Some subclasses of meromorphic multivalent functions involving a family of multiplier transforms

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Abstract. Making use of the principle of subordination between analytic functions and a family of multiplier transforms defined on the space of meromorphic functions, we introduce and investigate some new subclasses of meromorphic multivalent functions. Such results as inclusion relationships and integral-preserving properties associated with these subclasses are proved. Several subordination and superordination results involving this family of multiplier transforms are also investigated.

Keywords: analytic functions, meromorphic multivalent functions, subordination and superordination between analytic functions, Hadamard product (or convolution), multiplier transforms.

1. Introduction and preliminaries

Let $\Sigma_p$ denote the class of functions of the form

\[(1.1) \quad f(z) = z^{-p} + \sum_{k=0}^{\infty} a_k z^k \quad (p \in \mathbb{N} := \{1, 2, 3, \ldots\}),\]

which are analytic in the punctured open unit disk

\[\mathbb{U}^* := \{ z : z \in \mathbb{C} \text{ and } 0 < |z| < 1 \} =: \mathbb{U} \setminus \{0\}.\]

Let $\mathcal{H}(\mathbb{U})$ be the linear space of all analytic functions in $\mathbb{U}$. For a positive integer number $n$ and $a \in \mathbb{C}$, we let

\[\mathcal{H}[a, n] := \{ f \in \mathcal{H}(\mathbb{U}) : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots \}.\]

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Let $f, g \in \Sigma_p$, where $f$ is given by (1.1) and $g$ is defined by

$$g(z) = z^{-p} + \sum_{k=0}^{\infty} b_k z^k.$$  

Then the Hadamard product (or convolution) $f \ast g$ of the functions $f$ and $g$ is defined by

$$(f \ast g)(z) := z^{-p} + \sum_{k=0}^{\infty} a_k b_k z^k := (g \ast f)(z).$$

Let $\mathcal{P}$ denote the class of functions of the form

$$p(z) = 1 + \sum_{k=1}^{\infty} p_k z^k;$$

which are analytic and convex in $U$ and satisfy the condition $\Re(p(z)) > 0$ ($z \in U$).

For two functions $f$ and $g$, analytic in $U$, we say that the function $f$ is subordinate to $g$ in $U$, and write $f(z) \prec g(z)$, $(z \in U)$, if there exists a Schwarz function $!$, which is analytic in $U$ with $! (0) = 0$ and $|! (z)| < 1$ ($z \in U$) such that $f(z) = g (\omega(z))$, $(z \in U)$. Indeed, it is known that $f(z) \prec g(z)$, $(z \in U) \implies f(0) = g(0)$ and $f(U) \subset g(U)$. Furthermore, if the function $g$ is univalent in $U$, then we have the following equivalence:

$$f(z) \prec g(z) \quad (z \in U) \iff f(0) = g(0) \text{ and } f(U) \subset g(U).$$

In a recent paper, El-Ashwah [6] defined the multiplier transform $D_{n,l}^{n,l}$ of functions $f \in \Sigma_p$ by

$$(1.2) \quad D_{n,l}^{n,l} f(z) := \lambda^{-p} + \sum_{k=0}^{\infty} \left( \frac{\lambda + l(k + p)}{\lambda} \right)^n a_k z^k$$

($z \in U^*; \lambda > 0; l \geq 0; n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}; p \in \mathbb{N}$).

It should be remarked that the operators $D_{n,1}^{n,1}$ and $D_{1,1}^{n,1}$ are the multiplier transforms introduced and investigated, respectively, by Sarangi and Uralegaddi [14], and Uralegaddi and Somanatha [18, 19]. Analogous to $D_{\lambda,p}^{n,l}$, we here define a new multiplier transform $I_{\lambda,p,\mu}^{n,l}$ as follows.

By setting

$$(1.3) \quad I_{\lambda,p,\mu}^{n,l} f(z) := \lambda^{-p} + \sum_{k=0}^{\infty} \left( \frac{\lambda + l(k + p)}{\lambda} \right)^n z^k,$$

($z \in U^*; n, l \geq 0; \lambda > 0; p \in \mathbb{N}$),

we define a new function $f_{\lambda,p,\mu}^{n,l}(z)$ in terms of the Hadamard product (or convolution):
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\[ f_{n; l; p}^n(z) = \frac{1}{z(1-z)^\mu} \quad (z \in \mathbb{U}; \; \lambda, \mu > 0; \; n, l \geq 0; \; p \in \mathbb{N}). \]

Then, analogous to \( D_{n; l; p}^n \), we have

\[ I_{n; l; p}^n f(z) := f_{n; l; p}^n(z) = f(z) ; \quad (z \in \mathbb{U}), \]

where (and throughout this paper unless otherwise mentioned) the parameters \( n, l, p, \lambda \) and \( \mu \) are constrained as follows:

\[ n \geq 0; \; l \geq 0; \; p \in \mathbb{N}, \; \lambda > 0 \text{ and } \mu > 0. \]

We can easily find from (1.3), (1.4) and (1.5) that

\[ I_{n; l; p}^n f(z) = z^{-p} + \sum_{k=0}^{\infty} \left( \frac{(\mu)_{k+1}}{(k+1)!} \left( \frac{\lambda}{\lambda + l(k+p)} \right)^n a_k z^k, \quad (z \in \mathbb{U}), \]

where \((\mu)_k\) is the Pochhammer symbol defined by

\[ (\mu)_k := \begin{cases} 1, & (k = 0), \\ \mu(\mu + 1) \cdots (\mu + k - 1), & (k \in \mathbb{N}). \end{cases} \]

Clearly, the operator \( I_{n; l; p}^n f(z) \) is the well-known Cho-Kwon-Srivastava operator (see, for more details, [2, 3, 8, 13, 15]).

