Characterization of some linear groups by their conjugacy class sizes

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Abstract. Let G be a group and denote by $N(G)$ the set of conjugacy class sizes of G. In this paper, we proved that if $Z(G) = 1$ and $N(G) = N(PGL(3,\ q))$, where $q \in \{2, 3, 4, 5, 7, 8, 9, 11\}$, then $G \cong PGL(3,\ q)$.

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1. Introduction

All groups considered in this paper are finite and simple groups are nonabelian. Let G be a group. For $x \in G$ we denote by x^G the conjugacy class of x, and by $|x^G|$ the size of x^G. Then set $N(G) = \{|x^G| \mid x \in G\}$. It is a well-established topic to investigate the relationship between the arithmetical properties of $N(G)$ and the structural properties of group G. More recently, there have appeared a number of papers addressing this research field. This paper is also a contribution along this line, which is related to an open conjecture of John G. Thompson (ref. to [15, Problem 12.38]):

Thompson's conjecture. *If S is a simple group and G is a group satisfying that $Z(G) = 1$ and $N(G) = N(S)$, then $G \cong S$.*

The *prime graph* of a group G is a simple graph whose vertices are the prime divisors of $|G|$ and where two distinct primes p and q are joined by an edge if and only if G contains an element of order pq. Using the prime graph of simple group, the second author proved that Thompson's conjecture holds for all simple groups with disconnected prime graph in 1994 (see [1], also ref. to [2, 3, 4]). For the simple groups with connected prime graph, the conjecture has made considerable progress in recent years. Several mathematicians had proved the conjecture is true for the following simple groups: A_{10}, A_{16}, A_{22}, $U_4(4)$, $U_4(5)$, $A_n(q)$, $B_n(q)$, $C_n(q)$, $D_n(q)$, ${}^2D_n(q)$, and $E_7(q)$(see [5, 6, 7, 8, 9, 10, 11, 12, 13, 14]).

A group M is said to be an *almost simple* related to S if and only if $S \leq M \leq$Aut(S) for some simple group S. Naturally, one can put forward the following question: what are almost simple groups we can generalize Thompson's conjecture to? Some authors have generalized the conjecture to almost sporadic simple groups except Aut(J_2) and Aut(McL), symmetric groups S_n, where $n = p, p{+}1$, and p is an odd prime number, projective general linear groups $PGL(2, q)$, the automorphism groups of Suzuki-Ree groups (see[15, 16, 17, 18]). But they still used the second author's method, which is only valid for the groups with the disconnected prime graph.

In this paper, using Vasil'ev and Gorshkov's methods, we generalized Thompson's conjecture to projective general linear groups $PGL(3,\ q)$, where $q \in \{2, 3, 4, 5, 7, 8, 9, 11\}$. Note that $PGL(3,\ 4)$ and $PGL(3,\ 7)$ have the connected prime graphs.

Our main result is the following theorem:

Main Theorem. *Let G be a group with $Z(G) = 1$ and M one of groups $PGL(3,\ q)$, where $q \in \{2, 3, 4, 5, 7, 8, 9, 11\}$. If $N(G) = N(M)$, then $G \cong M$.*

By [19], we get $PGL(3,\ q) = L_3(q).d,\ d = gcd(3,\ q-1)$. Hence

$$M = PGL(3,\ q) = \begin{cases} L_3(q), & q = 2,\ 3,\ 5,\ 8,\ 9,\ 11, \\ L_3(q).3, & q = 4,\ 7. \end{cases}$$

Since simple groups $L_3(2)$, $L_3(3)$, $L_3(5)$, $L_3(8)$, $L_3(9)$, $L_3(11)$ have disconnected prime graphs, the second author in [1] has proved that Thompson's conjecture is right for these groups. Therefore, it is enough to prove Main Theorem for $PGL(3,\ 4)$ and $PGL(3,\ 7)$. We shall give the proofs on $PGL(3,\ 4)$ in Section 3 and $PGL(3,\ 7)$ in Section 4.

For convenience, we denote by $\pi(n)$ the set of all primes dividing n where n is a positive integer, and then n_π to denote π-part of n for $\pi \subseteq \pi(n)$. In addition, for a group G, we also denote by $\pi(G) = \pi(|G|)$, and $Soc(G)$ the socle of G which is a subgroup generated by all minimal normal subgroups of G. The other notation and terminologies in this paper are standard and the reader is referred to [19] and [21] if necessary.

2. Preliminaries

First, we cite here some known results which are useful in the sequel.

