Upper class functions on a controlled contraction principle in partial S-metric spaces

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Abstract. In this paper, we prove the existence and uniqueness of a fixed point of a self mapping on partial S-metric spaces under the partially α-contractive condition.

Keywords: common fixed point.

1. Introduction and mathematical preliminaries

The existence and uniqueness of a fixed point for a self mapping on different types of metric spaces were the main topic for many research papers [4-18]. The notion of S-metric space was introduced by Sedghi [3]. A generalization of S-metric space was given by Nabil in [1], where he introduced partial S-metric spaces. Moreover, he proved the existence of a fixed point for a self mapping in partial S-metric space. In this paper, we generalize the results in [1] by adding a control function to the contraction principle, which makes the results in [1] a direct consequences of our theorems.

Before proceeding to the main results, we set forth some definitions that will be used in the sequel.

Definition 1.1 ([4]). Let X be a nonempty set and \( p : X \times X \rightarrow [0, +\infty) \). We say that \((X, p)\) is a partial metric space if for all \( x, y, z \in X \) we have:

1. \( x = y \) if and only if \( p(x, y) = p(x, x) = p(y, y) \);
2. \( p(x, x) \leq p(x, y) \);

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3. \( p(x, y) = p(y, x) \);

4. \( p(x, z) \leq p(x, y) + p(y, z) - p(y, y) \).

**Definition 1.2** ([3]). Let \( X \) be a nonempty set. An \( S \)-metric space on \( X \) is a function \( S : X^3 \to [0, \infty) \) that satisfies the following conditions, for all \( x, y, z, a \in X \):

1. \( S(x; y; z) \geq 0 \),
2. \( S(x; y; z) = 0 \) if and only if \( x = y = z \),
3. \( S(x; y; z) \leq S(x; x; a) + S(y; y; a) + S(z; z; a) \).

The pair \((X; S)\) is called an \( S \)-metric space.

Next, we give the definition of partial \( S \)-metric space.

**Definition 1.3** ([1]). Let \( X \) be a nonempty set. A partial \( S \)-metric space on \( X \) is a function \( S_p : X^3 \to [0, \infty) \) that satisfies the following conditions, for all \( x, y, z, t \in X \):

(i) \( x = y \) if and only if \( S_p(x, x, x) = S_p(y, y, y) = S_p(x, x, y) \);
(ii) \( S_p(x, y, z) \leq S_p(x, x, t) + S_p(y, y, t) + S_p(z, z, t) - S_p(t, t, t) \);
(iii) \( S_p(x, x, x) \leq S_p(x, y, z) \);
(iv) \( S_p(x, x, y) = S_p(y, y, x) \).

The pair \((X; S_p)\) is called a partial \( S \)-metric space.

**Definition 1.4.** A sequence \( \{x_n\}_{n=0}^{\infty} \) of elements in \((X, S_p)\) is called \( p \)-Cauchy if the limit \( \lim_{n,m \to \infty} S_p(x_n, x_m, x_m) \) exists and finite. The partial \( S \)-metric space \((X, S_p)\) is called complete if for each \( p \)-Cauchy sequence \( \{x_n\}_{n=0}^{\infty} \) there exists \( z \in X \) such that \( S_p(z, z, z) = \lim_n S_p(z, z, x_n) = \lim_{n,m} S_p(x_n, x_m, x_m) \).

Moreover, \((X, S_p)\) is a complete partial \( S \)-metric space if and only if \((X, S_p)\) is a complete \( S \)-metric space. A sequence \( \{x_n\}_n \) in a partial \( S \)-metric space \((X, S_p)\) is called 0-Cauchy if \( \lim_{n,m \to \infty} S_p(x_n, x_n, x_m) = 0 \). We say that \((X, S_p)\) is 0-complete if every 0-Cauchy in \( X \) converges to a point \( x \in X \) such that \( S_p(x, x, x) = 0 \).

One can easily construct an example of a partial \( S \)-metric space by using the ordinary partial metric space.

**Example 1.5** ([1]). Let \( X = [0, \infty) \) and \( p \) be the ordinary partial metric space on \( X \). Define the mapping on \( X^3 \) to be \( S_p(x, y, z) = p(x, z) + p(y, z) \). Then \( S_p \) defines a partial \( S \)-metric space.

Now we introduce the notion of partially \( \alpha \)-contractive.
Definition 1.6. Let \((X, S_p)\) be a partial S-metric space and \(T : X \to X\) be a given mapping. We say that \(T\) is partially \(\alpha\)-contractive if there exists a constant \(k \in [0, 1)\) and a function \(\alpha : X \times X \to [0, +\infty)\) such that for all \(x, y \in X\) we have

\[
\alpha(x, y)S_p(Tx, Tx, Ty) \leq \max\{kS_p(x, x, y), S_p(x, x, x), S_p(y, y, y)\}.
\]

Definition 1.7. Let \((X, S_p)\) be a partial S-metric space and \(T : X \to X\) be a given mapping. We say that \(T\) is \(R_\alpha\)-admissible if \(x, y \in X, \alpha(x, y) \geq 1\) implies that \(\alpha(x, Ty) \geq 1\). Also, we say that \(T\) is \(\alpha\)-admissible if \(x, y \in X, \alpha(x, y) \geq 1\) implies that \(\alpha(Tx, Ty) \geq 1\).