It is readily verified from (1.6) that

\[ l z \left( I_{n; l; p}^{n+1} f \right)'(z) = \lambda I_{n; l; p}^n f(z) - (\lambda + p l) I_{n; l; p}^{n+1} f(z), \]

and

\[ z \left( I_{n; l; p}^n f \right)'(z) = \mu I_{n; l; p}^n f(z) - (\mu + 1) I_{n; l; p}^{n+1} f(z). \]

By making use of the principle of subordination between analytic functions, we introduce the subclasses \( \mathcal{M}S_p^* (\eta; \phi), \mathcal{M}K_p (\eta; \phi), \mathcal{M}C_p (\eta; \delta; \phi, \psi) \) and \( \mathcal{MQ}C_p (\eta; \delta; \phi, \psi) \) of the class \( \Sigma_p \) which are defined by

\[ \mathcal{M}S_p^* (\eta; \phi) := \left\{ f \in \Sigma_p : \; \frac{1}{p-\eta} \left( -zf'(z) - \eta \right) \prec \phi(z), \left( \phi \in \mathcal{P} ; \; 0 \leq \eta < p ; \; z \in \mathbb{U} \right) \right\}, \]

\[ \mathcal{M}K_p (\eta; \phi) := \left\{ f \in \Sigma_p : \; \frac{1}{p-\eta} \left( -1 + \frac{zf''(z)}{f'(z)} - \eta \right) \prec \phi(z), \left( \phi \in \mathcal{P} ; \; 0 \leq \eta < p ; \; z \in \mathbb{U} \right) \right\}, \]

\[ \mathcal{M}C_p (\eta; \delta; \phi, \psi) := \left\{ f \in \Sigma_p : \; \exists g \in \mathcal{M}S_p^* (\eta; \phi) \right. \]

such that \( \frac{1}{p-\delta} \left( -zf'(z) + \delta \right) \prec \psi(z) \),

\[ (\phi, \psi \in \mathcal{P}; \; 0 \leq \eta, \delta < p; \; z \in \mathbb{U}) \right\}, \]
and

$$\mathcal{MQC}_p(\eta, \delta; \phi, \psi) := \left\{ f \in \Sigma_p : \exists g \in \mathcal{MK}_p(\eta; \phi) \right\},$$

such that

$$\begin{align*}
\frac{1}{p-\delta} \left( \frac{(zf'(z))'}{g'(z)} - \delta \right) &<& \psi(z), (\phi, \psi) \in \mathcal{P}; 0 \leq \eta, \delta < p; z \in \mathbb{U} \end{align*}.$$ 

Indeed, the above mentioned function classes are generalizations of the general meromorphic starlike, meromorphic convex, meromorphic close-to-convex and meromorphic quasi-convex functions in analytic function theory (see, for details, [1, 7, 11, 12, 16, 17, 20, 21, 22]).

Next, by using the operator defined by (1.6), we define the following subclasses $\mathcal{MS}^{n,l}_{\lambda, p, \mu}(\eta; \phi)$, $\mathcal{MK}^{n,l}_{\lambda, p, \mu}(\eta; \phi)$, $\mathcal{MC}^{n,l}_{\lambda, p, \mu}(\eta, \delta; \phi, \psi)$ and $\mathcal{MQC}^{n,l}_{\lambda, p, \mu}(\eta, \delta; \phi, \psi)$ of the class $\Sigma_p$:

$$\begin{align*}
\mathcal{MS}^{n,l}_{\lambda, p, \mu}(\eta; \phi) &:= \left\{ f \in \Sigma_p : T^{n,l}_{\lambda, p, \mu} f \in \mathcal{MS}^*_p(\eta; \phi) \right\}, \\
\mathcal{MK}^{n,l}_{\lambda, p, \mu}(\eta; \phi) &:= \left\{ f \in \Sigma_p : T^{n,l}_{\lambda, p, \mu} f \in \mathcal{MK}_p(\eta; \phi) \right\}, \\
\mathcal{MC}^{n,l}_{\lambda, p, \mu}(\eta, \delta; \phi, \psi) &:= \left\{ f \in \Sigma_p : T^{n,l}_{\lambda, p, \mu} f \in \mathcal{MC}_p(\eta, \delta; \phi, \psi) \right\}, \\
\mathcal{MQC}^{n,l}_{\lambda, p, \mu}(\eta, \delta; \phi, \psi) &:= \left\{ f \in \Sigma_p : T^{n,l}_{\lambda, p, \mu} f \in \mathcal{MQC}_p(\eta, \delta; \phi, \psi) \right\}.
\end{align*}$$

and

Clearly, we know that

$$f \in \mathcal{MK}^{n,l}_{\lambda, p, \mu}(\eta; \phi) \iff -zf' \in \mathcal{MS}^{n,l}_{\lambda, p, \mu}(\eta; \phi),$$

and

$$f \in \mathcal{MQC}^{n,l}_{\lambda, p, \mu}(\eta, \delta; \phi, \psi) \iff -zf' \in \mathcal{MC}^{n,l}_{\lambda, p, \mu}(\eta, \delta; \phi, \psi).$$

In order to prove our main results, we need the following definition and lemmas.

**Definition 1.** (See [10]) Denote by $Q$ the set of all functions $f$ that are analytic and injective on $\mathbb{U} - E(f)$, where

$$E(f) = \left\{ \varepsilon \in \partial \mathbb{U} : \lim_{z \to \varepsilon} f(z) = \infty \right\},$$

and such that $f'(\varepsilon) \neq 0$ for $\varepsilon \in \partial \mathbb{U} - E(f)$.