Lemma 2.1. *Let K be a normal subgroup of G and $\overline{G} = G/K$. Then*

(a) *If $\overline{x}$ is the image of an element x of G in the group $\overline{G}$, then $|x^K|\,\big|\,|x^G|$ and $|\overline{x}^{\overline{G}}|\,\big|\,|x^G|$.*

(b) *If $x \in G$ and $(|x|, |K|) = 1$, then $C_{\overline{G}}(\overline{x}) = C_G(x)K/K$.*

(c) *If $x,\ y \in G$, $(|x|, |y|) = 1$, and $xy = yx$, then $C_G(xy) = C_G(x) \cap C_G(y)$.*

Lemma 2.2 ([5, Lemma 4])**.** *Let G be a group with trivial center, $p \in \pi(G)$ and p^2 not divide n for any $n \in N(G)$. Then a Sylow p-subgroup of G is elementary abelian.*

Lemma 2.3 ([9, Lemma 1.10])**.** *Let a Sylow p-subgroup of G be of order p, x be an element of order p, and $|x^G|$ be a number that is maximal with respect to divisibility in $N(G)$. Then $C_G(x)$ is abelian.*

Lemma 2.4 ([9, Lemma 1.9])**.** *Let G be a group, and p and q be two numbers in $\pi(G)$. If G satisfies the following conditions:*

(a)*$N(G)$ contains no number divisible by p^2 or q^2;*

(b)*$N(G)$ contains no number except 1 co-prime to pq;*

(c)*$N(G)$ contains a number h_q such that any n in $N(G)$ not divisible by q does not divide h_q and $N(G)$ contains no number divisible by h_q and n;*

(d)*$N(G)$ contains a number h_p such that any l in $N(G)$ not divisible by p does not divide h_p and $N(G)$ contains no number divisible by h_p and l.*

Then Sylow p-subgroups and q-subgroups of G are cyclic groups of prime order. In addition, G has no element of order pq.

Lemma 2.5 ([9, Lemma 1.12])**.** *Let G be a group, K the soluble radical of G, and $G/K = S$ a simple group. Suppose that there exists a prime p such that $p \in \pi(G) \setminus \pi(K)$. Assume that an element g of order p of G satisfies the following conditions:*

(a) *$|g^G| = |\overline{g}^S|$, where $\overline{g}$ is the image of an element g in the group S ;*

(b) *the number $|g^G|$ is maximal with respect to divisibility in $N(G)$;*

(c) *the subgroup* $C_G(g)$ *is abelian.*
Then $K \le Z(G)$.

Let M be one of $PGL(3,\ 4)$ and $PGL(3,\ 7)$. Information on the set $N(M)$ and the order of M given in the next two lemmas is obtained via [19] or GAP [22].

Lemma 2.6. *Let* $M \cong PGL(3,\ 4)$. *Then:*
(1) $|M| = 2^6 \cdot 3^3 \cdot 5 \cdot 7$;
(2) $N(M) = \{n_1 = 1,\ n_2 = 3^2 \cdot 5 \cdot 7,\ n_3 = 2^4 \cdot 3 \cdot 7,\ n_4 = 2^6 \cdot 3 \cdot 5,\ n_5 = 2^6 \cdot 5 \cdot 7,\ n_6 = 2^6 \cdot 3^2 \cdot 5,$
$n_7 = 2^2 \cdot 3^3 \cdot 5 \cdot 7,\ n_8 = 2^6 \cdot 3^2 \cdot 7,\ n_9 = 2^4 \cdot 3^2 \cdot 5 \cdot 7\}$.
Especially,
(3) $N(M)$ *contains no number other than* n_1, n_4 *and* n_6 *not divisible by 7;*
(4) $N(M)$ *contains no number other than* n_1, n_3 *and* n_8 *not divisible by 5;*
(5) *For any* $n \in N(M)$ *and* $p \in \{5,\ 7\}$, *it follows that* $p^2 \nmid n$;
(6) $|x^M| = n_1,\ x \in M$ *if and only if* $x = 1$.

