Urysohn lemma in semi-linear uniform spaces

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Abstract. In topology Urysohn Lemma is widely applicable, where it is commonly used to construct continuous functions with various properties on normal space. In this paper we shall present Urysohn Lemma in semi linear uniform spaces, besides we shall give a characterization of the closure in semi-linear uniform space, then we shall use this characterization to answer the question which given in [12], by A.Tallafha and R. Khalil namely (If $\rho(x, A) = \Delta$, must $x \in A^l$).

Keywords: uniform spaces, Semi-linear uniform spaces, topological spaces, metric spaces, types of metric spaces.

1. Introduction

One of the most important generalizations of a metric spaces is uniform space. Uniform spaces is a concept lies between metric spaces and topological structure, that is used to define uniform properties such as completeness, uniform continuity and uniform convergence. The uniform spaces have been studied extensively through years. The notion of uniformity has been investigated by several mathematician such as Weil [18] ,[19] , and [20] . L.W. Cohen [4] , and [5] . Graves [7]. The theory of uniform spaces was given by Burbaki in [3] . Also Wiel's in his booklet [20] , defined the notion of uniformly continuous mapping.

In 2009, the notion of a uniform space led A. Tallafha and R. Khalil to define a beautiful space which is a mixture of analysis and topology, namely semi-linear uniform space [12], also they studied some cases of best approximation in such spaces, besides they defined a set valued map ρ, called metric type, on semi-linear uniform spaces that enables one to study analytical concepts on semi-linear uniform space.

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Semi-linear uniform space is weaker than metric space and stronger than topological space since in [12], [13] , [14] and [15] , A. Tallafha answered the question"Is there a semi-linear uniform space which is not metrizable?". Also he defined another set valued map called δ on $X \times X$, which is used with ρ to give more properties of semi-linear uniform spaces. Finally he studied the relation between ρ and δ, he showed that $\rho(x, y) = \rho(s, t)$ if and only if $\delta(x, y) = \delta(s, t)$, then he defined Lipschitz condition and contraction mapping on semi-linear uniform spaces, which enables one to study fixed point for such functions. In [16] and [17], A. Tallafha and S. Alhihi established another properties of semi-linear uniform spaces.

In [11], A. Rawshdeh and A. Tallafha answered the question (If f is a contraction from a complete semi-linear uniform space (X, Γ) to it self, is f has a unique fixed point) negatively, they gave an example of a complete semi-linear uniform space (X, Γ) and a contraction $f : (X, \Gamma) \to (X, \Gamma)$ which has infinitely many fixed points.

Other nice result was given in [2], by S. Alhihi and M. Fayyad, where they showed that every semi-linear uniform space induced a Tychonoff space (X, T_Γ).

2. Semi-linear uniform spaces

Let X be a none empty set and D_X be a collection of all relations on X such that each element V of D_X is reflexive and symmetric. D_X is called the family of all entourages of the diagonal.

Now the above discussion allow us to define the uniform space.

Definition 2.1 ([6]). *Let X be a set. A uniform space is the pair $(X, \mathcal{F})$ where $\mathcal{F}$ is a subfamily of D_X which satisfies the following conditions:*

1) *If $U \in \mathcal{F}$ and $U \subseteq W \in D_X$, then $W \in \mathcal{F}$.*
2) *If $U_1, U_2 \in \mathcal{F}$, then $U_1 \cap U_2 \in \mathcal{F}$.*
3) *There exists a $W \in \mathcal{F}$ such that $W \circ W \subseteq U$, for every $U \in \mathcal{F}$,*
4) $\bigcap_{U \in F} U = \Delta$.

The notion of a uniform space led A. Tallafha and R. Khalil in 2009 to define the semi-linear uniform space:

Definition 2.2 ([12]). *Let Γ be a subcollection of D_X such that:*

(i) *If V_1 and V_2 are in Γ, then $V \cap V_2 \in \Gamma$.*

(ii) *For every $V \in \Gamma$, there exists $U \in \Gamma$ such that $U \circ U \subset V$.*

(iii) $\bigcap_{V \in \Gamma} V = \Delta$.

(vi) $\bigcup_{V \in \Gamma} V = X \times X$.

(v) *Γ is a chain (a chain in $X \times X$ we mean a totally or linearly) ordered collection of subsets of $X \times X$, ordered by set inclusion.*

Then the pair (X, Γ) is called a semi-linear uniform space.

In [12] and [13] , the authors defined the set valued map ρ and δ which played an important rule in the theory of fixed point on semi-linear uniform spaces.