**Lemma 1** ([5]). Let $\kappa, \vartheta \in \mathbb{C}$. Suppose also that $m$ is convex and univalent in $\mathbb{U}$ with

$$m(0) = 1, \quad \text{and} \quad \Re(\kappa m(z) + \vartheta) > 0, \quad (z \in \mathbb{U}).$$
If \( u \) is analytic in \( U \) with \( u(0) = 1 \), then the subordination

\[
    u(z) + \frac{zu'(z)}{nu(z) + \vartheta} \prec m(z), \quad (z \in U)
\]

implies that

\[
    u(z) \prec m(z), \quad (z \in U).
\]

**Lemma 2 ([9]).** Let \( h \) be convex univalent in \( U \) and \( \zeta \) be analytic in \( U \) with

\[
    \Re(\zeta(z)) \geq 0, \quad (z \in U).
\]

If \( q \) is analytic in \( U \) and \( q(0) = h(0) \), then the subordination

\[
    q(z) + \zeta(z)q'(z) \prec h(z), \quad (z \in U)
\]

implies that

\[
    q(z) \prec h(z), \quad (z \in U).
\]

The main purpose of the present paper is to investigate some inclusion relationships and integral-preserving properties of the subclasses

\[
    \mathcal{MS}^{n,l}_{\lambda,p,\mu}(\eta; \phi), \quad \mathcal{MK}^{n,l}_{\lambda,p,\mu}(\eta; \phi), \quad \mathcal{MC}^{n,l}_{\lambda,p,\mu}(\eta; \delta; \phi, \psi) \quad \text{and} \quad \mathcal{MQC}^{n,l}_{\lambda,p,\mu}(\eta; \delta; \phi, \psi)
\]

of meromorphic functions involving the operator \( I^{n,l}_{\lambda,p,\mu} \). Several subordination and superordination results involving this operator are also investigated.

2. The main inclusion relationships

We begin by presenting our first inclusion relationship given by Theorem 1 below.

**Theorem 1.** Let \( 0 \leq \eta < p \) and \( \phi \in \mathcal{P} \) with

\[
    \max_{z \in U} \{ \Re(\phi(z)) \} < \min \left\{ \frac{\mu + 1 - \eta}{p - \eta}, \frac{\lambda + pl - \eta l}{(p - \eta)l} \right\}, \quad (z \in U).
\]

Then

\[
    \mathcal{MS}^{n,l}_{\lambda,p,\mu+1}(\eta; \phi) \subset \mathcal{MS}^{n,l}_{\lambda,p,\mu}(\eta; \phi) \subset \mathcal{MS}^{n+1,l}_{\lambda,p,\mu}(\eta; \phi).
\]

**Proof.** We first prove that

\[
    \mathcal{MS}^{n,l}_{\lambda,p,\mu+1}(\eta; \phi) \subset \mathcal{MS}^{n,l}_{\lambda,p,\mu}(\eta; \phi).
\]

Let \( f \in \mathcal{MS}^{n,l}_{\lambda,p,\mu+1}(\eta; \phi) \) and suppose that

\[
    b(z) := \frac{1}{p - \eta} \left( -\frac{z}{I^{n,l}_{\lambda,p,\mu}f(z)} - \eta \right),
\]

where

\[
    I^{n,l}_{\lambda,p,\mu}f(z) := \int_{0}^{z} \frac{(\lambda + n - 1)(\lambda + n + p - 2\eta)}{(\lambda + n + l - 2\eta)(\lambda + n + l + p - 2\eta)} f(w)^{p - \eta - 1} dw.
\]
where \( h \) is analytic in \( U \) with \( h(0) = 1 \). Combining (1.8) and (2.3), we find that
\[
I_{\nu, l}^{m, l} \frac{T_{\lambda, p, \mu + 1} f(z)}{I_{\lambda, p, \mu} f(z)} = -(p - \eta)h(z) - \eta + \mu + 1.
\]

Taking the logarithmical differentiation on both sides of (2.4) and multiplying the resulting equation by \( z \), we get
\[
1
\frac{z}{p - \eta} \left( - \frac{z T_{\lambda, p, \mu + 1} f(z)}{I_{\lambda, p, \mu} f(z)} - \eta \right)
\]
\[
= h(z) + \frac{zh'(z)}{-(p - \eta)h(z) - \eta + \mu + 1} < \phi(z).
\]

By virtue of (2.1), an application of Lemma 1 to (2.5) yields \( h < \phi \), that is \( f \in MS_{\lambda, p, \mu}^{n, l} (\eta; \phi) \). Thus, the assertion (2.2) of Theorem 1 holds.

To prove the second part of Theorem 1, we assume that \( f \in MS_{\lambda, p, \mu}^{n, l} (\eta; \phi) \) and set
\[
g(z) := \frac{1}{p - \eta} \left( - \frac{z T_{\lambda, p, \mu + 1} f(z)}{I_{\lambda, p, \mu} f(z)} - \eta \right),
\]
where \( g \) is analytic in \( U \) with \( g(0) = 1 \). Combining (1.7), (2.1) and (2.6) and applying the similar method of proof of the first part, we get \( g < \phi \), that is \( f \in MS_{\lambda, p, \mu}^{n + 1, l} (\eta; \phi) \). Therefore, the second part of Theorem 1 also holds. The proof of Theorem 1 is evidently completed.

**Theorem 2.** Let \( 0 \leq \eta < p \) and \( \phi \in \mathcal{P} \) with (2.1) holds. Then
\[
\mathcal{M}K_{\lambda, p, \mu + 1}^{n, l} (\eta; \phi) \subset \mathcal{M}K_{\lambda, p, \mu}^{n, l} (\eta; \phi) \subset \mathcal{M}K_{\lambda, p, \mu}^{n + 1, l} (\eta; \phi).
\]