Lemma 2.7. *Let* $M \cong PGL(3,\ 7)$. *Then:*
(1) $|M| = 2^5 \cdot 3^3 \cdot 7^3 \cdot 19$;
(2) $N(M) = \{n_1 = 1,\ n_2 = 2^4 \cdot 3^2 \cdot 19,\ n_3 = 3 \cdot 7^2 \cdot 19,\ n_4 = 2^5 \cdot 3 \cdot 7^3,\ n_5 = 2^3 \cdot 7^3 \cdot 19,$
$n_6 = 2^5 \cdot 3^2 \cdot 7^3,\ n_7 = 2^5 \cdot 3^3 \cdot 7 \cdot 19,\ n_8 = 2 \cdot 3^2 \cdot 7^3 \cdot 19,\ n_9 = 2^4 \cdot 3^2 \cdot 7^2 \cdot 19,\ n_{10} = 2^3 \cdot 3 \cdot 7^3 \cdot 19\}$.
In particular,
(3) $N(M)$ *contains no number other than* n_1, n_4 *and* n_6 *not divisible by 19;*
(4) $N(M)$ *contains no number divided by* 19^2;
(5) $|x^M| = n_1,\ x \in M$ *if and only if* $x = 1$.

Lemma 2.8. *If* M *is one of* $PGL(3,\ 4)$ *and* $PGL(3,\ 7)$, *and* G *is a group with* $Z(G) = 1$ *and* $N(G) = N(M)$, *then* $|M|\big||G|$ *and* $\pi(G) = \pi(M)$.

Proof. Since the number in $N(G)$ divides $|G|$, under the hypothesis we see that $|M|\big||G|$ by Lemma 2.6 and Lemma 2.7. $\pi(M) = \pi(G)$ is the result of Lemma 1.2.1 in [1] or Lemma 3 in [5].

Lemma 2.9. *Let* S *be a simple group.*
(*i*) *If* $\pi(S) \subseteq \{2,\ 3,\ 5,\ 7\}$, *then* S *is isomorphic to one of simple groups of Table 1.*
(*ii*) *If* $\pi(S) \subseteq \{2,\ 3,\ 7,\ 19\}$, *then* S *is isomorphic to one of simple groups of Table 2.*

Proof. This is an immediate consequence of Theorem 2 in [23].

For convenience, we list all the cases of S in Lemma 2.9 as well as the orders of S, the orders of the outer automorphism of S in Table 1 and Table 2.

Table 1. Non-abeian simple groups S with $\pi(S) \subseteq \{2,\ 3,\ 5,\ 7\}$

S	Order of S	$\lvert\mathrm{Out}(S)\rvert$	S	Order of S	$\lvert\mathrm{Out}(S)\rvert$
A_5	$2^2 \cdot 3 \cdot 5$	2	A_9	$2^6 \cdot 3^4 \cdot 5 \cdot 7$	2
$L_2(7)$	$2^3 \cdot 3 \cdot 7$	2	J_2	$2^7 \cdot 3^3 \cdot 5^2 \cdot 7$	2
A_6	$2^3 \cdot 3^2 \cdot 5$	2^2	$U_3(5)$	$2^4 \cdot 3^2 \cdot 5^3 \cdot 7$	$\lvert S_3\rvert$
$L_2(8)$	$2^3 \cdot 3^2 \cdot 7$	3	$S_6(2)$	$2^9 \cdot 3^4 \cdot 5 \cdot 7$	1
A_7	$2^3 \cdot 3^2 \cdot 5 \cdot 7$	2	$U_4(3)$	$2^7 \cdot 3^6 \cdot 5 \cdot 7$	$\lvert D_8\rvert$
$U_3(3)$	$2^5 \cdot 3^3 \cdot 7$	2	$S_4(7)$	$2^8 \cdot 3^2 \cdot 5^2 \cdot 7^4$	2
A_8	$2^6 \cdot 3^2 \cdot 5 \cdot 7$	2	A_{10}	$2^7 \cdot 3^4 \cdot 5^2 \cdot 7$	2
$L_3(4)$	$2^6 \cdot 3^2 \cdot 5 \cdot 7$	$\lvert D_{12}\rvert$	$O_8^+(2)$	$2^{12} \cdot 3^5 \cdot 5^2 \cdot 7$	$\lvert S_3\rvert$
$U_4(2)$	$2^6 \cdot 3^4 \cdot 5$	2	$L_2(49)$	$2^4 \cdot 3 \cdot 5^2 \cdot 7^2$	2^2

Table 2. Non-abeian simple groups S with $\pi(S) \subseteq \{2,\ 3,\ 7,\ 19\}$

S	Order of S	$\lvert\mathrm{Out}(S)\rvert$	S	Order of S	$\lvert\mathrm{Out}(S)\rvert$
$L_2(7)$	$2^3 \cdot 3 \cdot 7$	2	$L_2(8)$	$2^3 \cdot 3^2 \cdot 7$	3
$U_3(3)$	$2^5 \cdot 3^3 \cdot 7$	2	$L_3(7)$	$2^5 \cdot 3^2 \cdot 7^3 \cdot 19$	$\lvert S_3\rvert$
$U_3(8)$	$2^7 \cdot 3^4 \cdot 7 \cdot 19$	$\lvert 3 \times S_3\rvert$			

3. Proof of the main theorem for $PGL(3,\ 4)$

Theorem 3.1. *Let G be a group with trivial center. If $N(G) = N(PGL(3,\ 4))$, then $G \cong PGL(3,\ 4)$.*

Proof. We divide the proof of this theorem into six steps.