Definition 2.3 ([12]). *Let* (X,Γ) *be a semi-linear uniform space. For* $(x,y)\in X\times X$, *let* $\Gamma_{(x,y)}=\{V\in\Gamma:(x,y)\in V\}$. *Then, the set valued map* ρ *on* $X\times X$, *is defined by* $\rho(x,y)=\bigcap\{V:V\in\Gamma_{(x,y)}\}$.

Clearly for all $(x,y)\in X\times X$, we have $\rho(x,y)=\rho(y,x)$, and $\Delta\subseteq\rho(x,y)$.

The following definition given in [13] , the authors defined $\delta\,(x,x)=\phi$, now if we define $\delta(x,x)=\rho(x,x)=\Delta$, then all the results in the literature still valid and the new definition of $\delta(x,x)$ seems to be more convenient.

Definition 2.4 ([13]). *Let* (X,Γ) *be a semi-linear uniform space. For* $(x,y)\in X\times X$, *define* $\Gamma^{c}_{(x,y)}=\{V\in\Gamma:(x,y)\notin V\}$. *Then the set valued map* δ *on* $X\times X$, *is defined by*

$$\delta(x,y)=\left\{\begin{array}{c}\cup\{V:V\in\Gamma^{C}_{(x,y)}\}:\ x\neq y\\ \Delta:\ x=y\end{array}\right\}.$$

By using the set valued map ρ and δ, A.Tallafha gave some important properties of semi-linear uniform spaces, some of these properties are given in the following proposition.

Proposition 2.5 ([13]). *Let* (X,Γ) *be a semi-linear uniform space. Then:*
i) If $V\in\Gamma^{c}_{(x,y)}$, *then* $V\subsetneqq\rho(x,y)$.
ii) $\delta(x,y)\subseteq\rho(x,y)$ *for all* $(x,y)\in X\times X$.
iii) If $V\in\Gamma_{(x,y)}$, *then* $\delta(x,y)\subseteq V$.
iv) If $(x,y)\in\rho(s,t)$, *then* $\rho(x,y)\subseteq\rho(s,t)$.
v) If $(x,y)\in\delta(s,t)$, *then* $\delta(x,y)\subseteq\delta(s,t)$.

In [1] Alhihi gave more properties of semi-linear uniform spaces as:

Theorem 2.6. *Let* $A\in\mathbf{\Lambda}$, *and* σ *a sub collection of* $\mathbf{\Lambda}$. *For* $n\in\mathbb{N}$, *we have:*
(i) $n\left(\frac{1}{n}A\right)\subseteq A$.
(ii) If $B\in\Gamma$ *satisfies* $nB\subseteq A$, *then* $B\subseteq\frac{1}{n}A$.
(iii) $\frac{1}{n+1}A\subseteq\frac{1}{n}A$.
(iv) $\frac{1}{n}A\subseteq A$.
(v) $\frac{1}{n}\bigcap_{A\in\sigma}A=\bigcap_{A\in\sigma}\frac{1}{n}A$.
(vi) $\bigcup_{A\in\sigma}\frac{1}{n}A\subseteq\frac{1}{n}\bigcup_{A\in\sigma}A$.

It is known that every metric space endues a semi-linear uniform space, the following define the semi-linear uniform space which induced by a metric space (X,d).

Definition 2.7. [13]. *Let* (X,d) *be a metric space. Define* $V_\epsilon=\{(x,y):d(x,y)<\epsilon\}$.*Then* (X,Γ) *is a semi-linear uniform space induced by* (X,d) *where* $\Gamma=\{V_\epsilon:\epsilon>0\}$.*This semi-linear uniform space will be denoted by* (X,Γ_d).

In [13], A. Tallafha gave the following example, which is a semi-linear uniform space but not metrizable.

Example 2.8. $(\mathbb{R},\Gamma)$ is a semi-linear uniform space, where $\Gamma=\{V_\epsilon,\epsilon>0\}, V_\epsilon=\{(x,y):x^2+y^2<\epsilon\}\cup\{\Delta\}$.

In [11], A. Rawshdeh and A. Tallafha, give the following lemma:

Lemma 2.9. *Let (X,Γ_d) be a semi-linear uniform space induced by the metric space (X,d). Then:*
(1) $\rho(x,y)=\{(s,t)\in X\times X : d(s,t)\le d(x,y)\}$.
(2) $\delta(x,y)=\{(s,t)\in X\times X : d(s,t)<d(x,y)\}$.