Example 1.8. Let \(X = [0, +\infty)\). Define \(T : X \to X\) by \(Tx = \sqrt{x}\) and \(\alpha : X \times X \to X\) by

\[
\alpha(x, y) = \begin{cases} 
   e^{x-y}, & \text{if } x \geq y \\
   0, & \text{if } x < y.
\end{cases}
\]

It is a straightforward to verify that \(T\) is \(\alpha\)-admissible and \(R_\alpha\)-admissible.

Now, we set

\[
\rho_{S_p}(\alpha) := \inf\{S_p(x, x, y) \mid x, y \in X : \alpha(x, y) \geq 1\}
\]

\[
= \inf\{S_p(x, x, x) \mid x \in X : \alpha(x, x) \geq 1\},
\]

\[
X_{S_p}(\alpha) = \{x \in X \mid S_p(x, x, x) = \rho_{S_p}(\alpha)\},
\]

\[
Z_{S_p}(\alpha) = \{x \in X_{S_p} \mid \alpha(x, x) \geq 1\}.
\]

Definition 1.9. Let \((X, S_p)\) be a partial S-metric space and \(T : X \to X\) be a given mapping. We say that \(T\) is \(R_\mu\)-subadmissible if \(x, y \in X, \mu(x, y) \leq 1\) implies that \(\mu(x, Ty) \leq 1\).

2. Main result

In this section, we prove the existence of a fixed point in a partial S-metric space. We prove relevant corollary. This next theorem is considered to be our main result.

Definition 2.1 ([?]). Let \(T : X \to X\) be a map and \(\mu : X \times X \to [0, +\infty)\) be a function. We say that \(T\) is \(\mu\)-subadmissible if \(x, y \in X, \mu(x, y) \leq 1\) implies that \(\mu(Tx, Ty) \leq 1\).

Definition 2.2. A map \(T : X \to X\) is said to be triangular \(\mu\)-subadmissible if the following holds:

\((T1)\) \(T\) is \(\mu\)-subadmissible,

\((T2)\) \(\mu(x, u) \leq 1\) and \(\mu(u, y) \leq 1\) implies that \(\mu(x, y) \leq 1\), \(x, u, y \in X\).
Lemma 2.3. Let $T : X \to X$ be a triangular $\mu$–suborbital admissible mapping. Assume that there exists $x_1 \in X$ such that $\mu(x_1, Tx_1) \leq 1$. Then there exists a sequence $\{x_n\}$ such that $\mu(x_n, x_m) \leq 1$ for all $m, n \in \mathbb{N}$ with $n < m$.

The letter $\mathbb{N}$ represent the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Further, the nonnegative real numbers will be denoted by $\mathbb{R}_0^+ = [0, \infty)$.

In 2014 the concept of pair $(F, h)$ is an upper class (see Definition 2.4 until 2.10) was introduced by A.H. Ansari in [19].

Definition 2.4 ([19, 20]). A function $h : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}$ is said to be a function of subclass of type I, if $x \geq 1 \implies h(1, y) \leq h(x, y)$ for all $y \in \mathbb{R}_0^+$.

Example 2.5 ([19, 20]). Define $h : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}$ by:

(a) $h(x, y) = (y + l)^x, l > 1$;
(b) $h(x, y) = (x + l)^y, l > 1$;
(c) $h(x, y) = x^n y, n \in \mathbb{N}$;
(d) $h(x, y) = y$;
(e) $h(x, y) = \frac{1}{n+1} \left( \sum_{i=0}^{n} x^i \right) y, n \in \mathbb{N}$;
(f) $h(x, y) = \left[ \frac{1}{n+1} \left( \sum_{i=0}^{n} x^i \right) + l \right]^y, l > 1, n \in \mathbb{N}$

for all $x, y \in \mathbb{R}_0^+$. Then $h$ is a function of subclass of type I.

Definition 2.6 ([19, 20]). Let $h, F : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}$, then we say that the pair $(F, h)$ is an upper class of type I and: (i) $0 \leq s \leq 1 \implies F(s, t) \leq F(1, t)$, (ii) $h(1, y) \leq F(1, t) \implies y \leq t$ for all $t, y \in \mathbb{R}_0^+$.