Step 1. Sylow 5-subgroups and Sylow 7-subgroups of G are cyclic groups of prime order and there are no elements of order 35 in G.

In view of $N(G) = N(PGL(3,\ 4))$ and Lemma 2.8, we can choose $p = 5$ and $q = 7$, and take $h_5 = n_6$ and $h_7 = n_8$ such that G satisfies the hypotheses of Lemma 2.4. Hence Sylow 5-subgroups and Sylow 7-subgroups of G are cyclic groups of prime order and there are no elements of order $5 \cdot 7$ in G.

Step 2. Let $g, h \in G$ be elements of orders 5 and 7, respectively. Then $|g^G| = n_8 = 2^6 \cdot 3^2 \cdot 7$ and $|h^G| = n_6 = 2^6 \cdot 3^2 \cdot 5$, and $C_G(g)$ and $C_G(h)$ are abelian.

Since the Sylow 5-subgroup of G is order of 5 by Step 1, one has that $5 \nmid |x^G|$ for any $1 \neq x \in C_G(g)$. Hence $|x^G| = n_3$ or n_8 by (4) and (6) of Lemma 2.6. Assume that $|g^G| = n_3 = 2^4 \cdot 3 \cdot 7$. Let H be a Sylow 3-subgroup of $C_G(g)$. Then H is a nontrivial group of order $|G|_3/3$ by Lemma 2.8. It follows that $Z(H) \neq 1$ and let $1 \neq y \in Z(H)$. Then $H \leq C_G(y)$, and so $|y^G|_3 \leq 3$. Thus $|y^G| = n_3$. Since $H \leq C_G(gy)$, we have that $|(gy)^G| = n_3$. In view of $C_G(gy) = C_G(g) \bigcap C_G(y)$, we see that $C_G(g) = C_G(y)$. The group $C_G(y)$

contains an element w from the center of a Sylow 3-subgroup of G, then $|w^G|$ is not divisible by 3, and so $|w^G| = n_5 = 2^6 \cdot 5 \cdot 7$ by (2) of Lemma 2.6. Thus $w \notin C_G(g)$, a contradiction. It follows that $|g^G| = n_8 = 2^6 \cdot 3^2 \cdot 7$. Since n_8 is maximal with respect to divisibility in $N(G)$, Lemma 2.3 implies that the group $C_G(g)$ is abelian.

In a similar way, we can show that $|h^G| = n_6 = 2^6 \cdot 3^2 \cdot 5$ and $C_G(h)$ is abelian.

In the following discussion, we assume that K is the soluble radical of a group G, and $\overline{G} = G/K$.

Step 3. G is non-soluble and has a unique composition factor S such that $5 \cdot 7\big||S|$ and $S \trianglelefteq \overline{G} \leq \mathrm{Aut}(S)$. Moreover, S may be isomorphic to one of the following groups:

$$A_7,\ A_8,\ A_9,\ L_4(3),\ S_6(2),\ U_4(3).$$

Assume that $5\big||K|$. Then $K/O_{\{5,7\}'}(K)$ has a normal subgroup T of order 5. Hence an element of $G/O_{\{5,7\}'}(K)$ of order 7 can act trivially on T, which implies that $G/O_{\{5,7\}'}(K)$ contains an element of order 35, so does G, contradicting with Step 1. Thus 5 does not divide $|K|$ and similarly we can prove that 7 does not divide $|K|$, and so G is not soluble.