Definition 2.10 ([9])**.** *Let (X,d) be a metric space. For $x\in X, r>0$, let $B[x,r]=\{t:d(x,t)\le r\}$. A metric space (X,d) is convex, if for all $x,y\in X$, $B[x,r_1]\cap B[y,r_2]\neq\phi$ whenever $r_1+r_2\ge d(x,y)$.*

In [10] and [11], A. Rawshdeh and A. Tallafha gave the following results.

Lemma 2.11. *Let (X,Γ_d) be a semi-linear uniform space induced by convex metric space (X,d), then:*
1) $n\rho(x,y)=\{(s,t)\in X\times X : d(s,t)\le nd(x,y)\}$.
2) $n\delta(x,y)\subseteq\{(s,t)\in X\times X : d(s,t)<nd(x,y)\}$.

Also in [12], A. Tallafha and R. Khalil, give the definition of continuos function, uniformly continuous function, converges of sequences in semi-linear uniform spaces and complete semi-linear space.

Definition 2.12. [12]. *Let $f:(X,\Gamma_X)\to(Y,\Gamma_Y)$, then:*

1- f is continuous at $x_\circ$ if for all $U\in\Gamma_Y$, there exists $V\in\Gamma_X$, such that if $(x,x_\circ)\in V$, then $(f(x),f(x_\circ))\in U$.

2- f is uniformly continuous if for all $U\in\Gamma_Y$, there exists $V\in\Gamma_X$, such that if $(x,y)\in V$, then $(f(x),f(y))\in U$.

Definition 2.13. [12]**.** *Let (X,Γ) be a semi-linear uniform space and (x_n) be a sequence in X. Then:*

1- (x_n) converges to x in X and denoted by $x_n\to x$, if for every $V\in\Gamma$ there exists k such that $(x_n,x)\in V$ for every $n\ge k$.

2- (x_n) is called Cauchy if for every $V\in\Gamma$ there exists k such that $(x_n,x_m)\in V$ for every $n,m\ge k$.

Definition 2.14. [12]**.** *Let (X,Γ) be a semi-linear uniform space. Then (X,Γ) is called complete, if every Cauchy sequence is convergent.*

3. Some topological properties of semi-linear uniform spaces

In this section, we will shed more light on the topology induced by a semi-linear uniform space,we shall state and discuss some important topological properties of semi-linear uniform spaces.

After A. Tallafha and R. Khalil in [12], gave the definition of semi-linear uniform space, they gave the the following definition:

Definition 3.1 ([12])**.** *Let (X,Γ) be a semi-linear uniform space. For $x \in X$ and $V \in \Gamma$, the open ball of center x and radius V to be $B(x,V) = \{y : (x,y) \in V\}$. Equivalently $B(x,V) = \{y : \rho(x,y) \subseteq V\}$.*

Also, they showed the following:

Theorem 3.2 ([12])**.** *Let (X,Γ)be a semi-linear uniform space. The family $\tau = \{G \subseteq X :$ for every $x \in G$ there exists a $U \in \Gamma$ such that $B(x,U) \subseteq G\}$ is topology on the set X.The topology which induced by (X,Γ) is denoted by τ_Γ.*

In [2], S. Alhihi and M. Fayyad, discussed the interior of a set and a neighborhood of $x \in X$ with respect to the topology induced by a semi-linear uniform space on the set X, they gave the following proposition and corollaries:

Proposition 3.3 ([2])**.** *Let (X,Γ)be a semi-linear uniform space and $A \subseteq X$. Then $int(A) = \{x \in X :$ there exists a $U \in \Gamma$ such that $B(x,U) \subseteq A\}$ with respect to τ_Γ.*

Corollary 3.4 ([2])**.** *If the topology of a space X is induced by a semi-linear uniform Γ, then for every $x \in X$ and any $U \in \Gamma$ the set int $B(x,U)$ is a neighborhood of x.*

Corollary 3.5 ([2])**.** *If the topology of a space X is induced by a semi-linear uniform Γ, then for every $x \in X$ and any $A \subseteq X$ we have $x \in \bar{A}$ if and only if $A \cap B(x,U) \neq \phi$ for every $U \in \Gamma$.*

Also, they gave the following definition:

Definition 3.6 ([2])**.** *A family $\mathcal{B} \subset \Gamma$ is called a base for the semi-linear uniform Γ if for every $U \in \Gamma$ there exists a $W \in \mathcal{B}$ such that $W \subset U$.*

Now we can conclude from above definitions the following remark and Theorem.