Example 2.7 ([19, 20]). Define $h, F : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}$ by:

(a) $h(x, y) = (y + l)^x, l > 1$ and $F(s, t) = st + l$;
(b) $h(x, y) = (x + l)^y, l > 1$ and $F(s, t) = (1 + l)^{st}$;
(c) $h(x, y) = x^m y, m \in \mathbb{N}$ and $F(s, t) = st$;
(d) $h(x, y) = y$ and $F(s, t) = t$;
(d) $h(x, y) = \frac{1}{n+1} \left( \sum_{i=0}^{n} x^i \right) y, n \in \mathbb{N}$ and $F(s, t) = st$;
(e) $h(x, y) = \left[ \frac{1}{n+1} \left( \sum_{i=0}^{n} x^i \right) + l \right]^y, l > 1, n \in \mathbb{N}$ and $F(s, t) = (1 + l)^{st}$

for all $x, y, s, t \in \mathbb{R}_0^+$. Then the pair $(F, h)$ is an upper class of type I.
Definition 2.8 ([19, 20]). A function \( h : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R} \) is said to be a function of subclass of type II, if for all \( x, y \geq 1 \), we have \( h(1, 1, z) \leq h(x, y, z) \), for all \( z \in \mathbb{R}^+ \).

Example 2.9 ([19, 20]). Define \( h : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R} \) by:

(a) \( h(x, y, z) = (z + l)^{xy}, l > 1; \)
(b) \( h(x, y, z) = (xy + l)^x, l > 1; \)
(c) \( h(x, y, z) = z; \)
(d) \( h(x, y, z) = x^m y^n z^p, m, n, p \in \mathbb{N}; \)
(e) \( h(x, y, z) = \frac{x^m + x^n y^p + y^n}{3} z^k, m, n, p, q, k \in \mathbb{N} \)

for all \( x, y, z \in \mathbb{R}^+ \). Then \( h \) is a function of subclass of type II.

Definition 2.10 ([19, 20]). Let \( h : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R} \) and \( F : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R} \), then we say that the pair \((F, h)\) is an upper class of type II, if \( h \) is a subclass of type II and the following holds:

(i) if \( 0 \leq s \leq 1 \) then we have \( F(s, t) \leq F(1, t) \),
(ii) if \( h(1, 1, z) \leq F(s, t) \) then we have \( z \leq st \) for all \( s, t, z \in \mathbb{R}^+ \).

Example 2.11 ([19, 20]). Define \( h : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R} \) and \( F : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R} \) by:

(a) \( h(x, y, z) = (z + l)^{xy}, l > 1, F(s, t) = st + l; \)
(b) \( h(x, y, z) = (xy + l)^x, l > 1, F(s, t) = (1 + l)^st; \)
(c) \( h(x, y, z) = z, F(s, t) = st; \)
(d) \( h(x, y, z) = x^m y^n z^p, m, n, p \in \mathbb{N}, F(s, t) = s^p t^p \)
(e) \( h(x, y, z) = \frac{x^m + x^n y^p + y^n}{3} z^k, m, n, p, q, k \in \mathbb{N}, F(s, t) = s^k t^k \)

for all \( x, y, z, s, t \in \mathbb{R}^+ \). Then the pair \((F, h)\) is an upper class of type II.

Notation.

\[ \rho_{S_p}(\alpha, \mu) := \inf \{ S_p(x, x, y) \mid x, y \in X : \alpha(x, y) \geq 1, \mu(x, y) \leq 1 \} \]
\[ = \inf \{ S_p(x, x, x) \mid x \in X : \alpha(x, x) \geq 1, \alpha(x, x) \leq 1 \}, \]
\[ X_{S_p}(\alpha, \mu) = \{ x \in X \mid S_p(x, x, x) = \rho_{S_p}(\alpha, \mu) \}, \]
\[ Z_{S_p}(\alpha, \mu) = \{ x \in X_{S_p} \mid \alpha(x, x) \geq 1, \mu(x, x) \leq 1 \}. \]
Definition 2.12. Let \((X, S_p)\) be a partial S-metric space and \(T: X \rightarrow X\) be a given mapping. We say that \(T\) is partially \((\mathcal{F}, h, \alpha, \mu)\)-contractive if there exists a constant \(k \in [0, 1)\) and a function \(\alpha, \mu: X \times X \rightarrow [0, +\infty)\) such that for all \(x, y \in X\) we have

\[
h(\alpha(x, y), S_p(Tx, Tx, Ty)) \leq \mathcal{F}(\mu(x, y), \max\{kS_p(x, x, y), S_p(x, x, y)\}),
\]

(2.1)

where the pair \((\mathcal{F}, h)\) is an upper class of type I.

