted that the converse of the Theorem 2.1 is also true. Thus we can say that an almost pseudo Z-symmetric spacetime is a natural example of an almost generalized pseudo Z-symmetric spacetime.

**Definition 2.1.** A non-flat Riemannian manifold $(M^n, g)(n > 3)$ is said to be a quasi-Einstein manifold [10] if its Ricci tensor $S$ of type $(0, 2)$ is not identically zero and satisfies the condition

$$ S(X, Y) = \lambda g(X, Y) + \mu \psi(X) \psi(Y), $$

where $\lambda, \mu \in \mathbb{R}$ and $\psi$ is a non-zero 1-form such that $g(X, U) = \psi(X)$, for all vector fields $X, U$ being a unit vector field of the 1-form.

Now, contracting $Y$ over $U$ in (2.1) we obtain

$$ (\nabla_X S)(Y, U) - (\nabla_U S)(X, Y) = B(X)S(Y, U) - B(U)S(X, Y) $$

$$ + D(X)g(Y, U) - D(U)g(X, Y) $$

after further contraction which leaves

$$ dr(X) = 2rB(X) - 2B(X) + 6D(X), $$

where $B(X) = S(X, g)$. It is known ([15], p, 41) that a conformally flat $(M^4, g)$ spacetime possesses the relation

$$ (\nabla_X S)(Y, U) - (\nabla_U S)(X, Y) = \frac{1}{6}[g(Y, U)dr(X) - g(X, Y)dr(U)]. $$
By virtue of (2.5), (2.6) and (2.7) we find

\[3[B(X)S(Y, U) - B(U)S(X, Y)]\]

(2.8) \[= [rB(X) - \tilde{B}(X)]g(Y, U) - [rB(U) - \tilde{B}(U)]g(X, Y)\]

which yields

(2.9) \[B(X)\tilde{B}(U) = B(U)\tilde{B}(X),\]

for \(Y = \varrho\). Assuming the Ricci tensor of the spacetime as codazzi type (in the sense of [4]) and then making use of (2.6), we obtain from (2.9) that

(2.10) \[B(X)D(U) = B(U)D(X) \forall X \text{ and } U.\]

This motivate us to state

**Proposition 2.1.** In a conformally flat \(A(GPRS)_{4}\)-spacetime with codazzi type of Ricci tensor, the 1-forms \(B\) and \(D\) are co-directional.

Again, for constant scalar curvature tensor (or codazzi type of Ricci tensor) by virtue of (2.6), (2.8) and (2.10), we can easily find out

(2.11) \[S(Y, U) = -\frac{D(\varrho)}{B(\varrho)}g(Y, U) + \frac{1}{B(\varrho)}[rB(Y) + nD(Y)]B(U),\]

where \(\frac{D(\varrho)}{B(\varrho)} = k \forall U\). If the 1-forms \(B\) and \(D\) are co-directional, then (2.11) takes the following form

(2.12) \[S(Y, U) = \alpha g(Y, U) + \beta B(Y)B(U).\]

This leads to the followings:

**Theorem 2.2.** A conformally flat \(A(GPRS)_{4}\)-spacetime with codazzi type of Ricci tensor, is a quasi-Einstein spacetime.

But, it is proved in ([8], Theorem 3.1) that a conformally flat \(A(GPRS)_{4}\)-spacetime is always quasi-Einstein spacetime. In consequence of Corollary 3.1 in [8], we can state the following:

**Corollary 2.1.** A conformally flat almost generalized pseudo-Ricci symmetric spacetime with constant scalar curvature can be considered as a model of the perfect fluid spacetime in general relativity.

**Corollary 2.2.** A conformally flat almost generalized pseudo-Ricci symmetric spacetime with constant scalar curvature is a space of quasi constant curvature.
3. Existence of almost pseudo-Ricci symmetric spacetime

**Example 3.1.** Let \((\mathbb{R}^4, g)\) be a 4-dimensional Lorentzian space endowed with the Lorentzian metric \(g\) given by

\[
ds^2 = g_{ij} dx^i dx^j = e^{-x^1} [(dx^1)^2 - (dx^2)^2 + 2 dx^3 dx^4],
\]

\((i, j = 1, 2, 3, 4)\).

The non-zero components of Riemannian curvature tensors, Ricci tensors (up to symmetry and skew-symmetry) and scalar curvature tensor are

\[
R_{2324} = \frac{1}{4} e^{-x^1} = R_{3434},
\]

\[
R_{22} = \frac{1}{2} = -R_{34},
\]

\[
r = -\frac{3}{2} e^{x^1}.
\]

Covariant derivatives of Ricci tensors (up to symmetry) is expressed as

\[
R_{12,2} = -R_{13,4} = -R_{14,3} = \frac{1}{4}
\]

\[
R_{22,1} = -R_{34,1} = \frac{1}{2}.
\]