Remark 3.1. Any base $\mathcal{B}$ for a semi-linear uniform space on a set X has the following properties:

1) If $U_1, U_2 \in \mathcal{B}$ there exists a $U \in \mathcal{B}$ such that $U \subseteq U_1 \cap U_2$.

2) For every $U \in \mathcal{B}$ there exists a $W \in \mathcal{B}$ such that $2W \subseteq U$.

3) $\bigcap \mathcal{B} = \Delta$.

Theorem 3.7. *Let (X,Γ)be a semi-linear uniform space. If $\mathcal{B} \subset \Gamma$ is a base for Γ then $\beta_x = \{B(x,W) : W \in \beta\}$ is a neighborhood base at x.*

Proof. Let (X,Γ)be a semi-linear uniform space and $\mathcal{B} \subset \Gamma$ is a base for Γ. To show $\beta_x = \{B(x,W) : W \in \beta\}$ is a neighborhood base at x we want to show the following:

1) $x \in B(x,W)$ for each $W \in \beta$ as $(x,x) \in \Delta \subseteq W$.

2) If $B(x,U), B(x,V) \in \beta_x$, then there exists $W \in \beta$ such that $W \subseteq U \cap V$. Hence $B(x,W) \subseteq B(x,U) \cap B(x,V)$.

3) If $B(x,U) \in \beta_x$, then $U \in \beta$.So there exists $V \in \beta$ such that $V \circ V \subseteq U$. Thus if $y \in B(x,V)$ and $z \in B(y,V)$ then $(x,z) \in V \circ V \subseteq U$, so $z \in B(x,U)$. Hence $B(y,V) \subseteq B(x,U)$,that is there exists $B(y,V) \in \beta_y$ and $B(y,V) \subseteq B(x,U)$. □

Proposition 3.8 ([2])**.** *If the topology of a space X is induced by a semi-linear uniform Γ and $\mathcal{B} \subset \Gamma$ is a base for Γ, then for every $x \in X$ and any $A \subseteq X$ we have $x \in \bar{A}$ if and only if $A \cap B(x,U) \neq \phi$ for every $U \in \beta$.*

In [12], A. Tallafha and R. Khalil, proved that the topology induced by semi-linear uniform space is T_2-space.Therefore the topology $(\mathbb{R}, \tau_{lef})$ where $\tau_{lef} = \{(-\infty, \alpha) : \alpha \in \mathbb{R}\} \cup \{\mathbb{R}, \phi\}$ is not induced by semi-linear uniform space. Moreover in [2], S. Alhihi and M. Fayyad, showed that every semi-linear uniform space induced a Tychonoff space (X, τ_Γ), where τ_Γ is the topology on X induced by local base B_x. For more topological properties of τ_Γ we refer the reader to [2].

Definition 3.9. *Let Γ_1,Γ_2 be two semi-linear uniform spaces on a set X. We say Γ_1 is weaker than Γ_2 if and only if for all $U \in \Gamma_1$ there exists $V \in \Gamma_2$ such that $V \subseteq U$.*

Definition3.9, lead us to give the following definition:

Definition 3.10. *Let Γ_1,Γ_2 be two semi-linear uniform spaces on a set X. We say Γ_1 is equivalent to Γ_2 if Γ_1 is weaker than Γ_2 and Γ_2 is weaker than Γ_1.*

In metric space any two equivalent metric spaces induce two equivalent topologies, but we can find two equivalent metric spaces give two different semi-linear uniform spaces, for example if $d_1$is Euclidean metric on $\mathbb{R}$ and define $d_2(x,y) = \left| \frac{x}{1+|x|} - \frac{y}{1+|y|} \right|$ for $x, y \in \mathbb{R}$.Then $\Gamma_{d_1} = \{U_\epsilon : \epsilon > 0\}$, where $U_\epsilon = \{(s,t) : d_1(s,t) < \epsilon\}$ is the induced semi-linear uniform space by d_1. Also $\Gamma_{d_2} = \{V_\epsilon : \epsilon > 0\}$, where $V_\epsilon = \{(s,t) : d_2(s,t) < \epsilon\}$ is the induced semi-linear uniform space by d_2. To show that Γ_{d_1} and Γ_{d_2} are different, if $\epsilon \geq 2$ then$V_\epsilon = \mathbb{R}^2$ and $\mathbb{R}^2 \notin \Gamma_{d_1}$.Thus $\Gamma_{d_1} \neq \Gamma_{d_2}$. Now the topology which induce by d_1 is the same as the topology which induce by d_2, since $f : \mathbb{R} \to (-1,1)$ which defined by $f(x) = \frac{x}{1+|x|}$ is homeomorphism and $d_1(f(x), f(y)) = d_2(x,y)$. But if we have two equivalent semi-linear uniform spaces we can get two equivalent topologies. For more details we have the following proposition:

Proposition 3.11. *Two equivalent semi-linear uniform spaces induce two equivalent topologies.*

Proof. Let Γ_1,Γ_2 be two equivalent semi-linear uniform spaces on a set X induce two topologies $\tau_{\Gamma_1}, \tau_{\Gamma_2}$ respectively. To show $\tau_{\Gamma_1}$and $\tau_{\Gamma_2}$are equivalent, we want to show $\tau_{\Gamma_1} \subseteq \tau_{\Gamma_2}$ and $\tau_{\Gamma_2} \subseteq \tau_{\Gamma_1}$. Now let $x \in G \in \tau_{\Gamma_1}$, then there exists $U \in \Gamma_1$ such that $B(x,U) \subseteq G$, as $\Gamma_1$is weaker than Γ_2, so there exists $V \in \Gamma_2$ such that $V \subseteq U$, which means $B(x,V) \subseteq B(x,U) \subseteq G$. Hence $G \in \tau_{\Gamma_2}$. The same proof for $\tau_{\Gamma_2} \subseteq \tau_{\Gamma_1}$. □

The following example is an example of two equivalent semi-linear uniform spaces:

Example 3.12. Let $X = \mathbb{R}$. Define $\Gamma_1 = \{V_t : 0 < t < \infty\}$ where $V_t = \{(x, y) : y - t < x < y + t$, and $-\infty < y < \infty\}$ and define $\Gamma_2 = \{V_t : 0 < t < \infty\}$ where $V_t = \{(x, y) : y - t \leq x \leq y + t$, and $-\infty < y < \infty\}$.Then $\Gamma_1 \neq \Gamma_2$, but $\Gamma_1$and Γ_2 are two equivalent semi-linear uniform spaces induce two topologies $\tau_{\Gamma_1}, \tau_{\Gamma_2}$ respectively. Clearly $\tau_{\Gamma_1}$and $\tau_{\Gamma_2}$are the usual topology $\mathbb{R}$.

Now we shall give some interesting facts about semi-linear uniform space:

Lemma 3.13. *Let (X, Γ)be a semi-linear uniform space. For $V \in \Gamma, U \subseteq X \times X$ we have $V \circ U \circ V = \bigcup\{B(x, V) \times B(y, V) : (x, y) \in U\}$.*

Proof. Let (X, Γ)be a semi-linear uniform space. If $(s, t) \in V \circ U \circ V$, then there exists $x, y \in X$ such that $(s, x) \in V, (x, y) \in U$ and $(y, t) \in V$, which means $s \in B(x, V)$ and $t \in B(y, V)$. Hence $(s, t) \in B(x, V) \times B(y, V)$ for $(x, y) \in U$. Now if $(s, t) \in B(x, V) \times B(y, V)$ for $(x, y) \in U$, then $(s, x), (t, y) \in V$. Thus $(s, t) \in V \circ U \circ V$. □

The proof of the following proposition is clear from above discussion:

Proposition 3.14. *If the topology of a space X is induced by a semi-linear uniform Γ, and $M \subseteq X \times X$, then for $(x, y) \in X \times X$we have $(x, y) \in \overline{M}$ if and only if $M \cap B(x, U) \times B(y, U) \neq \phi$ for every $U \in \Gamma$.*

The closure of a subset $M \subseteq X \times X$ is defined as the smallest closed subset of $X \times X$ that is contain M. The following theorem gives a useful and a new characterization of closure in term of member of semi-linear uniform space:

Theorem 3.15. *Let (X, Γ)be a semi-linear uniform space. If $M \subseteq X \times X$ then $\overline{M} = \cap\{V \circ M \circ V : V \in \Gamma\}$.*

Proof. Let (X, Γ)be a semi-linear uniform space. Let $(x, y) \in \overline{M}$ and $V \in \Gamma$,then $B(x, V) \times B(y, V) \cap M \neq \phi$, so there exists $(s, t) \in B(x, V) \times B(y, V) \cap M$, which means $(x, y) \in B(s, V) \times B(t, V)$ for $(s, t) \in M$, by Lemma 3.13, $(x, y) \in V \circ M \circ V$. Hence $\overline{M} \subseteq \cap\{V \circ M \circ V : V \in \Gamma\}$. The converse is clear by Proposition 3.14. □