Let $L = S_1 \times S_2 \times \cdots \times S_k$ be the socle of $\overline{G}$, where $S_1, S_2, \ldots,$ and S_k are simple groups. Let g be an element of order 5 of G and suppose that $5 \notin \pi(L)$. Then $\overline{g}$ is of order 5 in $\overline{G}$ and induces a nontrivial outer automorphism of the group L. Suppose that there exists i such that $S_i^{\overline{g}} \neq S_i$. Without loss of generality, we assume that $i = 1$. Let $H = \langle s \mid s = s_1 s_1^{\overline{g}} s_1^{\overline{g}^2} s_1^{\overline{g}^3} s_1^{\overline{g}^4},\ s_1 \in S_1\rangle$. Then H lies in the centralizer of the element $\overline{g}$ and is isomorphic to S_1, but the centralizer of g is abelian by Step 2, a contradiction. Hence $\overline{g}$ induces a nontrivial outer automorphism of the group S_i such that $5\big||\mathrm{Out}(S_i)|$. In view of $\pi(S_i) \subseteq \pi(G) = \{2, 3, 5, 7\}$ and by Table 1, the prime divisors of $|\mathrm{Out}(S_i)|$ are less than 5, a contradiction. Therefore $5\big||L|$ and similarly we can get $7\big||L|$.

If $k > 1$ and $\overline{g} \in S_i$, then $S_j < C_{\overline{G}}(\overline{g})$ for any $1 \leq j \leq k,\ j \neq i$, but $C_G(g)$ is abelian by Step 2, a contradiction. Thus $k = 1$. Let $S = S_1 = L$, and we get that G has a unique composition factor S such that $5 \cdot 7\big||S|$ and $S \trianglelefteq \overline{G} \leq \mathrm{Aut}(S)$. Since $\{5,\ 7\} \subseteq \pi(S) \subseteq \{2,\ 3,\ 5,\ 7\}$, $5 \parallel |S|$, and $7 \parallel |S|$, we can easily get that S can be isomorphic to one of the groups: $A_7,\ A_8,\ A_9,\ L_3(4),\ S_6(2),\ U_4(3)$ by Table 1.

Step 4. $S \cong L_3(4)$.

By Step 3, S may be isomorphic to one of groups $A_7, A_8, A_9, L_3(4), S_6(2), U_4(3)$. Recall that $S \trianglelefteq \overline{G} \leq \mathrm{Aut}(S)$.

If $S \cong A_7$, then $A_7 \trianglelefteq \overline{G} \leq \mathrm{Aut}(A_7) = S_7$ by Table 1. Since $2^6 \cdot 3^3\big||G|$, $2^4 \parallel |S_7|$, and $3^2 \parallel |S_7|$, we have $\pi(K) = \{2,\ 3\}$. Let $g, h \in G$ be elements of orders 5 and 7, and $\overline{g},\ \overline{h} \in \overline{G}$ be the image of the element g and h, respectively. If $\overline{G} \cong A_7$, then

$$|g^G| = 2^6 \cdot 3^2 \cdot 7,\ |h^G| = 2^6 \cdot 3^2 \cdot 5,$$

$$|\overline{g}^{\overline{G}}| = 2^3 \cdot 3^2 \cdot 7,\ |\overline{h}^{\overline{G}}| = 2^3 \cdot 3^2 \cdot 5.$$

Set $x = g,\ h$. Then $|x^G| = |\overline{x}^{\overline{G}}||x^K|$, and so $|K : C_K(x)| = 2^3$. It follows that g, h centralize every 3−element of K, and thus there exists a 3−element y in K such that $35\big||C_G(y)|$. By Lemma 2.6, one has that $|y^G| = n_1 = 1$, and so $y = 1$, a contradiction. If $\overline{G} \cong S_7$, then we also can get a contradiction in a similar way. Hence S is not isomorphic to A_7.

If $S \cong A_8$, then $A_8 \trianglelefteq \overline{G} \leq \mathrm{Aut}(A_8) = S_8$. By [19], S has an element x of order 6 satisfying with $|x^S| = 2^5 \cdot 3 \cdot 5 \cdot 7$ which does not divide any element of $N(G)$. Thus it is impossible that S is isomorphic to A_8.

Let x be an element of order 7 in G and $\overline{x}$ be its image in $\overline{G}$. If S is one of $A_9, S_6(2)$ and $U_4(3)$, then by [19], $|\overline{x}^S|$ is a multiple of 3^4, so are $|\overline{x}^{\overline{G}}|$ and $|x^G|$. This contradicts with (2) of Lemma 2.6, and so $S \cong L_3(4)$.

Step 5. $\overline{G} = G/K \cong PGL(3,\ 4)$.

By virtue of $L_3(4) \leq G/K \leq \mathrm{Aut}(L_3(4))$, $\overline{G}$ may be isomorphic to one of the following groups: $L_3(4), L_3(4).2_1, L_3(4).2_2, L_3(4).2_3, L_3(4).3 = PGL(3,\ 4)$, $L_3(4).2_2.2_3$, $L_3(4).3.2_1, L_3(4).3.2_2, L_3(4).D_{12}$. Let $g, h \in G$ be elements of orders 5 and 7, and $\overline{g},\ \overline{h} \in \overline{G}$ be the image of the element g and h, respectively.