**Proof.** In view of (1.9) and Theorem 1, we find that
\[
f \in \mathcal{M}K_{\lambda, p, \mu + 1}^{n, l} (\eta; \phi) \iff I_{\lambda, p, \mu + 1}^{n, l} f \in \mathcal{M}K_{\lambda, p, \mu}^{n, l} (\eta; \phi)
\]
\[
\iff -z I_{\lambda, p, \mu + 1}^{n, l} f' \in \mathcal{M}S_{\lambda, p, \mu}^{n, l} (\eta; \phi)
\]
\[
\iff I_{\lambda, p, \mu + 1}^{n, l} (-z f') \in \mathcal{M}S_{\lambda, p, \mu}^{n, l} (\eta; \phi)
\]
\[
\iff -zf' \in \mathcal{M}S_{\lambda, p, \mu + 1}^{n, l} (\eta; \phi)
\]
\[
\iff -zf' \in \mathcal{M}S_{\lambda, p, \mu}^{n, l} (\eta; \phi)
\]
\[
\iff I_{\lambda, p, \mu}^{n, l} (zf') \in \mathcal{M}S_{\lambda, p, \mu}^{n, l} (\eta; \phi)
\]
\[
\iff I_{\lambda, p, \mu}^{n, l} f \in \mathcal{M}K_{\lambda, p, \mu}^{n, l} (\eta; \phi)
\]
\[
\iff f \in \mathcal{M}K_{\lambda, p, \mu}^{n, l} (\eta; \phi),
\]
\[(2.7)\]
and

\[ f \in \mathcal{M}^{n,l}_{\lambda,p,\mu}(\eta; \phi) \iff -zf' \in \mathcal{M}^{n,l}_{\lambda,p,\mu}(\eta; \phi) \]
\[ \iff -zf' \in \mathcal{M}^{n+1,l}_{\lambda,p,\mu}(\eta; \phi) \]
\[ \iff \mathcal{I}_{\lambda,p,\mu}^{n+1,f}(-zf') \in \mathcal{M}^{n,l}_{\lambda,p,\mu}(\eta; \phi) \]
\[ \iff \mathcal{I}_{\lambda,p,\mu}^{n+1,f} \in \mathcal{M}^{n,l}_{\lambda,p,\mu}(\eta; \phi) \]
\[ \iff f \in \mathcal{M}^{n+1,l}_{\lambda,p,\mu}(\eta; \phi). \]

(2.8)

Combining (2.7) and (2.8), we deduce that the assertion of Theorem 2 holds.

**Theorem 3.** Let \( 0 \leq \eta < p, \ 0 \leq \delta < p \) and \( \phi \in \mathcal{P} \) with (2.1) holds. Then

\[ \mathcal{M}^{n,l}_{\lambda,p,\mu+1}(\eta; \delta; \phi, \psi) \subset \mathcal{M}^{n,l}_{\lambda,p,\mu}(\eta; \delta; \phi, \psi) \subset \mathcal{M}^{n+1,l}_{\lambda,p,\mu}(\eta; \delta; \phi, \psi). \]

**Proof.** We begin by proving that

(2.9)

\[ \mathcal{M}^{n,l}_{\lambda,p,\mu+1}(\eta; \delta; \phi, \psi) \subset \mathcal{M}^{n,l}_{\lambda,p,\mu}(\eta; \delta; \phi, \psi). \]

Let \( f \in \mathcal{M}^{n,l}_{\lambda,p,\mu+1}(\eta; \delta; \phi, \psi) \). Then, by definition, we know that

(2.10)

\[ \frac{1}{p - \delta} \left( -z \frac{\mathcal{I}^{n,l}_{\lambda,p,\mu+1}(z)}{\mathcal{I}^{n,l}_{\lambda,p,\mu+1}(z)} - \delta \right) \prec \psi(z), \quad (z \in \mathbb{U}) \]

with \( g \in \mathcal{M}^{n,l}_{\lambda,p,\mu+1}(\eta; \phi) \).

Moreover, by Theorem 1, we know that \( g \in \mathcal{M}^{n,l}_{\lambda,p,\mu}(\eta; \phi) \), which implies that

(2.11)

\[ q(z) := \frac{1}{p - \eta} \left( -z \frac{\mathcal{I}^{n,l}_{\lambda,p,\mu}(z)}{\mathcal{I}^{n,l}_{\lambda,p,\mu}(z)} - \eta \right) \prec \phi(z), \quad (z \in \mathbb{U}). \]

We now suppose that

(2.12)

\[ p(z) := \frac{1}{p - \delta} \left( -z \frac{\mathcal{I}^{n,l}_{\lambda,p,\mu}(z)}{\mathcal{I}^{n,l}_{\lambda,p,\mu}(z)} - \delta \right), \quad (z \in \mathbb{U}), \]

where \( p \) is analytic in \( \mathbb{U} \) with \( p(0) = 1 \). Combining (1.8) and (2.12), we find that

(2.13)

\[-(p - \delta)p(z) + \delta \mathcal{I}^{n,l}_{\lambda,p,\mu}(z) = \mu \mathcal{I}^{n,l}_{\lambda,p,\mu+1}(z) - (\mu + 1) \mathcal{I}^{n,l}_{\lambda,p,\mu}(z). \]
Differentiating both sides of (2.13) with respect to \( z \) and multiplying the resulting equation by \( z \), we get

\[
-(p - \delta)z\frac{\partial f}{\partial z} - [(p - \delta)p(z) + \delta]-(p - \eta)q(z) - \eta + \mu + 1
\]

(2.14)

\[
\mu \left( \frac{T^{n,l}_{\lambda,p+1}(z)}{T^{n,l}_{\lambda,p+1}(z)} \right) \frac{\partial f}{\partial z}.
\]

In view of (1.8), (2.11) and (2.14), we conclude that

\[
\frac{1}{p - \delta} \left( -z \left( \frac{T^{n,l}_{\lambda,p+1}(z)}{T^{n,l}_{\lambda,p+1}(z)} \right) - \delta \right)
\]

(2.15)

\[
p(z) + \frac{z\frac{\partial f}{\partial z}}{-(p - \eta)q(z) - \eta + \mu + 1} \prec \psi(z), \quad (z \in \mathbb{U}).
\]

By noting that (2.1) holds and

\[
q(z) \prec \phi(z), \quad (z \in \mathbb{U}),
\]

we know that

\[
\Re(-(p - \eta)q(z) - \eta + \mu + 1) > 0, \quad (z \in \mathbb{U}).
\]

Thus, an application of Lemma 2 to (2.15) yields

\[
p(z) \prec \psi(z), \quad (z \in \mathbb{U}),
\]

that is, that \( f \in \mathcal{MC}^{n,l}_{\lambda,p+1}(\eta, \delta; \phi, \psi) \), which implies that the assertion (2.9) of Theorem 3 holds.