Theorem 2.13. Let \((X, S_p)\) be a complete partial S-metric space, \(T\) be a self mapping on \(X\) and assume that \(T\) is partially \((\mathcal{F}, h, \alpha, \mu)\)-contractive. If \(T\) is \(\alpha\)-admissible, \(\mu\)-subadmissible and \(R_\alpha\)-admissible, \(R_\mu\)-subadmissible and if \(X_{S_p}(\alpha, \mu)\) is nonempty, then \(Z_{S_p}(\alpha, \mu)\) is nonempty. Also, assume that there exists \(x_0 \in X\) such that \(\alpha(x_0, x_0) \geq 1, \mu(x_0, x_0) \leq 1\), then there exists \(a \in Z_{S_p}(\alpha)\) such that \(Ta = a\).

Moreover, if for all \(u, v\) in \(Z_{S_p}(\alpha, \mu)\) with the property \(Tu = u\) and \(Tv = v\) we have \(\alpha(u, v) \geq 1, \mu(u, v) \leq 1\), then \(T\) has a unique fixed point in \(Z_{S_p}(\alpha, \mu)\).

Proof. Let \(x_0 \in X\) such that \(\alpha(x_0, x_0) \geq 1\). Define a sequence \(\{x_n\}\) for all \(n \geq 0\) in \(X\) such that \(x_1 = Tx_0, x_2 = Tx_1, \ldots, x_{n+1} = Tx_n, \ldots\). Since \(T\) is \(\alpha\)-admissible, \(\mu\)-subadmissible and \(R_\alpha\)-admissible, \(R_\mu\)-subadmissible, we have \(\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geq 1, \mu(x_0, x_1) = \mu(x_0, Tx_0) \leq 1\), and hence \(\alpha(x_1, x_2) = \alpha(Tx_0, Tx_1) \geq 1, \mu(x_1, x_2) = \mu(Tx_0, Tx_1) \leq 1\). So, by induction on \(n\) we get

\[
\alpha(x_n, x_{n+1}) \geq 1, \mu(x_n, x_{n+1}) \leq 1,
\]

for all \(n \geq 0\). Also, since \(T\) is \(R_\alpha\)-admissible and \(R_\mu\)-subadmissible; \(\alpha(x_0, x_0) \geq 1, \mu(x_0, x_0) \leq 1\) implies \(\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geq 1, \mu(x_0, x_1) = \mu(x_0, Tx_0) \leq 1\). By induction on \(n\), we also conclude that

\[
\alpha(x_0, x_n) \geq 1, \mu(x_0, x_n) \leq 1
\]

for all \(n \geq 0\). Also, given the fact that \(T\) is \(\alpha\)-admissible and \(\alpha(x_0, x_0) \geq 1\), it not difficult to prove that \(\alpha(x_n, x_n) \geq 1\) for all \(n \geq 0\). Hence,

\[
\begin{align*}
&h(1, S_p(x_1, x_1, x_1)) = h(1, S_p(Tx_0, Tx_0, Tx_0)) \\
&\leq h(\alpha(x_0, x_0), S_p(Tx_0, Tx_0, Tx_0)) \\
&\leq \mathcal{F}(\mu(x_0, x_0), \max\{kS_p(x_0, x_0, x_0), S_p(x_0, x_0, x_0)\}, S_p(x_0, x_0, x_0)) \\
&\leq \mathcal{F}(1, \max\{kS_p(x_0, x_0, x_0), S_p(x_0, x_0, x_0)\}, S_p(x_0, x_0, x_0)) \}
\end{align*}
\]

This implies that

\[
S_p(x_1, x_1, x_1) \leq \max\{kS_p(x_0, x_0, x_0), S_p(x_0, x_0, x_0), S_p(x_0, x_0, x_0)\}
\]

\[
= S_p(x_0, x_0, x_0).
\]
By induction on \( n \), we obtain:

\[ S_p(x_{n+1}, x_{n+1}, x_{n+1}) \leq S_p(x_n, x_n, x_n). \]

Therefore, \( \{S_p(x_n, x_n, x_n)\}_{n \geq 0} \) is a nonincreasing sequence. Define

\[ r_0 := \lim_n S_p(x_n, x_n, x_n) = \inf_n S_p(x_n, x_n, x_n) \geq 0 \]

and

\[ M_0 := \frac{2}{1-k} S_p(x_0, x_0, x_1) + S_p(x_0, x_0, x_0). \]

Next, we show that \( S_p(x_0, x_0, x_n) \leq M_0 \), for any \( n \geq 0 \). If \( n = 0 \) the case is trivial. For \( n = 1 \) and using the fact that \( k \in [0, 1] \) we deduce that

\[ S_p(x_0, x_0, x_1) \leq \frac{2}{1-k} S_p(x_0, x_0, x_1) \leq \frac{2}{1-k} S_p(x_0, x_0, x_1) + S_p(x_0, x_0, x_0) = M_0. \]

So, we may assume that it is true for all \( n \leq n_0 - 1 \) and prove it for \( n = n_0 \geq 2 \).