For the following choice of the 1-forms

\[
A_i = \begin{cases} 
\frac{1}{2}, & \text{for } i = 1 \\
0, & \text{otherwise}
\end{cases}
\]

\[
B_i = \begin{cases} 
\frac{1}{2}, & \text{for } i = 1 \\
0, & \text{otherwise},
\end{cases}
\]

one can easily verify the followings

\[
R_{12,k} = (A_k + B_k) R_{12} + A_1 R_{k2} + A_2 R_{1k},
\]

\[
R_{13,k} = (A_k + B_k) R_{13} + A_1 R_{k3} + A_3 R_{1k},
\]

\[
R_{14,k} = (A_k + B_k) R_{14} + A_1 R_{k4} + A_4 R_{1k},
\]

\[
R_{23,k} = (A_k + B_k) R_{23} + A_2 R_{k3} + A_3 R_{2k},
\]

\[
R_{24,k} = (A_k + B_k) R_{24} + A_2 R_{k4} + A_4 R_{2k},
\]

\[
R_{34,k} = (A_k + B_k) R_{34} + A_3 R_{k4} + A_4 R_{3k},
\]

\[
R_{11,k} = (A_k + B_k) R_{11} + A_1 R_{k1} + A_1 R_{1k},
\]

\[
R_{22,k} = (A_k + B_k) R_{22} + A_2 R_{k2} + A_2 R_{2k},
\]

\[
R_{33,k} = (A_k + B_k) R_{33} + A_3 R_{k3} + A_3 R_{3k},
\]

\[
R_{44,k} = (A_k + B_k) R_{44} + A_4 R_{k4} + A_4 R_{4k},
\]

where \(k = 1, 2, 3, 4\).
In consequence of the above, one can say that

**Theorem 3.1.** There exists a spacetime \((\mathbb{R}^4, g)\) which is an almost pseudo-Ricci symmetric spacetime with the above mentioned choice of the 1-forms.

It is obvious that the spacetime bearing the metric given by (3.1) can not be Ricci symmetric, Ricci recurrent, generalized Ricci recurrent as well as almost generalized pseudo-Ricci symmetric spacetime.

4. Existence of \(A(GPRS)_n\)- spacetime

**Example 4.1.** Let \((\mathbb{R}^4, g)\) be a 4-dimensional Lorentzian space endowed with the Lorentzian metric \(g\) given by

\[(4.1) \quad ds^2 = g_{ij}dx^idx^j = (x^4)^{4/3}[(dx^1)^2 + (dx^2)^2 + (dx^3)^2] - (dx^4)^2,\]

\((i, j = 1, 2, 3, 4)\). The non-zero components of Ricci tensors (up to symmetry)

\[
R_{11} = \frac{2}{3(x^4)^{2/3}} = R_{22} = R_{33}, \quad R_{44} = \frac{2}{3(x^4)^{2/3}},
\]

Covariant derivative (up to symmetry) \(R_{ik;j}\) of Ricci tensors is expressed by

\[
R_{11,4} = -\frac{4}{3(x^4)^{5/3}} = R_{22,4} = R_{33,4}, \quad R_{44,4} = -\frac{4}{3(x^4)^{3}}
\]

\[
R_{14,1} = -\frac{8}{9(x^4)^{5/3}} = R_{24,2} = R_{34,3}.
\]

For following choice of the 1-forms

\[
A_i = \begin{cases} \frac{1}{x^4}, & \text{for } i = 4, \\ 0, & \text{otherwise} \end{cases}
\]

\[
B_i = \begin{cases} -\frac{19}{3x^4}, & \text{for } i = 4, \\ 0, & \text{otherwise} \end{cases}
\]

\[
C_i = \begin{cases} -\frac{14}{9(x^4)^3}, & \text{for } i = 4, \\ 0, & \text{otherwise} \end{cases}
\]

\[
D_i = \begin{cases} \frac{34}{9(x^4)^3}, & \text{for } i = 4, \\ 0, & \text{otherwise} \end{cases}
\]
one can easily verify the followings

\[
\begin{align*}
R_{12,k} &= (A_k + B_k) R_{12} + A_1 R_{k2} + A_2 R_{1k} + (C_k + D_k) g_{12} + C_{1g2} + C_{2g1}, \\
R_{13,k} &= (A_k + B_k) R_{13} + A_1 R_{k3} + A_3 R_{1k} + (C_k + D_k) g_{13} + C_{1g3} + C_{3g1}, \\
R_{14,k} &= (A_k + B_k) R_{14} + A_1 R_{k4} + A_4 R_{1k} + (C_k + D_k) g_{14} + C_{1g4} + C_{4g1}, \\
R_{23,k} &= (A_k + B_k) R_{23} + A_2 R_{k3} + A_3 R_{2k} + (C_k + D_k) g_{23} + C_{2g3} + C_{3g2}, \\
R_{24,k} &= (A_k + B_k) R_{24} + A_2 R_{k4} + A_4 R_{2k} + (C_k + D_k) g_{24} + C_{2g4} + C_{4g2}, \\
R_{34,k} &= (A_k + B_k) R_{34} + A_3 R_{k4} + A_4 R_{3k} + (C_k + D_k) g_{34} + C_{3g4} + C_{4g3}, \\
R_{11,k} &= (A_k + B_k) R_{11} + A_1 R_{k1} + A_1 R_{1k} + (C_k + D_k) g_{11} + C_{1g1} + C_{1gk}, \\
R_{22,k} &= (A_k + B_k) R_{22} + A_2 R_{k2} + A_2 R_{2k} + (C_k + D_k) g_{22} + C_{2g2} + C_{2gk}, \\
R_{33,k} &= (A_k + B_k) R_{33} + A_3 R_{k3} + A_3 R_{3k} + (C_k + D_k) g_{33} + C_{3g3} + C_{3gk}, \\
R_{44,k} &= (A_k + B_k) R_{44} + A_4 R_{k4} + A_4 R_{4k} + (C_k + D_k) g_{44} + C_{4g4} + C_{4gk},
\end{align*}
\]

where \( k = 1, 2, 3, 4 \).

In consequence of the above, one can say that

**Theorem 4.1.** There exists a spacetime \((\mathbb{R}^4, g)\) which is an almost generalized pseudo-Ricci symmetric for the above mentioned choice of the 1-forms.

It is obvious that the spacetime bearing the metric given by (4.1) cannot be Ricci symmetric, Ricci recurrent, generalized Ricci recurrent as well as pseudo-Ricci symmetric.

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**References**


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