Corollary 3.16. *Let (X, Γ)be a semi-linear uniform space. $\{\overline{V} : V \in \Gamma\}$ is a base for Γ.*

Proof. Let $U \in \Gamma$. Then there exists $V \in \Gamma$ such that $V \circ V \circ V \subseteq U$, by Theorem 3.15, $\overline{V} \subseteq V \circ V \circ V \subseteq U$. □

Now we will give the following definition:

Definition 3.17. *Let (X, Γ)be a semi-linear uniform space and $A \subseteq X$. For $U \in \Gamma$ define $B(A, U) = \bigcup_{x \in A} B(x, U)$.*

Proposition 3.18. *Let* (X, Γ)*be a semi-linear uniform space with a base* β *and* $A \subseteq X$*. Then* $\bar{A} = \bigcap\{B(A, V) : V \in \beta\}$.

Proof. Let (X, Γ)be a semi-linear uniform space and $A \subseteq X$. Let $x \in \bar{A}$. For $V \in \beta$ there exists $W \in \Gamma$such that $W \circ W \subseteq V$, so by Corollary 3.5, there exists $y \in A \cap B(x, W) \neq \phi$. As $y \in A$ and $y \in B(x, W)$, so $x \in B(y, W) \subseteq B(y, V) \subseteq B(A, V)$. Hence $x \in \bigcap\{B(A, V) : V \in \beta\}$. Now let $x \in \bigcap\{B(A, V) : V \in \beta\}$ then $x \in B(A, V)$ for all $V \in \beta$. Let $V \in \beta$, since $x \in B(A, V)$, then there exists $t \in A$ such that $x \in B(t, V)$, so $t \in B(x, V) \cap A$ which implies $x \in \bar{A}$. □

Corollary 3.19. *Let* (X, Γ)*be a semi-linear uniform and* $A \subseteq X$*. If* $B(A, U) = A$ *for some* $U \in \Gamma$ *then* A *is both open and closed.*

Proof. Let (X, Γ)be a semi-linear uniform and $A \subseteq X$. If $x \in \bar{A}$, then $x \in \bigcap\{B(A, V) : V \in \Gamma\}$, so $x \in B(A, U) = A$. To show that A is open, suppose $x \in A$, then $B(x, U) \subseteq A$. Hence A is open. □

In [12], A. Tallafha and R. Khalil, proved that if $x \in A^l$ then $\rho(x, A) = \Delta$. Then they asked the following question: If $\rho(x, A) = \Delta$, must $x \in A^l$. Now we shall answer this question:

Proposition 3.20. *If* $\rho(x, A) = \Delta$ *then* $x \in \bar{A}$.

Proof. To show that $x \in \bar{A}$, we want to show $A \bigcap B(x, U) \neq \phi$ for every $U \in \Gamma$. Suppose not, i.e., there exists $U \in \Gamma$ such that $A \bigcap B(x, U) = \phi$, which means $(x, \alpha) \notin U$ for all $\alpha \in A$, by Proposition 2.5, $U \subsetneq \rho(x, \alpha)$ for all $\alpha \in A$. Thus $U \subsetneq \bigcap_{\alpha \in A} \rho(x, \alpha) \subseteq \Delta$, which is contradiction. Hence $A \bigcap B(x, U) \neq \phi$ for every $U \in \Gamma$, and so $x \in \bar{A}$. □

So we have the following corollary:

Corollary 3.21. $x \in \bar{A}$ *if and only if* $\rho(x, A) = \Delta$.

Now we shall discuss some concept related to continuous function:

Theorem 3.22. *Let* $(X, \tau_\Gamma), (Y, \Gamma_{\tau_Y})$*be two topologies which induced by the semi-linear uniform spaces* (X, Γ_X) *and* (Y, Γ_Y) *respectively. Then* $f : (X, \Gamma_X) \to (Y, \Gamma_Y)$ *is continuous if and only if* $f : (X, \tau_{\Gamma_X}) \longrightarrow (Y, \tau_{\Gamma_Y})$ *is continuous.*

