Generalization of $T$-small submodules

Sahira M. Yaseen
Mathematics Department
College of Science
University of Baghdad
Iraq
Sahira.mahmood@gmail.com

Abstract. Let $R$ be an associative ring with identity and let $M$ be a unitary left $R$-module. A submodule $N$ of $M$ is called, $T$-small in $M$ denoted by $N \ll_T M$, in case for any submodule $X \subseteq M$, $T \subseteq N + X$ implies that $T \subseteq X$. In this paper, we introduce the concept of $GT$-small submodule in $M$. A submodule $N$ of an $R$-module $M$ is called $GT$-small submodule, denoted by $N \ll_{GT} M$, in case for every essential submodule $X$ of $M$, $T \subseteq N + X$ implies that $T \subseteq X$. We introduce and study the concepts $GT$-hollow module, $GT$-lifting modules and $GT$-supplement submodules as a generalization of $T$-hollow module, $T$-lifting modules and $T$-supplement submodules respectively we supply some examples and properties of these modules.

Keywords: $GT$-hollow module, $GT$-lifting module, $T$-small submodule, $GT$-supplement submodules.

1. Introduction

Throughout this paper all rings will be associative with identity and all modules will be unital left modules. Let $R$ be a ring and $M$ be an $R$-module. We will denote a submodule $N$ of $M$ by $N \leq M$. Let $M$ be an $R$-module and $N \leq M$. If $L = M$ for every submodule $L$ of $M$ such that $M = N + L$, then $N$ is called a small submodule of $M$ and denoted by $N \ll M$ [1]. Let $M$ be an $R$-module and $N \leq M$. If there exists a submodule $K$ of $M$ such that $M = N + K$ and $N \cap K = 0$, $N$ is called a direct summand of $M$ and it denoted by $N \ll M$. A submodule $N$ of an $R$-module $M$ is called an essential submodule and denoted by $N \leq_e M$ in case $K \cap N \neq 0$ for every submodule $K \neq 0$. Let $M$ be an $R$-module and $K$ be a submodule of $M$. $K$ is called a $G$-small submodule of $M(K \leq_G M)$ if for every essential submodule $T$ of $M$ with the property $M = K + T$ implies that $T = M$. There are some important properties of $G$-small submodules in [6], [8]. The concept of small submodule has been generalized by some researchers, for this see [7, 2, 8].

In [3] the authors introduced the concept of small submodule with respect to an arbitrary submodule. Recall that a submodule $N$ of $M$ is called, $T$-small in $M$ denoted by $N \ll_T M$, in case for any submodule $X \leq M$, $T \subseteq N + X$ implies that $T \subseteq X$. 
In this paper, we introduce the concept of \( GT \)-small submodule in \( M \) as generalization of \( T \)-small submodule. A submodule \( N \) of an \( R \)-module \( M \) is called \( GT \)-small submodule in \( M \), denoted by \( N \ll_{GT} M \), in case for essential submodule \( X \) of \( M \), \( T \subseteq N + X \) implies that \( T \subseteq X \). It is clear that every \( T \)-small submodule is \( GT \)-small. We show by example that \( GT \)-small submodule of \( M \) need not be \( T \)-small submodule see(1.2). Let \( M \) be a non-zero module and \( T \) be a submodule of \( M \). \( M \) is a \( T \)-hollow module if every submodule \( K \) of \( M \) such that \( T \not\subseteq K \) is a \( T \)-small submodule of \( M \) [3]. We introduce and study the concept of \( GT \)-hollow module as a generalization of \( T \)-hollow module.

\begin{definition}
Let \( T \) be a submodule of an \( R \)-module \( M \). A submodule \( N \) of an \( R \)-module \( M \) is called \( GT \)-small submodule in \( M \), denoted by \( N \ll_{GT} M \), in case for essential submodule \( X \) of \( M \), \( T \subseteq N + X \) implies that \( T \subseteq X \).
\end{definition}

\begin{examples}
1. If \( T = 0 \), then every submodule of \( M \) is \( GT \)-small in \( M \). And if \( T = M \), then \( N \ll_{GM} M \) if and only if \( N \ll_G M \).