By virtue of (1.7) and (2.1), and making use of the similar arguments of the details above, we deduce that

\[
\mathcal{MC}^{n,l}_{\lambda,p+1}(\eta, \delta; \phi, \psi) \subset \mathcal{MC}^{n+1,l}_{\lambda,p+1}(\eta, \delta; \phi, \psi).
\]

The proof of Theorem 3 is thus completed.

**Theorem 4.** Let \( 0 \leq \eta < p \), \( 0 \leq \delta < p \) and \( \phi, \psi \in \mathcal{P} \) with (2.1) holds. Then

\[
\mathcal{MQC}^{n,l}_{\lambda,p+1}(\eta, \delta; \phi, \psi) \subset \mathcal{MC}^{n,l}_{\lambda,p+1}(\eta, \delta; \phi, \psi) \subset \mathcal{MC}^{n+1,l}_{\lambda,p+1}(\eta, \delta; \phi, \psi).
\]

**Proof.** In view of (1.10) and Theorem 3, and by similarly applying the method of proof of Theorem 2, we conclude that the assertion of Theorem 4 holds.
3. A set of integral-preserving properties

In this section, we derive some integral-preserving properties involving two families of integral operators.

**Theorem 5.** Let \( f \in \mathcal{MS}^{n,l}_{\lambda,p,\mu}(\eta; \phi) \) with \( \phi \in \mathcal{P} \) and

\[
\mathcal{R}(\phi(z)) < \frac{\mathcal{R}(\nu) - \eta}{p - \eta}, \quad (z \in \mathbb{U}; \mathcal{R}(<p).
\]

Then the integral operator \( F_\nu(f) \) defined by

\[
F_\nu(f)(z) := \frac{\nu - p}{z^\nu} \int_0^z t^{\nu-1} f(t) dt, \quad (z \in \mathbb{U}; \mathcal{R}(\nu) > p)
\]

belongs to the class \( \mathcal{MS}^{n,l}_{\lambda,p,\mu}(\eta; \phi) \).

**Proof.** Let \( f \in \mathcal{MS}^{n,l}_{\lambda,p,\mu}(\eta; \phi) \). Then, from (3.2), we find that

\[
z \left( T^{n,l}_{\lambda,p,\mu} F_\nu(f) \right)'(z) + \nu T^{n,l}_{\lambda,p,\mu} F_\nu(f)(z) = (\nu - p) T^{n,l}_{\lambda,p,\mu} f(z).
\]

By setting

\[
P(z) := \frac{1}{p - \eta} \left( -z \left( T^{n,l}_{\lambda,p,\mu} F_\nu(f) \right)'(z) - \eta \right),
\]

we observe that \( P \) is analytic in \( \mathbb{U} \) with \( P(0) = 1 \). It follows from (3.3) and (3.4) that

\[
-(p - \eta)P(z) - \eta + \nu = (\nu - p) \frac{T^{n,l}_{\lambda,p,\mu} f(z)}{T^{n,l}_{\lambda,p,\mu} F_\nu(f)(z)}.
\]

Differentiating both sides of (3.5) with respect to \( z \) logarithmically and multiplying the resulting equation by \( z \), we get

\[
P(z) + \frac{zP'(z)}{(p - \eta)P(z) - \eta + \nu} = \frac{1}{p - \eta} \left( -\frac{z \left( T^{n,l}_{\lambda,p,\mu} f(z) \right)'}{T^{n,l}_{\lambda,p,\mu} f(z)} - \eta \right) - \phi(z), \quad (z \in \mathbb{U}).
\]

Since (3.1) holds, an application of Lemma 1 to (3.6) yields

\[
\frac{1}{p - \eta} \left( -\frac{z \left( T^{n,l}_{\lambda,p,\mu} F_\nu(f) \right)'(z)}{T^{n,l}_{\lambda,p,\mu} F_\nu(f)(z)} - \eta \right) - \phi(z),
\]

which implies that the assertion of Theorem 5 holds. \( \Box \)
Theorem 6. Let $f \in \mathcal{M}^{n, l}_{\lambda, \mu}(\eta; \phi)$ with $\phi \in \mathcal{P}$ and (3.1) holds. Then the integral operator $F_\nu(f)$ defined by (3.2) belongs to the class $\mathcal{M}^{n, l}_{\lambda, \mu}(\eta; \phi)$.

Proof. By virtue of (1.9) and Theorem 5, we easily find that

$$f \in \mathcal{M}^{n, l}_{\lambda, \mu}(\eta; \phi) \iff -zf' \in \mathcal{M}^{n, l}_{\lambda, \mu}(\eta; \phi)$$

$$\Rightarrow F_\nu(-zf') \in \mathcal{M}^{n, l}_{\lambda, \mu}(\eta; \phi)$$

$$\iff -z(F_\nu(f))' \in \mathcal{M}^n_p(\eta; \phi)$$

$$\iff F_\nu(f) \in \mathcal{M}^{n, l}_{\lambda, \mu}(\eta; \phi).$$

The proof of Theorem 6 is evidently completed.

Theorem 7. Let $f \in \mathcal{M}^{n, l}_{\lambda, \mu}(\eta, \delta; \phi, \psi)$ with $\phi \in \mathcal{P}$ and (3.1) holds. Then the integral operator $F_\nu(f)$ defined by (3.2) belongs to the class $\mathcal{M}^{n, l}_{\lambda, \mu}(\eta, \delta; \phi, \psi)$.