Proof. Let$f : (X, \Gamma_X) \to (Y, \Gamma_Y)$be continuous at $x_\circ$. For $V \in \Gamma_Y$ take $B(f(x_\circ), V)$. Since $f : (X, \Gamma_X) \to (Y, \Gamma_Y)$is continuous at $x_\circ$ then there exists $U \in \Gamma_X$ such that $(x, x_\circ) \in U$ then $(f(x), f(x_\circ)) \in V$. If $z \in f(B(x_\circ, U))$ then there exists $x \in B(x_\circ, U)$ such that $z = f(x)$, which means $(x, x_\circ) \in U$, and so $(f(x), f(x_\circ)) \in V$. Hence $f(x) \in B(f(x_\circ), V)$.To proof the converse, suppose $f : (X, \tau_{\Gamma_X}) \longrightarrow (Y, \tau_{\Gamma_Y})$ is continuous at $x_\circ$. For $V \in \Gamma_Y$ take $B(f(x_\circ), V)$. As $f : (X, \tau_{\Gamma_X}) \longrightarrow (Y, \tau_{\Gamma_Y})$ is continuous at $x_\circ$, so there exists open set $O \in \tau_{\Gamma_X}$ such that $x_\circ \in O$ and $f(O) \subseteq B(f(x_\circ), V)$, but as $x_\circ \in O \in \tau_{\Gamma_X}$, there exists $W \in \Gamma_X$ such that $B(x_\circ, W) \subseteq O$. So if $(x, x_\circ) \in W$ then $x \in B(x_\circ, W) \subseteq O$, which means $f(x) \in f(O) \subseteq B(f(x_\circ), V)$. Hence $(f(x), f(x_\circ)) \in V$. □

Let (X, Γ) be a semi-linear uniform space. For $A \subseteq X$, we can define a semi-linear uniform space on A by $\Gamma_A = \{(A \times A) \cap U : U \in \Gamma\}$.

Theorem 3.23. *Let (X, Γ) be a semi-linear uniform space and $A \subseteq X$. Then (A, Γ_A) can be defined by $\Gamma_A = \{(A \times A) \cap U : U \in \Gamma\}$ is a semi-linear uniform space on A and (A, Γ_A) is called the subspace of the semi-linear uniform space (X, Γ).*

Proof. Let (X, Γ)be a semi-linear uniform and $A \subseteq X$. For $U \in \Gamma$ define $A_U = (A \times A) \cap U$. From the definition of Γ_A, we note that for all $A_U \in \Gamma_A$ we have $\{(\alpha, \alpha) : \alpha \in A\} \subseteq A_U$ and $A_U = A_U^{-1}$. Now to complete the proof we need to prove the following:

(i) If A_{V_1} and A_{V_2} are in Γ_A, then $A_{V_1} \cap A_{V_2} = (A \times A) \cap (V_1 \cap V_2) \in \Gamma_A$ as $V_1 \cap V_2 \in \Gamma$. (ii) For every $A_V \in \Gamma_A$, there exists $U \in \Gamma$ such that $U \circ U \subset V$. Let $(s, t) \in [(A \times A) \cap U] \circ [(A \times A) \cap U]$. Then there exists $y \in X$ such that $(s, y) \in [(A \times A) \cap U]$ and $(y, t) \in [(A \times A) \cap U]$, so (s, y),$(y, t) \in U$ which means $(s, t) \in U \circ U \subset V$. Thus $[(A \times A) \cap U] \circ [(A \times A) \cap U] \subseteq (A \times A) \cap V = A_V$. (iii) $\bigcap_{V \in \Gamma} A_V \subseteq (\bigcap_{V \in \Gamma} V) \cap A \times A = \Delta_A$.(vi) Since (X, Γ) is a semi-linear uniform, so $\bigcup_{V \in \Gamma} V = X \times X$. Thus $\bigcup_{V \in \Gamma} A_V = A \times A$.(v) Since Γ is a chain, so Γ_A is a chain. □

From the definition of (A, Γ_A) we can note, if τ_Γ is the topology on X induced by Γ, then (A, τ_A) is the subspace topology on A.

Theorem 3.24. *If $f : (X, \Gamma_X) \to (Y, \Gamma_Y)$ is continuous, then $f : (A, \Gamma_A) \to (Y, \Gamma_Y)$ is continuous.*

Proof. Let $f : (X, \Gamma_X) \to (Y, \Gamma_Y)$ be continuous. Let $V \in \Gamma_Y$. Then there exists $U \in \Gamma_X$, such that $(x, x_\circ) \in U$, implies $(f(x), f(x_\circ)) \in V$. So $A_U = (A \times A) \cap U$ is the set required. □

Theorem 3.25. *Let $f : (X, \Gamma_X) \to (Y, \Gamma_Y)$ be uniform continuous. Then $f : (A, \Gamma_A) \to (Y, \Gamma_Y)$ is uniformly continuous.*