2. It is clear that if \( N \) is \( T \)-small submodule of \( M \) then \( N \) is \( GT \)-small submodule in \( M \), but the converse is not true in general. For example, in the \( Z \)-module \( Z_{24} \), let \( T = \{0, 8, 16\} \) and the only essential submodules in \( Z_{24} \) are \( Z_{24}, 2Z_{24} \) and \( 4Z_{24} \), let \( N = 6Z_{24} \) then \( T \subseteq 6Z_{24} + 2Z_{24} \) and \( T \subseteq 2Z_{24} \) also \( T \subseteq 6Z_{24} + 4Z_{24} \) and \( T \subseteq 4Z_{24} \). Then the submodule \( 6Z_{24} \) is \( GT \)-small submodule. But is not \( T \)-small, since if \( X = 3Z_{24}, T \subseteq 6Z_{24} + 3Z_{24} \) but \( T \) is not submodule of \( 3Z_{24} \).

3. Let \( Z \) be the ring of integers. It is easy to see that \( (0) \) is the only small submodule of \( Z \) and also for any nonzero integer \( m \), the submodule \( (0) \) is the only \( GmZ \)-small submodule of \( Z \).
\end{examples}

\begin{proposition}
Let \( M \) be an \( R \)-module, \( K \leq L \leq M \) and \( L \leq_e M \) if \( K \ll_{GT} M \), then \( K \ll_{GT} L \).
\end{proposition}

\begin{proof}
Let \( T \subseteq K + X, X \leq_e L \) and \( L \leq_e M \) then \( X \leq_e M \) [9], \( K \ll_{GT} M \), then \( T \subseteq X \) so \( K \ll_{GT} L \).
\end{proof}
Proposition 2.4. Let $M$ be an $R$-module with submodules $N \leq K \leq M$ and $T \leq K$. If $N \ll_{GT} K$, then $N \ll_{GT} M$.

Proof. Suppose that $T \subseteq N + X$, for some $X \leq_e M$. Then $T \subseteq (N + X) \cap K = N + (X \cap K)$. Since $N \ll_{GT} K$, $X \leq_e M$ and $K \leq_e K$, we have $T \subseteq X \cap K \subseteq X$ so $N \ll_{GT} M$. □

Proposition 2.5. Let $M$ be an $R$-module with submodules $N_1 , N_2$ and $T$. Then $N_1 \ll_{GT} M$ and $N_2 \ll_{GT} M$ if and only if $N_1 + N_2 \ll_{GT} M$.

Proof. Clear. □

Proposition 2.6. Let $M$ be an $R$-module with submodules $K \leq N \leq M$ and $K \subseteq T$. If $N \ll_{GT} M$, then $K \ll_{GT} M$ and $\frac{N}{K}$ is $G_{T/K}$ - small in $\frac{M}{K}$.

Proof. Suppose that $N \ll_{GT} M$ and $T \subseteq K + X$ for some $X \leq_e M$. Then $T \subseteq N + X$ and by our assumption $T \subseteq X$. Thus $K \ll_{GT} M$. Now assume that $\frac{T}{K} \subseteq \frac{N}{K} + \frac{X}{K} = \frac{N + X}{K}$ for some $K \subseteq X \subseteq M$ and $\frac{X}{K} \leq \frac{M}{K}$. Then $T \subseteq N + X$ and $X \leq_e M$ [9], so $T \subseteq X$ and $\frac{T}{K} \subseteq \frac{X}{K}$. □

Proposition 2.7. Let $M$ be an $R$-module with $K_1 \leq M_1 \leq M$ and $K_2 \leq M_2 \leq M$ such that $T \subseteq M_1 \cap M_2$. Then $K_1 \ll_{GT} M_1$ and $K_2 \ll_{GT} M_2$ if and only if $K_1 + K_2 \ll_{GT} M_1 + M_2$.