If $\overline{G} \cong L_3(4)$, then $3\big||K|$, and so $K \neq 1$. By Lemma 2.6 (2), Step 2 and [19], we have that n_8 is maximal in $N(G)$, $C_G(g)$ is abelian, and $|g^G| = |\overline{g}^S| = n_8$. Thus by Lemma 2.5, $K \leq Z(G) = 1$, a contradiction.

If $\overline{G}$ is one of $L_3(4).2_2,\ L_3(4).2_2.2_3,\ L_3(4).3.2_1,\ L_3(4).3.2_2$ and $L_3(4).D_{12}$, then by [22], there exists an element g of order 5 in G such that $|\overline{g}^S| = 8064 \nmid |g^G| = 4032$, a contradiction with Lemma 2.1.

If $\overline{G}$ is one of $L_3(4).2_1$ and $L_3(4).2_3$, also by [22], we can find an element h of order 7 in G such that $|\overline{h}^S| = 5760 \nmid |h^G| = 2880$, a contradiction again. Hence $\overline{G} = G/K \cong PGL(3,\ 4)$.

Step 6. K is a trivial group such that $G \cong PGL(3,\ 4)$.

Let $h \in G,\ |h| = 7$, and $\overline{h} \in G/K = \overline{G}$ be the image of the element h. In view of Lemma 2.6 (2), Step 2 and [19], we see that n_6 is maximal in $N(G)$, $C_G(h)$ is abelian and $|h^G| = |\overline{h}^S| = |\overline{h}^{\overline{G}}| = n_6$. Thus $K \leq C_G(h)$. If $K \neq 1$, then h centralizes an element from the center of a Sylow p-subgroup of G for some prime $p \in \pi(K)$, which is impossible by Lemma 2.6. Hence K is a trivial group, and so $G \cong PGL(3,\ 4)$.

4. Proof of the main theorem for $PGL(3,\ 7)$

Theorem 4.1. *Let G be a group with trivial center. If $N(G) = N(PGL(3,\ 7))$, then $G \cong PGL(3,\ 7)$.*

Proof. We divide the proof of this theorem into eight steps.

Step 1. The Sylow 19-subgroup P of G is order of 19.

Using Lemma 2.2 and (2) of Lemma 2.7, we derive that P is elementary abelian. Assume that 19^2 divides G. Since $N(G) = N(PGL(3,\ 7))$, the centralizer of every element of G contains an element of order 19 by (4) of Lemma 2.7. Considering an element y of G such that $|y^G| = n_2 = 2^4 \cdot 3^2 \cdot 19$.

Suppose that 19 does not divide $|y|$. Let x be an element of order 19 in $C_G(y)$. Then by (3) of Lemma 2.1, $C_G(xy) = C_G(x) \bigcap C_G(y)$, and so $lcm(|x^G|, |y^G|)$ divides $|(xy)^G|$. Since P is abelian, $C_G(x)$ includes P up to conjugacy. Hence 19 does not divide $|x^G|$. It follows that $|x^G|$ is equal to $n_4 = 2^5 \cdot 3 \cdot 7^3$ or $n_6 = 2^5 \cdot 3^2 \cdot 7^3$ by (3) and (5) of Lemma 2.7. In both cases, $2^5 \cdot 3^2 \cdot 7^3 \cdot 19$ divides $|(xy)^G|$, which is impossible by (2) of Lemma 2.7.

Suppose that 19 divides $|y|$. Let $|y| = 19t$. Since P is elementary abelian, one has that $gcd(19,\ t) = 1$. Put $u = y^{19}$ and $v = y^t$. Then $y = uv$, and so $C_G(uv) = C_G(u) \bigcap C_G(v)$ by Lemma 2.1. Therefore, $|v^G|$ divides $|y^G| = n_2 = 2^4 \cdot 3^2 \cdot 19$. On the other hand, the element v is order of 19, and thus $|v^G|$ is equal to $n_4 = 2^5 \cdot 3 \cdot 7^3$ or $n_6 = 2^5 \cdot 3^2 \cdot 7^3$ by Lemma 2.7, a contradiction. Hence P has order of 19.

Step 2. For an element $x \in G$ of order 19, it follows that $|x^G| = n_6 = 2^5 \cdot 3^2 \cdot 7^3$ and $C_G(x)$ is abelian.