Proof. Let $f : (X, \Gamma_X) \to (Y, \Gamma_Y)$ be uniform continuous. For $V \in \Gamma_Y$, there exists $U \in \Gamma_X$, such that $(x, y) \in U$, implies $(f(x), f(y)) \in V$. So $A_U = (A \times A) \cap U$ is the set required. □

Now we will present a new definition:

Definition 3.26. *Let (X, Γ)be a semi-linear uniform space and $A, B \subseteq X$, $U \in \Gamma$. If $(A \times B) \cap U = \phi$, then we say A and B are $U-$apart and A, B are apart if there exists $U \in \Gamma$such that A, B are $U-$apart.*

Direct application on the previous definition is the following proposition:

Proposition 3.27. *Let (X, Γ)be a semi-linear uniform space and A, B be $U-$apart. If $V \circ V \circ V \subseteq U$ for $V \in \Gamma$, then $B(A, V)$ and $B(B, V)$are $V-$apart.*

Proof. Let (X, Γ)be a semi-linear uniform space and $A, B \subseteq X$ be $U-$apart. if $(s,t) \in (B(A,V) \times B(B,V)) \cap V$, then $s \in B(A,V)$ and $t \in B(B,V)$, which means $(a,s) \in V$ and $(b,t) \in V$ for $a \in A$ and $b \in B$, but $(s,t) \in V$ so we have $(a,b) \in V \circ V \circ V \subseteq U$. Which contradicts $A, B \subseteq X$ are $U-$apart. □

In topology Urysohn Lemma is widely applicable, where it is commonly used to construct continuous functions with various properties on normal space. So we shall present one of the most important facts in semi$-$linear uniform spaces which is called by Urysohn Lemma in semi-linear uniform spaces. Urysohn Lemma in semi-linear uniform spaces is the surprising fact in this work, the proof of this lemma will increase the strength and beauty of the previous definitions that have been defined in this paper or in the other research related to semi$-$linear uniform spaces. Now we will present the lemma:

Lemma 3.28 (Urysohn Lemma in semi-linear uniform spaces.)**.** *Let $A, C \subseteq X$ be apart.Then there exists a uniformly continuos function $f : (X, \Gamma) \to [0,1]$, such that $f(x) = 0$ for $x \in A$ and $f(x) = 1$ for $x \in C$.*

Proof. Let $A, C \subseteq X$ be apart. There exists $U \in \Gamma$ such that $(A \times C) \cap U = \phi$, and as (X, Γ) is semi-linear uniform space, so for all $U \in \Gamma$ there exists $V \in \Gamma$ such that $V \circ V \subseteq U$, so we can define the following sequence $(U_0, U_1, U_2.....)$ as the following: $U_0 = U$, $U_n \circ U_n \subseteq U_{n-1}$, for every $n = 1, 2, 3....$ For $n = 1, 2, ...$and $k = 0, 1, ..., 2^n$, define $A_r, C_r \subseteq X$ for every $r = \frac{k}{2^n}$ as the following: $A_\circ = A, A_1 = X, C_\circ = X$, $C_1 = C$, where $A_{\frac{2k-1}{2^n}} = X - B(C_{\frac{k}{2^{n-1}}}, U_n)$ and $C_{\frac{2k-1}{2^n}} = X - B(A_{\frac{k-1}{2^{n-1}}}, U_n)$. It is clear that $A_{\frac{m-1}{2^n}}$ and $C_{\frac{m}{2^n}}$ are U_n-apart $n = 0, 1, 2, ...$ and $m = 1, ..., 2^n$. Note that $X = A_r \cup C_r$, now define the following increasing assignment: $f : r \to A_r$ by $f(x) = inf\{r : x \in A_r\}$. It is clear $f(x) = 0$ for $x \in A$ and $f(x) = 1$ for $x \in C$. For every $\epsilon > 0$ then there is $n > 0$ such that $\epsilon > \frac{1}{2^{n-1}}$. Now for $x, y \in X$ with $f(x) + \epsilon \leq f(y)$ there exists $m \in \{1, 2, 3, ...2^n\}$ such that $f(x) < \frac{m-1}{2^n} < \frac{m}{2^n} < f(y)$. Hence if $x \in A_{\frac{m-1}{2^n}}$ and $y \notin A_{\frac{m}{2^n}}$ we have $y \in C_{\frac{m}{2^n}}$, so $U_n \subsetneq \rho(x,y)$. Thus for every $(x,y) \in U_n$, we have $(x,y) \notin A_{\frac{m-1}{2^n}} \times C_{\frac{m}{2^n}}$, which means $|(f(x) - f(y)| < \epsilon$. □

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