Proof. Assume that $K_1 \ll_{GT} M_1$ and $K_2 \ll_{GT} M_2$. Then By Proposition 2.4 $K_1 \ll_{GT} M_1 + M_2$ and $K_2 \ll_{GT} M_1 + M_2$. And by Proposition 2.5, $K_1 + K_2 \ll_{GT} M_1 + M_2$. The other direction is clear. □

Proposition 2.8. Let $M$ and $N$ be an $R$-modules and $f : M \rightarrow N$ be an $R$-homomorphism. If $K$ and $T$ are submodules of $M$ such that, $K \ll_{GT} M$, then $f(K) \ll_{GF(T)} N$. In particular, if $K \ll_{GT} M$ , $M \subseteq N$, then $K \ll_{GT} N$.

Proof. Let $f(T) \neq 0$ and $f(T) \subseteq f(K) + X$, for some $X \leq_e N$. It is clear that $T \subseteq K + f^{-1}(X)$ and $f^{-1}(X) \leq_e M$. But Since $K \ll_{GT} M$, then $T \subseteq f^{-1}(X)$ and hence $f(T) \subseteq X$. □

Proposition 2.9. Let $T_1$ and $T_2$ be submodules of an $R$-module $M$ and $K$ be a submodule of $M$. If $K \ll_{GT_1} M$, and $K \ll_{GT_2} M$, then $K \ll_{G(T_1 + T_2)} M$.

Proof. Since $K \ll_{GT_1} M$, then if $T_1 \subseteq N + X$ for some $X \leq_e M$, then $T_1 \subseteq X$ and $K \ll_{GT_2} M$, then if $T_2 \subseteq N + X$ for some $X \leq_e M$, then $T_2 \subseteq X$. Thus $T_1 + T_2 \subseteq N + X$ and $T_1 + T_2 \subseteq X$ So $K \ll_{G(T_1 + T_2)} M$. □

Proposition 2.10. Let $M = H_1 \oplus H_2$ be a module with $R = ann(H_1) + ann(H_2)$. If $H_1 \ll_{GT_1} M$, and $H_2 \ll_{GT_2} M$, then $H_1 \oplus H_2 \ll_{G(T_1 \oplus T_2)} M$. 

Proof. Let $T_1 \oplus T_2 \subseteq H_1 \oplus H_2 + X$, for some $X \leq_e M$ Since $R = \text{ann}(H_1) + \text{ann}(H_2)$ then $X = X_1 \oplus X_2$. By [10] $X_1 \leq_e H_1$ and $X_2 \leq_e H_2$ and $T_1 \oplus T_2 \subseteq H_1 \oplus H_2 + X_1 \oplus X_2 = (H_1 + X_1) \oplus (H_2 + X_2)$ it is clear that $T_1 \subseteq H_1 + X_1$ and $T_2 \subseteq H_2 + X_2$. Since $H_1 \ll_{GT_1} M$ and $H_2 \ll_{GT_2} M$, then $T_1 \subseteq X_1$ and $T_2 \subseteq X_2$. Thus $T_1 \oplus T_2 \subseteq X_1 \oplus X_2 \subseteq X$ and $H_1 \oplus H_2 \ll_{G(T_1 \oplus T_2)} M$. \hfill \square

Proposition 2.11. Let $M$ be finitely generated, faithful and multiplication module, and let $I, J$ be ideals in $R$. Then $I \ll_{GJ} R$ if and only if $IM \ll_{GJ} M$.

Proof. Assume; that $I \ll_{GJ} R$. Let $I$ be an ideal of $R$. Then $IM$; is a submodule of $M$, Let $JM \subseteq IM + X$ for some essential submodule $X$ of $M$, $M$ is multiplication module then $X = KM$ for some ideal $K$ of $R$ by. Then $JM \subseteq IM + KM = (I + K)M$. Since $M$ is finitely generated, faithful and multiplication module then by [4], $J \subseteq (I + K)$, since $KM \leq_e M$ then by [4, th.2.13] $K \leq_e R$. Since $I \ll_{GJ} R$ then $J \subseteq K$ thus $JM \subseteq KM = X$. Then $IM \ll_{GJ} M$.