Proof. Let $f \in \mathcal{M}^{n, l}_{\lambda, \mu}(\eta, \delta; \phi, \psi)$. Then, by definition, we know that there exists a function $g \in \mathcal{M}^{n, l}_p(\eta; \phi)$ such that

$$H(z) := \frac{1}{p - \eta} \left( -z \left( T^{n, l}_{\lambda, \mu} f \right)'(z) - \eta \right) < \psi(z), \quad (z \in U).$$

Since $g \in \mathcal{M}^{n, l}_p(\eta; \phi)$, by Theorem 5, we easily find that $F_\nu(g) \in \mathcal{M}^{n, l}_p(\eta; \phi)$, which implies that

$$H(z) := \frac{1}{p - \eta} \left( -z \left( T^{n, l}_{\lambda, \mu} F_\nu(g) \right)'(z) - \eta \right) < \phi(z).$$

We now set

$$Q(z) := \frac{1}{p - \delta} \left( -z \left( T^{n, l}_{\lambda, \mu} F_\nu(f) \right)'(z) - \delta \right),$$

where $Q$ is analytic in $U$ with $Q(0) = 1$. From (3.3) and (3.9), we get

$$-[(p - \delta)Q(z) + \delta]T^{n, l}_{\lambda, \mu} F_\nu(g)(z) + \nu T^{n, l}_{\lambda, \mu} F_\nu(f)(z) = (\nu - p)T^{n, l}_{\lambda, \mu} f(z).$$

Combining (3.8), (3.9) and (3.10), we find that

$$-(p - \delta)zQ'(z) - [(p - \delta)Q(z) + \delta][-(p - \eta)H(z) - \eta + \nu]$$

$$= (\nu - p) z \left( T^{n, l}_{\lambda, \mu} f \right)'(z) \frac{1}{T^{n, l}_{\lambda, \mu} F_\nu(g)(z)}. $$
By virtue of (1.8), (3.8) and (3.11), we deduce that
\[
\frac{1}{p - \delta} \left( -z \left( \frac{I_{n, l}^{\nu, \eta} g(z)}{I_{n, l}^{\nu, \eta} f(z)} \right)'(z) \right)
\]
\[
= Q(z) + \frac{zQ'(z)}{(p - \eta)H(z) - \eta + \nu} < \psi(z), \quad (z \in \mathbb{U}).
\]  
(3.12)

The remainder of the proof of Theorem 7 is much akin to that of Theorem 3. We, therefore, choose to omit the analogous details involved. We thus find that
\[
Q(z) < \psi(z), \quad (z \in \mathbb{U}),
\]
which implies that \( F_\nu(f) \in MC_{\nu, \eta; \phi}^n \). The proof of Theorem 7 is thus completed.

**Theorem 8.** Let \( f \in MQC_{\nu, \eta; \phi}^n \) with \( \nu \in \mathcal{P} \) and (3.1) holds. Then the integral operator \( F_\nu(f) \) defined by (3.2) belongs to the class \( MQC_{\nu, \eta; \phi}^n \).

**Proof.** In view of (1.10) and Theorem 7, and by similarly applying the method of proof of Theorem 6, we deduce that the assertion of Theorem 8 holds.

**Theorem 9.** Let \( f \in MS_{\nu, \eta; \phi}^n \) with \( \phi \in \mathcal{P} \) and
\[
\Re(\sigma - \eta \xi - (p - \eta)\xi \phi(z)) > 0, \quad (z \in \mathbb{U}; \ \xi \neq 0).
\]  
(3.13)

Then the function \( I_{n, l}^{\nu, \eta} K_{\xi}^n f(z) \in \Sigma_\eta \) defined by
\[
I_{n, l}^{\nu, \eta} K_{\xi}^n f(z) := \frac{1}{z^\sigma} \int_0^z t^{\sigma - 1} (I_{n, l}^{\nu, \eta} f(t))^{\xi} dt, \quad (z \in \mathbb{U}^*; \ \xi \neq 0)
\]  
(3.14)

belongs to the class \( MS_{\nu, \eta; \phi}^n \).

**Proof.** Let \( f \in MS_{\nu, \eta; \phi}^n \) and suppose that
\[
\mathcal{M}(z) := \frac{1}{p - \eta} \left( -z \left( \frac{I_{n, l}^{\nu, \eta} K_{\xi}^n f(z)}{I_{n, l}^{\nu, \eta} K_{\xi}^n f(z)} \right)'(z) \right), \quad (z \in \mathbb{U}).
\]  
(3.15)

Combining (3.14) and (3.15), we have
\[
\sigma - \eta \xi - (p - \eta)\xi \mathcal{M}(z) = (\sigma - p \xi) \left( \frac{I_{n, l}^{\nu, \eta} f(z)}{I_{n, l}^{\nu, \eta} K_{\xi}^n f(z)} \right)^\xi.
\]  
(3.16)
Making use of (3.14), (3.15) and (3.16), we get

\[
M(z) + \frac{z M'(z)}{\sigma - \eta \xi - (p - \eta) \xi M(z)} = \frac{1}{p - \eta} \left( -z \left( \frac{T^{n,l}_{\lambda,p,\mu} f'(z)}{T^{n,l}_{\lambda,p,\mu} f(z)} - \eta \right) \right) < \phi(z), \quad (z \in \mathbb{U}).
\]

(3.17)

Since (3.13) holds, an application of Lemma 1 to (3.17) yields

\[
M(z) < \phi(z), \quad (z \in \mathbb{U}),
\]

that is, that \( T^{n,l}_{\lambda,p,\mu} K^\sigma_\xi (f) \in MS^{n,l}_{\lambda,p,\mu}(\eta; \phi) \). We thus complete the proof of Theorem 9.

\[ \square \]

**Theorem 10.** Let \( f \in MK^{n,l}_{\lambda,p,\mu}(\eta; \phi) \) with \( \phi \in \mathcal{P} \) and (3.13) holds. Then the function \( T^{n,l}_{\lambda,p,\mu} K^\sigma_\xi (f) \in \Sigma_p \) defined by (3.14) belongs to the class \( MK^{n,l}_{\lambda,p,\mu}(\eta; \phi) \).

**Proof.** By virtue of (1.9) and Theorem 9, and by similarly applying the method of proof of Theorem 6, we conclude that the assertion of Theorem 10 holds.

\[ \square \]

**Theorem 11.** Let \( f \in MC^{n,l}_{\lambda,p,\mu}(\eta, \delta; \phi, \psi) \) with \( \phi \in \mathcal{P} \) and (3.13) holds.