By Step 1, for any $1 \neq y \in C_G(x)$ one has that $19 \nmid |y^G|$, and hence $|y^G| = n_4$ or n_6 by (3) and (5) of Lemma 2.7. Assume that $|x^G| = n_4 = 2^5 \cdot 3 \cdot 7^3$ and let H be a Sylow 3-subgroup of $C_G(x)$. Then H is a nontrivial group of order $|G|_3/3$ by Lemma 2.8. Hence there exists a 3-subgroup K of G such that H is a normal subgroup of K and $|K/H| = 3$. Then $1 \neq H \bigcap Z(K) \leq C_G(x)$. Taking $1 \neq h \in H \bigcap Z(K)$, we have that $K \leq C_G(h)$, and so $|h^G|_3 = 1$. But $|h^G| = n_4$ or n_6, one has that $3 \big| |h^G|$, a contradiction. It follows that $|x^G| = n_6 = 2^5 \cdot 3^2 \cdot 7^3$.

Since n_6 is maximal with respect to divisibility in $N(G)$, Lemma 2.3 implies that the group $C_G(x)$ is abelian.

Step 3. Suppose that $q \in \{2,\ 3,\ 7\}$, Q is a Sylow q-subgroup of G. Then $19 \nmid |C_G(x)|$, $x \in Z(Q)$.

Let $1 \neq x \in Z(Q)$. Then q does not divide $|x^G|$, and by Lemma 2.7, $|x^G| = n_3 = 3 \cdot 7^2 \cdot 19$ while $q = 2$, $|x^G| = n_5 = 2^3 \cdot 7^2 \cdot 19$ while $q = 3$, and $|x^G|$ is equal to $n_2 = 2^4 \cdot 3^2 \cdot 19$ while $q = 7$. The Step 3 follows.

Step 4. G is non-soluble and $O_{2,\ 2'}(G) = O_2(G)$.

Let $K = O_2(G)$, $\overline{G} = G/K$, and denote by $\overline{x}$ the images of an element x of G in $\overline{G}$. If the statement is false, then there exists $r \in \{3,\ 7,\ 19\}$ such that $O_r(\overline{G}) \neq 1$.

If $O_{19}(\overline{G}) \neq 1$, then $|O_{19}(\overline{G})| = 19$ by Step 1. Let Q be a Sylow 7-subgroup of G and $y \in Z(Q)$ be an element of order 7. Obviously, the subgroup $O_{19}(\overline{G})\langle\overline{y}\rangle$ is cyclic. Hence 19 divides $|C_{\overline{G}}(\overline{y})|$. Since $(7,\ |K|) = 1$, Lemma 2.1(2) implies that 19 divides $|C_G(y)|$, which is impossible by Step 3. Thus, $O_{19}(\overline{G}) = 1$.

If $O_7(\overline{G}) \neq 1$, then $V = Z(O_7(\overline{G}))$ is a nontrivial normal subgroup of G. If x is an element of order 19 in G, then $V = C_V(\overline{x}) \times [V,\ \overline{x}]$ such that $\overline{x}$ acts fixed-point freely on $[\overline{x}, V]$, and then $|[\overline{x}, V]| - 1$ is divisible by 19. Lemma 2.1 (1) implies that $|\overline{x}^{\overline{G}}|$ is a divisor of $2^5 \cdot 3^2 \cdot 7^3$, and hence $|[V,\ \overline{x}]| = |V : C_V(\overline{x})|$ divides 7^3, which implies $[V,\ \overline{x}] = 1$ and $V = C_V(\overline{x})$. Let Q be a Sylow 7-subgroup of $\overline{G}$. Then $Z(Q)$ has a nontrivial intersection with V and let $\overline{z}$ be of order 7 from this intersection. Since $(|K|,\ 7) = 1$, there exists a pre-image z of $\overline{z}$ in G such that z lies in the center of a Sylow 7-subgroup of G by Lemma 2.1 (2). Further, the centralizer of z also contains an element of order 19, which contradicts Step 3. Therefore, $O_7(\overline{G}) = 1$.

Similarly, we can prove $O_3(\overline{G}) = 1$. The Step 4 holds.

Step 5. Let $K = O_2(G)$, $\overline{G} = G/K$. Then every minimal normal subgroup of $\overline{G}$ is non-soluble. Especially, $Soc(\overline{G}) \trianglelefteq \overline{G} \lesssim \mathrm{Aut}(Soc(\overline{G}))$.