Conversely, assume; that $IM \ll_{GJM} M$. Let $J$ be an ideal of $R$ such that $J \subseteq I + K$, $K \leq_e R$, $M$ is multiplication module then $JM \subseteq IM + KM$ and by [4, th.2.13] $KM \leq_e M$, $IM \ll_{GJM} M$ thus $JM \subseteq KM$ so $J \subseteq K$. Then $I \ll_{GJ} R$. \hfill \square

3. The $GT$-hollow module

Let $M$ be a non-zero module and $T$ be a submodule of $M$. $M$ is a $T$-hollow module if every submodule $K$ of $M$ such that $T \nsubseteq K$ is a $T$-small submodule of $M$. And that $M$ is a $G$-hollow module if every submodule of $M$ a $G$-small submodule of $M$.

Definition 3.1. Let $M$ be a non-zero module and $T$ be a submodule of $M$. We say that $M$ is a $GT$-hollow module if every submodule $K$ of $M$ such that $T \nsubseteq K$ is a $GT$-small submodule of $M$.

Remark 3.2. (a) Let $M$ be a non-zero module. Then $M$ is $GM$-hollow module if and only if $M$ is $G$-hollow module. $Z$ as $Z$-module is not $Z$-hollow module and not $GZ$-hollow module.

(b) A $GT$-hollow module need not to be hollow module as the following example shows : Consider the module $Z_6$ as $Z$-module. If $T = \{0, 3\}$, then one can easily show $Z_6$ is $GT$-hollow module. But $Z_6$ is not hollow module.

(c) If $M$ is uniform $R$-module. Then $M$ is $GM$-hollow module if and only if $M$ is hollow module.

(d) Every $T$-hollow module is $GT$-hollow module.

(e) The $Z$-module $Z_{24}$ is not $GT$-hollow module.
Proposition 3.3. Let $M$ be a $GT$-hollow module then every essential submodule $N$ of $M$ such that $T \subseteq N$ is a $GT$-hollow module.

Proof. Let $M$ be a $GT$-hollow module and $N$ any essential submodule of $M$, $T \subseteq N$. To show that $N$ is $GT$-hollow module, let $L$ be a proper submodule of $N$ such that $T \not\subseteq L$. Since $M$ is a $GT$-hollow module, then $L \ll_{GT} M$. By proposition 2.3, then $L \ll_{GT} N$. Thus $N$ is $GT$-hollow module.

Proposition 3.4. Let $M$ be a $GT$-hollow module and let $f : M \rightarrow N$ be an epimorphism, where $N$ is a non-zero module. Then $N$ is $Gf(T)$-hollow module.

Proof. Suppose that $M$ is a $GT$-hollow module and let $f : M \rightarrow N$ be an epimorphism. To show that $N$ is $Gf(T)$-hollow. Let $K \not\subseteq N$ such that $f(T) \not\subseteq K$. To show that $K \ll f(T)N$. Let $f(T) \subseteq K + X$, for some $X \leq_e N$. Then $f^{-1}(f(T)) \subseteq f^{-1}(K + X)$. Therefore $\ker f \subseteq f^{-1}(K) + f^{-1}(X)$. Thus $T \subseteq f^{-1}(K) + f^{-1}(X)$. To show that $T \not\subseteq f^{-1}(K)$. Assume $T \subseteq f^{-1}(K)$. Then $f(T) \subseteq K$ which is a contradiction. Thus $T \not\subseteq f^{-1}(K)$. Since $M$ is $GT$-hollow module, then $f^{-1}(K) \ll_{GT} M$ and hence $T \not\subseteq f^{-1}(X)$. Therefore $f(T) \subseteq X$. Thus $N$ is $f(T)$-hollow module.

Proposition 3.5. Let $T$ and $K$ be submodules of a module $M$ such that $K \subseteq T$. If $K$ is $GT$-small submodule of $M$ and $\frac{M}{K}$ is $\frac{GT}{K}$-hollow module, then $M$ is $GT$-hollow.