\[
S_p(x_0, x_0, x_{n_0}) \leq S_p(x_0, x_0, x_1) + S_p(x_0, x_0, x_{n_0}) - S_p(x_{n_1}, x_{n_1}, x_{n_1})
\leq 2S_p(x_0, x_0, x_1) + S_p(x_{n_0}, x_{n_0}, x_{n_0}) - S_p(x_{n_1}, x_{n_1}, x_{n_1})
\leq 2S_p(x_0, x_0, x_1) + \alpha(x_{n_0}, x_{n_0-1}) S_p(Tx_0, Tx_0, Tx_{n_0-1})
\leq 2S_p(x_0, x_0, x_1) + \max\{kS_p(x_0, x_0, x_{n_0-1}), S_p(x_0, x_0, x_0), S_p(x_{n_0-1}, x_{n_0-1}, x_{n_0-1})\}
\leq 2S_p(x_0, x_0, x_1) + \max\{kS_p(x_0, x_0, x_{n_0-1}), S_p(x_0, x_0, x_0)\}.
\]

Also, by induction assumption, we have \( S_p(x_0, x_0, x_{n_0-1}) \leq \frac{2}{1-k} S_p(x_0, x_0, x_1) + S_p(x_0, x_0, x_0) \). So, we have

\[
S_p(x_0, x_0, x_{n_0}) \leq 2S_p(x_0, x_0, x_1) + \max\{ \frac{2k}{1-k} S_p(x_0, x_0, x_1) + kS_p(x_0, x_0, x_0), S_p(x_0, x_0, x_0) \}
\leq 2S_p(x_0, x_0, x_1) + \frac{2k}{1-k} S_p(x_0, x_0, x_1) + S_p(x_0, x_0, x_0)
= \frac{2}{1-k} S_p(x_0, x_0, x_1) + S_p(x_0, x_0, x_0) = M_0.
\]

Hence, we conclude that \( S_p(x_0, x_0, x_n) \leq M_0 \). Next, we need to show that

\[
\lim_{n,m} S_p(x_n, x_n, x_m) = r_0.
\]

For all \( n, m \) we have \( S_p(x_n, x_n, x_m) \geq S_p(x_n, x_n, x_n) \geq r_0 \). Let \( \epsilon > 0 \) find a natural number \( r_0 \) such that \( S_p(x_{n_0}, x_{n_0}, x_{n_0}) < r_0 + \epsilon \) and \( 2M_0 k^{n_0} < r_0 + \epsilon \). Now for any \( n, m \geq 2n_0 \), since \( T \) is \( R_\alpha \)-admissible and using the fact that \( \alpha(x_n, x_{n+1}) \geq 1, \mu(x_n, x_{n+1}) \leq 1 \) we deduce that \( \alpha(x_n, x_m) \geq 1, \mu(x_n, x_m) \leq 1 \). Hence,

\[
h(1, S_p(x_n, x_n, x_m)) \leq h(\alpha(x_n, x_n), S_p(x_n, x_n, x_m)) \leq \mathcal{F}(\mu(x_n, x_m), \theta) \leq \mathcal{F}(1, \theta),
\]