Then the function \( T^{n,l}_{\lambda,p,\mu} K^\sigma_\xi (f) \in \Sigma_p \) defined by (3.14) belongs to the class \( MC^{n,l}_{\lambda,p,\mu}(\eta, \delta; \phi, \psi) \).

**Proof.** Let \( f \in MC^{n,l}_{\lambda,p,\mu}(\eta, \delta; \phi, \psi) \). Then, by definition, we know that there exists a function \( g \in MS^*_p(\eta; \phi) \) such that (3.7) holds. Since \( g \in MS^*_p(\eta; \phi) \), by Theorem 9, we easily find that \( T^{n,l}_{\lambda,p,\mu} K^\sigma_\xi (g) \in MS^*_p(\eta; \phi) \), which implies that

\[
\Re(z) := \frac{1}{p - \eta} \left( -z \left( \frac{T^{n,l}_{\lambda,p,\mu} K^\sigma_\xi (g)'(z)}{T^{n,l}_{\lambda,p,\mu} K^\sigma_\xi (g)(z)} - \eta \right) \right) < \phi(z).
\]

(3.18)

We now set

\[
\mathbb{L}(z) := \frac{1}{p - \delta} \left( -z \left( \frac{T^{n,l}_{\lambda,p,\mu} K^\sigma_\xi (f)'(z)}{T^{n,l}_{\lambda,p,\mu} K^\sigma_\xi (f)(z)} - \delta \right) \right),
\]

(3.19)

where \( \mathbb{L} \) is analytic in \( \mathbb{U} \) with \( \mathbb{L}(0) = 1 \). From (3.14) and (3.19), we get

\[
-\xi [(p - \delta) \mathbb{L}(z)] + \delta \mathbb{L}(z) + \delta T^{n,l}_{\lambda,p,\mu} K^\sigma_\xi (g)(z) + \delta T^{n,l}_{\lambda,p,\mu} K^\sigma_\xi (f)(z) = (\delta - p \xi) T^{n,l}_{\lambda,p,\mu} f(z).
\]

(3.20)
Combining (3.18), (3.19) and (3.20), we find that

\[
\frac{\xi(p - \delta)zL'(z) - [(p - \delta)L(z) + \delta][-\eta + \xi + \delta]}{\sigma - p\xi} = \sigma - p\xi.
\]

(3.21)

By virtue of (1.8), (3.18) and (3.21), we deduce that

\[
\frac{1}{p - \delta} \left( \frac{z}{1 + \frac{L'(z)}{L(z)}} - \delta \right)
\]

(3.22)

which implies that \( L(z) \prec \psi(z), \quad (z \in \mathbb{U}), \)

where

\[
(4.2) \quad \varphi := \frac{l^2 + \lambda^2 - |l^2 - \lambda^2|}{4l\lambda},
\]

The remainder of the proof of Theorem 11 is similar to that of Theorem 3. We, therefore, choose to omit the analogous details involved. We thus find that

\[
L(z) \prec \psi(z), \quad (z \in \mathbb{U}),
\]

which implies that \( I_{\lambda, p, \mu}^{n, l} \in \mathcal{MC}_{\lambda, p, \mu}(\eta, \delta; \phi, \psi). \) The proof of Theorem 11 is thus completed.

**Theorem 12.** Let \( f \in \mathcal{MQC}_{\lambda, p, \mu}^{n, l}(\eta, \delta; \phi, \psi) \) with \( \phi \in \mathcal{P} \) and (3.13) holds. Then the function \( I_{\lambda, p, \mu}^{n, l} \in \Sigma_p \) defined by (3.14) belongs to the class \( \mathcal{MC}_{\lambda, p, \mu}^{n, l}(\eta, \delta; \phi, \psi). \)

**Proof.** By virtue of (1.10) and Theorem 11, and by similarly applying the method of proof of Theorem 6, we deduce that the assertion of Theorem 12 holds.

**4. Subordination and superordination results**

Finally, we derive some subordination and superordination results associated with the operator \( I_{\lambda, p, \mu}^{n, l}. \) The proofs are much akin to that of the results obtained by Cho et al. [4], we here choose to omit the details involved.

**Corollary 1.** Let \( f, g \in \Sigma_p \) and \( l > 0. \) If

\[
(4.1) \quad \Re\left(1 + \frac{z\varphi''(z)}{\varphi'(z)}\right) > -\varrho, \quad (z \in \mathbb{U}; \varphi(z) := z^nI_{\lambda, p, \mu}^{n, l}g(z)),
\]

where

\[
(4.2) \quad \varrho := \frac{l^2 + \lambda^2 - |l^2 - \lambda^2|}{4l\lambda},
\]
then the following subordination relationship
\[ z^p T_{\lambda,p,\mu}^n f(z) \prec z^p T_{\lambda,p,\mu}^n g(z), \quad (z \in \mathbb{U}) \]
implies that
\[ z^p T_{\lambda,p,\mu}^{n+1} f(z) \prec z^p T_{\lambda,p,\mu}^{n+1} g(z), \quad (z \in \mathbb{U}). \]
Furthermore, the function \( z^p T_{\lambda,p,\mu}^{n+1} g \) is the best dominant.

**Corollary 2.** Let \( f, g \in \Sigma_p \). If
\[ \Re \left( 1 + \frac{z \chi''(z)}{\chi'(z)} \right) > -\varpi, \quad (z \in \mathbb{U}; \chi(z) := z^p T_{\lambda,p,\mu}^n g(z)), \]
where
\[ (4.3) \quad \varpi := \frac{1 + \mu^2 - |1 - \mu^2|}{4\mu}, \]
then the following subordination relationship
\[ z^p T_{\lambda,p,\mu}^n f(z) \prec z^p T_{\lambda,p,\mu}^n g(z), \quad (z \in \mathbb{U}) \]
implies that
\[ z^p T_{\lambda,p,\mu}^n f(z) \prec z^p T_{\lambda,p,\mu}^n g(z), \quad (z \in \mathbb{U}). \]
Furthermore, the function \( z^p T_{\lambda,p,\mu}^n g \) is the best dominant.