Proof. Assume that $K \ll_{GT} M$ and $\frac{M}{K}$ is $\frac{GT}{K}$-hollow module. Let $N \leq M$ such that $T \not\subseteq N$ and let $T \subseteq N + X$ for some $X \leq_e M$. Then $\frac{T}{K} \subseteq \frac{N + X}{K}$ and hence $\frac{T}{K} \subseteq \frac{(N + X)}{K}$. To show that $\frac{T}{K} \not\subseteq \frac{(N + K)}{K}$. Assume that $T/K = (N + K)/K$. Then $T = N + K$ and hence $T \subseteq N + K$. Since $K \ll_{GT} M$, then $T \subseteq N$ which is a contradiction. Thus $T/K \not\subseteq (N + K)/K$. Since $M/K$ is a $GT/K$-hollow module, then $(N + K)/K \ll_{GT/K} M/K$. Therefore $T/K \subseteq (X + K)/K$. Thus $T \subseteq X + K$. Since $K \ll_{GT} M$, then $T \subseteq X$. Thus $M$ is $GT$-hollow module.

Proposition 3.6. Let $T$ be a non-zero submodule of a module $M$. If $M$ is $GT$-hollow module. Then $T$ is indecomposable.

Proof. Suppose that there are proper submodules $K$ and $L$ of $T$ such that $T = K \oplus L$. Therefore $T \not\subseteq K$. Since $M$ is $GT$-hollow module, then $K \ll_{GT} M$. But $T \subseteq K \oplus L$, therefore $T \subseteq L$ and hence $T = L$. This is a contradiction. Thus $T$ is indecomposable.

4. $GT$-lifting module

$M$ is $G$-lifting; module if for any submodule $N$ of $M$ there exist; submodules $L$, $K$ of $M$ such that $N = L \oplus K$ with $L \leq N$ where $L$ is direct summand of $M$; and $K \ll_G N$ [5]. $M$ is called; $T$-lifting module if for; any submodule $N$ of $M$
there exists a direct summand $D$ of $M$ and $H \ll_T M$ such that $N = D + H$.

In this section we introduce the notion of $GT$-lifting modules and discuss some properties of this kind of modules.

**Definition 4.1.** Let $T$ be a submodule of a module $M$. $M$ is called $GT$-lifting module if for any submodule $N$ of $M$ there exists a direct summand $D$ of $M$ and $H \ll_{GT} M$ such that $N = D + H$.

**Examples and remarks 4.2.**

1. Let $M$ be a module. $M$ is $GM$-lifting module if and only if $M$ is $G$-lifting module.

   **Proof.** Let $M$ be $GM$-lifting module. Let $N$ submodule of a module $M$. Then there exists a direct summand $D$ of $M$ and $H \ll_{GT} M$ such that $N = D + H$. Thus $N = N \cap (D \oplus L) = D \oplus (N \cap L)$. Let $H = N \cap L$ then $H \ll_G M$ by (2.2) thus $M$ is $G$-lifting module. Other direction is clear.

2. Let $M$ be a module. If $M$ is $T$-lifting module then $M$ is $GT$-lifting module.

3. Let $Z_8$ as $Z$-module, $T = \{0, 4\}$ and, $N = \{0, 4\}$ then $Z_8$ is not $GT$-lifting module.

4. If $M$ is indecomposable module. then $M$ is not $GT$-lifting module for every non trivial submodule $T$ of $M$.

   **Proof.** Let $T$ be non trivial submodule of $M$. If $M$ is $GT$-lifting module then $T = D + H$ where $D$ is direct summand $D$ of $M$ and $H \ll_{GT} M$ but $M$ is indecomposable module, then $D = 0$. Thus $T = H \ll_{GT} M$ which is a contradiction then $M$ is not $GT$-lifting module.

5. Le $M$ be a $GT$-lifting module then every essential submodule $N$ of $M$ such that $T \subseteq N$ is also $GT$-lifting.

   **Proof.** Let $M$ be $GT$-lifting module and $N$ a essential submodule of $M$ such that $T \subseteq N$ and $X \subseteq N$ then $X = D + H$ where $D$ is direct summand $D$ of $M$ and $H \ll_{GT} M$. It is clear that $D$ is direct summand $D$ of $N$, $T \subseteq N$ and $N \leq_e M$ then $H \ll_{GT} N$ by (prop 2.3). Thus $N$ is $GT$-lifting.