where

\[
\mathcal{F}(\mu(x_n, x_m), \theta) = \frac{\mu(x_n, x_m)}{1-k}.
\]
where
\[
\theta = \max \{kS_p(x_{n-1}, x_{n-1}, x_{m-1}), S_p(x_{n-1}, x_{n-1}, x_{n-1}), S_p(x_{m-1}, x_{m-1}, x_{m-1}) \}.
\]
This implies that
\[
S_p(x_n, x_n, x_m) \\
\leq \max \{kS_p(x_{n-1}, x_{n-1}, x_{m-1}), S_p(x_{n-1}, x_{n-1}, x_{n-1}), S_p(x_{m-1}, x_{m-1}, x_{m-1}) \} \\
\leq \max \{k^2 S_p(x_{n-2}, x_{n-2}, x_{m-2}), S_p(x_{n-2}, x_{n-2}, x_{m-2}), S_p(x_{m-2}, x_{m-2}, x_{m-2}) \} \\
\leq \cdots \leq \max \{k^n S_p(x_{n-n_0}, x_{n-n_0}, x_{m-n_0}), S_p(x_{n-n_0}, x_{n-n_0}, x_{n-n_0}) \} \\
\leq r_0 + \epsilon.
\]
Hence,
\[
\lim_{n,m} S_p(x_n, x_n, x_m) = r_0.
\]
Since \((X, p)\) is a complete partial S-metric space; there exists \(\bar{x} \in X\) such that
\[
r_0 = S_p(\bar{x}, \bar{x}, \bar{x}) = \lim_{n} S_p(\bar{x}, \bar{x}, x_n) = \lim_{n,m} S_p(x_n, x_n, x_m).
\]
Now, we show that \(S_p(\bar{x}, \bar{x}, \bar{x}) = S_p(\bar{x}, \bar{x}, T\bar{x})\). For each natural number \(n\) we have
\[
S_p(\bar{x}, \bar{x}, T\bar{x}) \leq 2S_p(\bar{x}, \bar{x}, x_n) - S_p(x_n, x_n, x_n) + S_p(T\bar{x}, T\bar{x}, x_n).
\]
Using the property that \(T\) is \(\alpha\)-contractive, we deduce that there exists a subsequence of natural numbers \(\{n_l\}\) such that
\[
h(1, S_p(T\bar{x}, T\bar{x}, x_{n_l})) \leq h(\alpha(\bar{x}, x_{n_l-1}), S_p(T\bar{x}, T\bar{x}, x_{n_l})) \\
\leq \mathcal{F}(\mu(\bar{x}, x_{n_l-1}), \max \{kS_p(\bar{x}, \bar{x}, x_{n_l-1}), S_p(\bar{x}, \bar{x}, x_{n_l-1}), S_p(x_{n_l-1}, x_{n_l-1}, x_{n_l-1}) \}) \\
\leq \mathcal{F}(1, \max \{kS_p(\bar{x}, \bar{x}, x_{n_l-1}), S_p(\bar{x}, \bar{x}, x_{n_l-1}), S_p(x_{n_l-1}, x_{n_l-1}, x_{n_l-1}) \}),
\]
and thus
\[
S_p(T\bar{x}, T\bar{x}, x_{n_l}) \leq \max \{kS_p(\bar{x}, \bar{x}, x_{n_l-1}), S_p(\bar{x}, \bar{x}, x_{n_l}), S_p(x_{n_l-1}, x_{n_l-1}, x_{n_l-1}) \}.
\]
So, for \(l \geq 1\), we have either \(S_p(T\bar{x}, T\bar{x}, x_{n_l}) \leq kS_p(\bar{x}, \bar{x}, x_{n_l-1})\) or less than or equal \(S_p(\bar{x}, \bar{x}, \bar{x})\) or less than or equal \(S_p(x_{n_l-1}, x_{n_l-1}, x_{n_l-1})\). In all of these three cases, if we take the limit as \(l \to \infty\) we get \(S_p(\bar{x}, \bar{x}, T\bar{x}) \leq S_p(\bar{x}, \bar{x}, \bar{x})\). But, we know by the property \((ii)\) of the partial S-metric space definition that \(S_p(\bar{x}, \bar{x}, \bar{x}) \leq S_p(\bar{x}, \bar{x}, T\bar{x})\). Therefore, \(S_p(\bar{x}, \bar{x}, \bar{x}) = S_p(\bar{x}, \bar{x}, T\bar{x})\).
Now, we show that \(X_{S_p}(\alpha, \mu)\) is nonempty. For each natural number \(l\) pick \(x_l \in X\) with \(\alpha(x_l, x_l) \geq 1\) and \(S_p(x_l, x_l, x_l) < \rho_{S_p}(\alpha, \mu) + \frac{1}{l}\) and show that
\[
\lim_{n,m} S_p(\bar{x}_n, \bar{x}_n, \bar{x}_m) = \rho_{S_p}(\alpha, \mu).
\]
Let \( \epsilon > 0 \) put \( n_0 := \left( \frac{3}{\epsilon(1-k)} \right) + 1 \) if \( l \geq n_0 \) then we have: 
\[
\rho_{S_p}(\alpha, \mu) \leq S_p(x_l, x_l, T x_l) \\
\leq S_p(x_l, x_l, T x_l) - r_{x_l} \leq S_p(x_l, x_l, T x_l) < \rho_{S_p}(\alpha, \mu) + \frac{1}{l} \leq \rho_{S_p}(\alpha, \mu) + \frac{\epsilon(1-k)}{3}
\]
Hence, we deduce that: 
\[
U_l := S_p(x_l, x_l, T x_l) - S_p(T x_l, T x_l, T x_l) < \frac{\epsilon(1-k)}{3},
\]
for \( i \geq n_0 \).

Also, if \( l \geq n_0 \), then \( S_p(x_l, x_l, T x_l) = r_{x_l} \leq S_p(x_l, x_l, x_l) < \rho_{S_p}(\alpha) + \frac{1}{n_0} \).