Let N be any minimal normal subgroup of $\overline{G}$ and assume that N is soluble. Then N is an elementary ablelian p-group for some $p \in \{3,\ 7,\ 19\}$, and so $N \leq O_p(\overline{G})$. It follows that $O_p(\overline{G})$ is nontrivial, contradicts Step 4. Hence every minimal normal subgroup of $\overline{G}$ is non-soluble. Let $N_1, N_2, \ldots, N_s$ be all minimal normal subgroups of $\overline{G}$, where s is a positive integer. Then $Soc(\overline{G}) = N_1 \times N_2 \times \cdots \times N_s$. We assert that $C_{\overline{G}}(Soc(\overline{G})) = 1$. Otherwise, $1 \neq C_{\overline{G}}(Soc(\overline{G})) \trianglelefteq \overline{G}$. But $C_{\overline{G}}(Soc(\overline{G})) \bigcap Soc(\overline{G})$=1 because $N_i (1 \leq i \leq s)$ are a direct product of some isomorphic simple groups. Hence $C_{\overline{G}}(Soc(\overline{G}))$ is soluble, a contradiction. By N/C theorem, we have $Soc(\overline{G}) \trianglelefteq \overline{G} = \overline{G}/C_{\overline{G}}(Soc(\overline{G})) \lesssim \mathrm{Aut}(Soc(\overline{G}))$.

Step 6. Let $L = Soc(\overline{G})$. Then L is a non-ableian simple group and $19 \big| |L|$.

By Step 5, we have that L is a direct product of non-ableian simple groups of $S_1, S_2, \ldots,$ and S_k. Let g be an element of order 19 of G and suppose that $19 \notin \pi(L)$. Then $\overline{g}$ is of order 19 in $\overline{G}$ and induces a nontrivial outer automorphism of the group L. Suppose that there exists a positive integer i satisfying $S_i^{\overline{g}} \neq S_i$. Without loss of generality, we assume that $i = 1$. Let $H = \langle s\ |s = s_1 s_1^{\overline{g}} s_1^{\overline{g}^2} \cdots s_1^{\overline{g}^{18}},\ s_1 \in S_1\rangle$. Then H lies in the centralizer of the element $\overline{g}$ and is isomorphic to S_1, but the centralizer of g is abelian by Step 2, a contradiction. Hence $\overline{g}$ induces a nontrivial outer automorphism of the group S_i such that $19 \big| |\mathrm{Out}(S_i)|$. In view of $\pi(S_i) \subseteq \pi(G) = \{2, 3, 7, 19\}$ and Table 2, the prime divisors of $|\mathrm{Out}(S_i)|$ are not greater than 5, a contradiction. Therefore $19 \big| |L|$.

If $k > 1$ and $\overline{g} \in S_j$, then $S_i \leq C_{\overline{G}}(\overline{g})$ for any $1 \leq i \leq k,\ i \neq j$. On the other hand, $C_{\overline{G}}(\overline{g})$ is abelian by Step 2, a contradiction. Therefore $k = 1$, and so L is a non-ableian simple group and $19 \big| |L|$.

Step 7. $L \cong L_3(7)$.

By Step 6 and Step 1, we have that L is a non-ableian simple group satisfying $19 \parallel |M|$. Then by Table 2, L may be isomorphic to one of $U_3(8)$ and $L_3(7)$.

By Table 2 and Step 5, we see that $\pi(\mathrm{Out}(L)) \subseteq \{2,\ 3\}$ and $L \trianglelefteq \overline{G} \lesssim \mathrm{Aut}(L)$. In view of $K = O_2(G)$ and $7^3 \big| |G|$, we have that 7^3 divides $|L|$. Hence L must be isomorphic to $L_3(7)$.

Step 8. $G \cong PGL(3,\ 7)$.

Let x be an element of order 19 in G and $\overline{x}$ be its image in $\overline{G}$. It is clear that $\overline{x} \in L$. By Lemma 2.1 and [19], we have that $|\overline{x}^L| = |\overline{x}^{\overline{G}}| = |x^G| = 2^5 \cdot 3^2 \cdot 7^3$ such that $K \leq C_G(x)$. If $K \neq 1$, then x centralizes an element from the center of a Sylow 2-subgroup of G, which is impossible by Step 3. Hence $G = \overline{G}$ and $L_3(7) \trianglelefteq G \lesssim \mathrm{Aut}(L_3(7))$. By [22], we see that $|N(G)| \neq |N(T)|$ for any group T except $PGL(3,\ 7)$, where T satisfies with $L_3(7) \leq T \leq \mathrm{Aut}(L_3(7))$. Therefore $G \cong PGL(3,\ 7)$.

Proof of Main Theorem. It follows directly from Theorem 3.1 and Theorem 4.1.

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