   Let $H_1$ be $GT_1$-lifting and $H_2$ is $GT_2$-lifting modules, then $M = H_1 \oplus H_2$ need not be $GT_1 \oplus GT_2$-lifting module as the following example:

   Let $H_1 = Z_8$, $H_2 = Z_2$, each of $H_1$, $H_2$ is $GH_i$-lifting module but $M = Z_8 \oplus Z_2$ as $Z$-module, $M$ is not $GM$-lifting module by (Ex.4.2 (1)).

   Now we give a sufficient condition under which $M = H_1 \oplus H_2$ is $GT_1 \oplus GT_2$-lifting module.
Proposition 4.3. Let $M = H_1 \oplus H_2$ be a module with $R = \text{ann}(H_1) + \text{ann}(H_2)$. If $H_1$ is $GT_1$-lifting and $H_2$ is $GT_2$-lifting modules, then $M$ is $GT_1 \oplus GT_2$-lifting module.

Proof. Let $N$ submodule of $M$. Since $R = \text{ann}(H_1) + \text{ann}(H_2)$, then $N = N_1 \oplus N_2$ where $N_1 \subseteq H_1$ and $N_2 \subseteq H_2$. $H_1$ is $GT_1$-lifting and $H_2$ is $GT_2$-lifting modules, then for each $i \in \{1, 2\}$, there exists a direct summand $D_i$ of $H_i$, such that $N_i = D_i \oplus L_i$ with $D_i \leq N_i$ and $L_i \ll_{GT} H_i$ then, $N = N_1 \oplus N_2 = (D_1 \oplus L_1) \oplus (D_2 \oplus L_2) = (D_1 \oplus D_2) \oplus (L_1 \oplus L_2)$, we have $(D_1 \oplus D_2) \leq N$, then $(D_1 \oplus D_2)$ is direct summand of $M$ by (Prop:2.10) then $(L_1 \oplus L_2) \ll_{G(T_1+T_2)} M$. Thus $M$ is $GT_1 \oplus GT_2$-lifting module. \hfill \Box

Proposition 4.4. Let $M$ be finitely generated, faithful and multiplication module. Then $M$ is $GT$-lifting module if and only if $R$ is $[GT : M]$-lifting.

Proof. Assume that $M$ is $GT$-lifting module. Let $I$ be an ideal of $R$. $M$ is $GT$-lifting hence there exist submodules $D \leq \oplus M$ and $H \ll_{GT} M$ such that $N = D + H$. But $M$ is a multiplication $R$-module, so there are ideals $J$ and $K$ of $R$ such that $D = JM$ and $H = KM$. Then $JM = JM + KM = (J + K)M$. But $M$ is finitely generated, faithful and multiplication module then by [4] $I = J + K,$ Let $M = D + L$ and $L = J'M$ for some $J'$ of $R$. Then $RM = M = JM \oplus J'M = (J + J')M$ Then $R = J + J'$. Since $M$ is finitely generated, faithful and multiplication module then $0 = JM \cap J'M = (J \cap J')M$ thus $JJ' = 0$, and $J \leq \oplus R$ by (prop. 2.11) $K \ll_{G(T;M)} R$. Thus $R$ is $[GT : M]$-lifting. Conversely, let $R$ be $[GT : M]$-lifting and $N$ submodule of $M$. Since $M$ is finitely generated, faithful and multiplication module there exist $I$ an ideal of $R$ such that $N = IM$ and exist $J \leq \oplus R$ and $K \ll_{G(T;M)} R$ with $I = J + K$. Then $IM = JM + KM = (J + K)M$. Thus $N = JM + KM$, let $R = J \oplus J'$ for some $J'$ of $R$ then $M = RM = (J + J')M = JM \oplus J'M$. Since $M$ is finitely generated, faithful and multiplication module then $JM \cap J'M = (J \cap J')M = 0 = 0$. Then $JM \leq \oplus M$ by (prop.2.11), $k \ll_{GT} M$. Then $M$ is $GT$-lifting module. \hfill \Box

5. $GT$-supplemente submodule

Definition 5.1. Let $M$ be an $R$-module and $T, X, Y \leq M$. $Y$ is called a $GT$-supplement of $X$ in $M$, if $T \subset X + Y$ and $X \cap Y \ll_{GT} Y$. If every submodule of $M$ has a $GT$-supplement in $M$, then $M$ is called a $GT$-supplemented module.