Which implies that \( S_p(x_l, x_l, T x_l) \leq \rho_{S_p}(\alpha, \mu) + \frac{\epsilon(1-k)}{3} \) for all \( l \geq n_0 \). Now, if \( n, m \geq n_0 \), then 
\[
S_p(x_n, x_n, x_m) \leq 2S_p(x_n, x_n, T x_n) + S_p(T x_n, T x_n, T x_m) + 2S_p(T x_m, T x_m, x_m) - S_p(T x_n, T x_n, T x_n) - S_p(T x_m, T x_m, T x_m).
\]

We know that \( S_p(x, x, x) = S_p(x, x, T x) \) which implies that 
\[
h(1, S_p(T x_n, T x_n, T x_m) \leq h(\alpha(x_n, x_m), S_p(T x_n, T x_n, T x_m)) \\
\leq F(\mu(x_n, x_m), \max\{kS_p(x_n, x_n, x_m), S_p(x_n, x_n, x_n), S_p(x_n, x_n, x_m)\}) \\
\leq F(1, \max\{kS_p(x_n, x_n, x_m), S_p(x_n, x_n, x_n), S_p(x_m, x_m, x_m)\})
\]

Therefore,
\[
S_p(T x_n, T x_n, T x_m) \leq \max\{kS_p(x_n, x_n, x_m), S_p(x_n, x_n, x_n), S_p(x_n, x_n, x_m), \}
S_p(x_m, x_m, x_m) \leq U_n + U_m + S_p(T x_n, T x_n, T x_m)
< U_n + U_m + \max\{kS_p(x_n, x_n, x_m), S_p(x_n, x_n, x_n), S_p(x_m, x_m, x_m)\}.
\]

Hence,
\[
\rho_{S_p}(\alpha, \mu) \leq S_p(x_n, x_n, x_m) \\
\leq \max\{\frac{2}{3} \epsilon, \frac{2}{3} \epsilon(1-k) + S_p(x_n, x_n, x_m), \frac{2}{3} \epsilon(1-k) + S_p(x_m, x_m, x_m)\} \\
\leq \max\{\frac{2}{3} \epsilon, \rho_{S_p}(\alpha, \mu) + \epsilon(1-k)\} < \rho_{S_p}(\alpha, \mu) + \epsilon.
\]

Thus,
\[
\lim_{n,m} S_p(x_n, x_n, x_m) = \rho_{S_p}(\alpha, \mu).
\]

Since \( (X, S_p) \) is complete, there exists \( a \in X \) such that,
\[
S_p(a, a, a) = \lim_{n} S_p(a, a, a) = \lim_{n,m} S_p(x_n, x_n, x_m) = \rho_{S_p}(\alpha, \mu).
\]

Therefore, we have \( a \in X_{S_p}(\alpha, \mu) \) and thus \( X_{S_p}(\alpha, \mu) \) is nonempty. This implies that, \( Z_{S_p}(\alpha, \mu) \) is nonempty.

Now, let \( x_0 \in Z_{S_p}(\alpha, \mu) \) be arbitrary. Then by the above argument we have 
\[
\rho_{S_p}(\alpha, \mu) \leq S_p(T x, T x, T x) \leq S_p(x, x, T x) = S_p(x, x, x) = r_0 = \rho_{S_p}(\alpha, \mu).
\]
Thus, $T\bar{x} = \bar{x}$. Now, assume that $T$ has two fixed points $u, v \in Z_{S_p}(\alpha, \mu)$. By our hypothesis, we know that $\alpha(u, v) \geq 1, \mu(u, v) \leq 1$. Thus,
\[
\begin{align*}
    h(1, S_p(u, u, v)) &\leq h(\alpha(u, v), S_p(Tu, Tu, Tv)) \\
    &\leq F(\mu(u, v), \max\{kS_p(u, u, v), S_p(u, u, u), S_p(v, v, v)\}) \\
    &\leq F(1, \max\{kS_p(u, u, v), S_p(u, u, u), S_p(v, v, v)\}).
\end{align*}
\]
So we have,
\[
S_p(u, u, v) \leq \max\{kS_p(u, u, v), S_p(u, u, u), S_p(v, v, v)\}.
\]
Now, if $S_p(u, u, v) \leq kS_p(u, u, v)$ we deduce that $S_p(u, u, v) = 0$ and in this case $u = v$, or $S_p(u, u, v) \leq S_p(u, u, u) = S_p(v, v, v)$ and in this case by condition $(ii)$ of the definition of the partial $S$-metric space we obtain $S_p(u, u, v) = S_p(u, u, u) = S_p(v, v, v)$ and hence by condition $(i)$ of the same definition we conclude that $u = v$. Therefore, we obtain the uniqueness as desired.

As a consequence of the above result, the following corollary follows easily.

**Corollary 2.14.** Let $(X, S_p)$ be a $0$-complete partial $S$-metric space, $k \in [0, 1]$ and consider the map $T : X \rightarrow X$ to be $\alpha$-admissible and $R_\alpha$-admissible, and there exists $x_0 \in X$ such that $\alpha(x_0, x_0) \geq 1$, also for every $x, y \in X$ we have $\alpha(x, y)S_p(Tx, Tx, Ty) \leq kS_p(x, x, y)$. Then there exists $\bar{x} \in X$ such that $T\bar{x} = \bar{x}$.

