A note on a finite group with all non-nilpotent maximal subgroups being normal

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Abstract. In this paper we give an elementary proof to show that a finite group with all non-nilpotent maximal subgroups being normal is solvable.

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1. Introduction

It is known that one of the important characterizations of a nilpotent group is that a finite group $G$ is nilpotent if and only if all maximal subgroups of $G$ are normal. As a generalization of this result, it is interesting to characterize the finite group with all non-nilpotent maximal subgroups being normal. The second author, C. Zhang and S. Guo [4, Lemma 4] used a theorem of Ballester-Bolinshe and Shemetkov [1, Theorem 2] about the $p$-nilpotent group to show that a finite group with all non-nilpotent maximal subgroups being normal is solvable. In [5] the second author used a theorem of Rose [3, Theorem 1] about the non-solvable group with a nilpotent maximal subgroup of even order to give two distinct proofs of the solvability of such a group.

In this paper, without using neither [1, Theorem 2] nor [3, Theorem 1], we give an elementary proof of the solvability of such a group.

Theorem 1.1. A finite group $G$ with all non-nilpotent maximal subgroups being normal is solvable.

The following lemma is necessary in the proof of Theorem 1.1 which is given in Section 2.

Lemma 1.2 ([2, Theorem 9.1.10]). Let the finite group $G$ possess a nilpotent Hall $\pi-$subgroup $H$. Then all Hall $\pi-$subgroups of $G$ are conjugate.

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2. Proof of Theorem 1.1

Proof. Suppose that the theorem is not true. Let $G$ be a counterexample of minimal order.

If all maximal subgroups of $G$ are non-nilpotent. By the hypothesis of the theorem, one has that all maximal subgroups of $G$ are normal. Then $G$ is nilpotent, a contradiction. Thus $G$ has nilpotent maximal subgroups. It is clear that $G$ also must have non-nilpotent maximal subgroups. It follows that $G$ is not a simple group.

Let $N$ be a minimal normal subgroup of $G$. Since the hypothesis of the theorem holds for $G/N$ and $|G/N| < |G|$, one has that $G/N$ is solvable. It follows that $N$ is non-solvable. In particular, $G$ has no solvable non-trivial normal subgroup. Let $L$ be a nilpotent maximal subgroup of $G$ and $P$ be any Sylow subgroup of $L$. If $P$ is not a Sylow subgroup of $G$, then $L < N_G(P) \leq G$. One has $N_G(P) = G$ as $L$ is maximal in $G$. It follows that $G$ possesses a solvable non-trivial normal subgroup $P$, a contradiction. Then any Sylow subgroup of $L$ is also a Sylow subgroup of $G$. Thus we have that every nilpotent maximal subgroup of $G$ must be a nilpotent Hall subgroup of $G$.

Let $L_1$ be a nilpotent maximal subgroup of $G$ which is a Hall $\pi_1$–subgroup of $G$ for the set of prime divisors $\pi_1$ of $|G|$ and $L_2$ be a nilpotent maximal subgroup of $G$ which is a Hall $\pi_2$–subgroup of $G$ for the set of prime divisors $\pi_2$ of $|G|$. We will show that $\pi_1 = \pi_2$. Otherwise, assume $\pi_1 \neq \pi_2$. By [2, Theorem 10.4.2], one has $2 \in \pi_1$ and $2 \in \pi_2$. Let $R_1 \in \text{Syl}_2(L_1)$ and $R_2 \in \text{Syl}_2(L_2)$. Suppose $q \in \pi_1$ but $q \notin \pi_2$. Let $Q \in \text{Syl}_q(L_1)$. Then $Q \leq N_G(R_1)$. One has $q | |N_G(R_1)|$. Since $L_2 \leq N_G(R_2)$ and $G$ has no solvable non-trivial normal subgroup, one has $N_G(R_2) = L_2$. Note that $|N_G(R_1)| = |N_G(R_2)|$ since $R_1$ and $R_2$ are conjugate. It follows that $q | |L_2|$, a contradiction. Thus we have $\pi_1 = \pi_2$. That is, all nilpotent maximal subgroups of $G$ are nilpotent Hall $\pi$–subgroups of $G$ for a fixed $\pi$. By Lemma 1.2, we have that all nilpotent maximal subgroups of $G$ are conjugate.

Let $\pi_e(N) = \{p_1, p_2, \ldots, p_s\}$ be the set of prime divisors of $|N|$. For every prime divisor $p_i$ of $|N|$ with $1 \leq i \leq s$, let $P_i \in \text{Syl}_{p_i}(N)$. By Frattini argument, one has $G = N_G(P_i)N$. Since $G$ has no solvable non-trivial normal subgroup, one has $N_G(P_i) < G$. Note that every non-nilpotent maximal subgroup of $G$ is normal in $G$ and $N$ is non-solvable. It follows that every non-nilpotent maximal subgroup of $G$ must contain $N$. Thus $N_G(P_i)$ is contained in some nilpotent maximal subgroup of $G$.

Let $M_i$ be a nilpotent maximal subgroup of $G$ such that $N_G(P_i) \leq M_i$. Then $P_i \leq M_i$. Since all $M_i$ are conjugate for $1 \leq i \leq s$, there exists a $g_i \in G$ such that $M_i^{g_i} = M_1$ for every $2 \leq i \leq s$. Thus $P_i^{g_i} \leq M_i^{g_i} = M_1$. Note that $P_i^{g_i} \leq N^{g_i} = N$ and $N$ can be generated by all its Sylow subgroups, that is, $N = \langle P_1, P_2^{g_2}, \ldots, P_s^{g_s} \rangle$. Then we have $N \leq M_1$. It follows that $N$ is nilpotent, a contradiction.
So the counterexample of minimal order does not exist and then $G$ is solvable.

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References


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