Examples and remarks 5.2.

1. If $T = 0$, then every submodule of $M$ is $GT$-supplement in $M$.

2. and If $T = M$, then $M$ is $GM$-supplement in $M$ if and only if $M$ is $G$-supplement in $M$.

3. Let $Z$ be the ring of integers. It is easy to see that $(0)$ is the only $GmZ$-small submodule of $Z$. Now let $T = 0$, $X = 2Z$ and $Y = 3Z$ then
Proposition 5.3. Let $M$ be an $R$-module, $T$, $X$ and $Y \leq M$ such that $Y$ is $GT$-supplement of $X$ in $M$ if $T \subseteq K + Y$, for some submodule $K$ of $M$. Then $Y$ is a $GT$-supplement of $K$ in $M$.

Proof. Let $Y$ be is $GT$-supplement of $X$ in $M$, $K$ submodule of $M$ such that $T \subseteq K + Y$. Since $K \cap Y \subseteq X \cap Y \leq_{GT} Y$ by (Prop:2.8). Then $Y$ is a $GT$-supplement of $K$ in $M$. \hfill $\Box$

Proposition 5.4. Let $M$ be an $R$-module, $T$, $X$ and $Y \leq M$ and $Y$ be a $GT$-supplement of $X$ in $M$, $L \leq Y$ and $L \leq_{GT} Y$. Then $Y$ is a $GT$-supplement of $X + L$ in $M$.

Proof. Let $Y$ be a $GT$-supplement of $X$ in $M$ and $L \leq Y$ and $L \leq_{GT} Y$. Then $T \subseteq T + Y \subseteq T + Y + L$, $X \cap Y \leq_{GT} Y$. Then $T \subseteq T \cap Y \subseteq X \cap (X + L) + K$. Then $T \subseteq (X \cap Y) + L + K$, $K \subseteq L + K$ is essential submodule in $M$ hence $T \subseteq L + K$. Since $L \leq_{GT} Y$ thus $T \subseteq K$. Then $Y$ is a $GT$-supplement of $X + L$ in $M$. \hfill $\Box$

Proposition 5.5. Let $M$ and $N$ be $R$-modules, and let $f : M \rightarrow N$ be an epimorphism. If $M$ is $GT$-supplemented module. Then $N$ is $Gf(T)$-supplemented module.

Proof. Suppose that $M$ is a $GT$-supplemented module and let $f : M \rightarrow N$ be an epimorphism. Let $K$ be submodule of $N$, $M$ is a $GT$-supplemented module then $T \subseteq L + f^{-1}(K)$ and $f^{-1}(K) \cap L \leq_{GT} Y$. Then $f(T) \subseteq f(L + f^{-1}(K))$. Then $f(T) \subseteq f(L) + K$. Since $f^{-1}(K) \cap L \leq_{GT} Y$ then $K \cap f(L) = f(f^{-1}(K)) \cap L \leq_{Gf(T)} f(Y)$. Therefore by (Prop:2.8) $f(L)$ is $Gf(T)$-supplement submodule of $K$ in $M$. \hfill $\Box$

Proposition 5.6. Let $M$ be $GT$-lifting module and $Y$ be a $GT$-supplement of $X$ in $M$. Then $Y$ contains a $GT$-supplement of $X$ which is direct summand of $M$.

Proof. Suppose that $M$ is $GT$-lifting module and $Y$ be a $GT$-supplement of $X$ in $M$. Then $T \subseteq T + X = X \cap Y \leq_{GT} Y$. $M$ is $GT$-lifting then $Y = D + H$, where $D \leq \oplus M$ and $H \leq_{GT} M$. Since $T \subseteq T + X$, then $T \subseteq X + D + H$ thus $T \subseteq T + X$, now $X \cap D \subseteq X \cap Y \leq_{GT} Y$ by (Prop:2.6) $X \cap D \leq_{GT} Y$ then $D$ is a $GT$-supplement of $X$ in $M$. \hfill $\Box$
References


Accepted: 14.04.2018