**Proof.** Using the same technique and notation in the proof of Theorem 2.13, we deduce that $S_p(x_n, x_n, x_n) = \alpha(x_n, x_n)S_p(x_n, x_n, x_n) \leq k^n S_p(x_0, x_0, x_0)$. Thus,
\[
r_0 = S_p(\bar{x}, \bar{x}, \bar{x}) = \lim_{n} S_p(\bar{x}, \bar{x}, x_n) = \lim_{n} S_p(x_n, x_n, x_n) = 0.
\]
This implies that $S_p(\bar{x}, \bar{x}, \bar{x}) = 0$. Since $S_p(\bar{x}, \bar{x}, \bar{x}) = S_p(\bar{x}, \bar{x}, T\bar{x}) = 0$, we have $\bar{x} = T\bar{x}$ as required.

In closing, we change the contraction principle in Theorem 2.13, to show that there exist a unique fixed point in the whole space $X$.

**Theorem 2.15.** Let $(X, S_p)$ be a complete partial $S$-metric space, $k \in [0, 1]$ and assume the there exists $x_0 \in X$ such that $\alpha(x_0, x_0) \geq 1$. Consider the map $T : X \rightarrow X$ to be $\alpha-$admissible and $R_{\alpha}-$admissible. Assume that for every $x, y \in X$ we have
\[
\alpha(x, y)S_p(Tx, Tx, Ty) \leq \max\{kS_p(x, x, y), \frac{S_p(x, x, x) + S_p(y, y, y)}{2}\},
\]
then there exists a unique $u \in X$ such that $Tu = u$.

**Proof.** Note that, for every $x, y \in X$ we have:
\[
\begin{align*}
\alpha(x, y)S_p(Tx, Tx, Ty) &\leq \max\{kS_p(x, x, y), \frac{S_p(x, x, x) + S_p(y, y, y)}{2}\} \\
&\leq \max\{kS_p(x, x, y), S_p(x, x, x), S_p(y, y, y)\}.
\end{align*}
\]
Thus, all conditions of Theorem 2.13 are satisfied. Hence, there exists $u \in X$ such that $Tu = u$. Assume that there exist two fixed points $u, v \in X$ for $T$ such that $\alpha(u, v) \geq 1$. Hence,

$$S_p(u, u, v) = S_p(Tu, Tu, Tv) \leq \alpha(u, v)S_p(Tu, Tu, Tv)$$

$$\leq \max\{kS_p(u, u, v), \frac{S_p(u, u, u) + S_p(v, v, v)}{2}\}.$$

Thus, we either have $S_p(u, u, v) \leq kS_p(u, u, v)$ which implies that $S_p(u, u, v) = 0$ and hence $u = v$, or $0 = 2S_p(u, u, v) - S_p(u, u, u) - S_p(v, v, v)$ which also implies that $u = v$ as desired.

**Example 3.** Let $(X, S_p)$ be a partial $S$-metric space, where $X = [0, 1] \cup [2, 3]$ and the partial $S$-metric space $S_p : X^3 \rightarrow [0, +\infty)$ is defined by

$$S_p(x, y, z) = \begin{cases} \| \max\{x, y\} - z \|, & \text{if } \{x, y, z\} \cap [2, 3] \neq \emptyset \\ |x - y - z|, & \text{if } \{x, y, z\} \subset [0, 1]. \end{cases}$$

Define the functions $T : X \rightarrow X$ and $\alpha : X \times X \rightarrow [0, +\infty)$ as follows $Tx = \frac{x + 1}{2}$ if $0 \leq x \leq 1$, $T2 = 1$, and $Tx = \frac{x + 2}{2}$ if $2 < x \leq 3$,

$$\alpha(x, y) = \begin{cases} e^{x-y}, & \text{if } x \geq y \\ 0, & \text{if } x < y. \end{cases}$$

It is easy to see that $T$ is $\alpha$-admissible and $R_\alpha$-admissible. Note that, we can always pick our $x$, $y$ and $z$ such that $\max\{x, y\} > z$. Also $T$ is an increasing function. So, for every $x, y \in X$ we have:

$$S_p(Tx, Tx, Ty) \leq \alpha(x, y)S_p(Tx, Tx, Ty) \leq \frac{1}{2}S_p(x, x, y),$$

if $x, y \in [0, 1]$, and

$$S_p(Tx, Tx, Ty) \leq \alpha(x, y)S_p(Tx, Tx, Ty) \leq \frac{S_p(x, x, x) + S_p(y, y, y)}{2},$$

$\{x, y\} \cap [2, 3] \neq \emptyset$.

One can verify that the function $T$ in this example satisfies the conditions of Theorem 2.15 and the unique fixed point will be 1.

**References**


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