If \( f \) is subordinate to \( \mathcal{F} \), then \( \mathcal{F} \) is superordinate to \( f \). We now derive the following superordination results.

**Corollary 3.** Let \( f, g \in \Sigma_p \) and \( l > 0 \). If
\[ \Re \left( 1 + \frac{z \varphi''(z)}{\varphi'(z)} \right) > -\varrho, \quad (z \in \mathbb{U}; \varphi(z) := z^p T_{\lambda,p,\mu}^n g(z)), \]
where \( \varrho \) is given by (4.2), also let the function \( z^p T_{\lambda,p,\mu}^n f \) is univalent in \( \mathbb{U} \) and \( z^p T_{\lambda,p,\mu}^{n+1} f \in Q \), then the following subordination relationship
\[ z^p T_{\lambda,p,\mu}^n g(z) \prec z^p T_{\lambda,p,\mu}^n f(z), \quad (z \in \mathbb{U}) \]
implies that
\[ z^p T_{\lambda,p,\mu}^{n+1} g(z) \prec z^p T_{\lambda,p,\mu}^{n+1} f(z), \quad (z \in \mathbb{U}). \]
Furthermore, the function \( z^p T_{\lambda,p,\mu}^{n+1} g \) is the best subordinant.
Corollary 4. Let \( f, g \in \Sigma_p \). If
\[
\Re \left( 1 + \frac{z \chi''(z)}{\chi'(z)} \right) > -\varpi, \quad (z \in \mathbb{U}; \; \chi(z) := z^p \mathcal{T}^n_{\lambda,p;+1} g(z)),
\]
where \( \varpi \) is given by (4.3), also let the function \( z^p \mathcal{T}^n_{\lambda,p;+1} f \) is univalent in \( \mathbb{U} \) and \( z^p \mathcal{T}^n_{\lambda,p;+1} f \in \mathbb{Q} \), then the following subordination relationship
\[
z^p \mathcal{T}^n_{\lambda,p;+1} g(z) < z^p \mathcal{T}^n_{\lambda,p;+1} f(z), \quad (z \in \mathbb{U})
\]
implies that
\[
z^p \mathcal{T}^n_{\lambda,p;+1} g(z) < z^p \mathcal{T}^n_{\lambda,p;+1} f(z), \quad (z \in \mathbb{U}).
\]
Furthermore, the function \( z^p \mathcal{T}^n_{\lambda,p;+1} g \) is the best subordinant.

Combining the above mentioned subordination and superordination results involving the operator \( \mathcal{T}^n_{\lambda,p;+1} \), we get the following “sandwich-type results”.

Corollary 5. Let \( f, g_k \in \Sigma_p \) (\( k = 1, 2 \)) and \( l > 0 \). If
\[
\Re \left( 1 + \frac{z \varphi''_k(z)}{\varphi'_k(z)} \right) > -\varrho, \quad (z \in \mathbb{U}; \; \varphi_k(z) := z^p \mathcal{T}^n_{\lambda,p;+1} g_k(z) \; (k = 1, 2)),
\]
where \( \varrho \) is given by (4.2), also let the function \( z^p \mathcal{T}^n_{\lambda,p;+1} f \) is univalent in \( \mathbb{U} \) and \( z^p \mathcal{T}^n_{\lambda,p;+1} f \in \mathbb{Q} \), then the subordination relationship
\[
z^p \mathcal{T}^n_{\lambda,p;+1} g_1(z) < z^p \mathcal{T}^n_{\lambda,p;+1} f(z) < z^p \mathcal{T}^n_{\lambda,p;+1} g_2(z), \quad (z \in \mathbb{U})
\]
implies that
\[
z^p \mathcal{T}^n_{\lambda,p;+1} g_1(z) < z^p \mathcal{T}^n_{\lambda,p;+1} f(z) < z^p \mathcal{T}^n_{\lambda,p;+1} g_2(z), \quad (z \in \mathbb{U}).
\]
Furthermore, the functions \( z^p \mathcal{T}^n_{\lambda,p;+1} g_1 \) and \( z^p \mathcal{T}^n_{\lambda,p;+1} g_2 \) are, respectively, the best subordinant and the best dominant.

Corollary 6. Let \( f, g_k \in \Sigma_p \) (\( k = 1, 2 \)). If
\[
\Re \left( 1 + \frac{z \chi''_k(z)}{\chi'_k(z)} \right) > -\varpi, \quad (z \in \mathbb{U}; \; \chi_k(z) := z^p \mathcal{T}^n_{\lambda,p;+1} g_k(z) \; (k = 1, 2)),
\]
where \( \varpi \) is given by (4.3), also let the function \( z^p \mathcal{T}^n_{\lambda,p;+1} f \) is univalent in \( \mathbb{U} \) and \( z^p \mathcal{T}^n_{\lambda,p;+1} f \in \mathbb{Q} \), then the subordination relationship
\[
z^p \mathcal{T}^n_{\lambda,p;+1} g_1(z) < z^p \mathcal{T}^n_{\lambda,p;+1} f(z) < z^p \mathcal{T}^n_{\lambda,p;+1} g_2(z), \quad (z \in \mathbb{U})
\]
implies that
\[
z^p \mathcal{T}^n_{\lambda,p;+1} g_1(z) < z^p \mathcal{T}^n_{\lambda,p;+1} f(z) < z^p \mathcal{T}^n_{\lambda,p;+1} g_2(z), \quad (z \in \mathbb{U}).
\]
Furthermore, the functions \( z^p \mathcal{T}^n_{\lambda,p;+1} g_1 \) and \( z^p \mathcal{T}^n_{\lambda,p;+1} g_2 \) are, respectively, the best subordinant and the best dominant.
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