Fuzzy sets in fully UP-semigroups

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Abstract. In this paper, we introduce several types of subsets and of fuzzy sets of fully UP-semigroups, and investigate the algebraic properties of fuzzy sets under the operations of intersection and union. Further, we discuss the relation between t-characteristic fuzzy sets and UP_s-subalgebras (resp., UP_i-subalgebras, UP_s-filters, UP_i-filters, UP_s-ideals, UP_i-ideals, strongly UP_s-ideals and strongly UP_i-ideals).
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1. Introduction and preliminaries

Among many algebraic structures, algebras of logic form important class of algebras. Examples of these are BCK-algebras [7], BCI-algebras [8], B-algebras [17], UP-algebras [4] and others. They are strongly connected with logic. For example, BCI-algebras introduced by Iseki [8] in 1966 have connections with BCI-logic being the BCI-system in combinatory logic which has application in the language of functional programming. BCK and BCI-algebras are two classes of logical algebras. They were introduced by Imai and Iseki [7, 8] in 1966 and have been extensively investigated by many researchers. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras.

Several researches introduced a new class of algebras related to logical algebras and semigroups such as: In 1993, Jun et al. [10] introduced the notion of BCI-semigroups. In 1998, Jun et al. [14] renamed the BCI-semigroup as the IS-algebra. In 2006, Kim [15] introduced the notion of KS-semigroups. In 2015, Endam and Vilela [2] introduced the notion of JB-semigroups. In 2018, Iampan [5] introduced the notion of fully UP-semigroups.

The concept of a fuzzy subset of a set was first considered by Zadeh [23] in 1965. The fuzzy set theories developed by Zadeh and others have found many applications in the domain of mathematics and elsewhere. After the introduction of the concept of fuzzy sets by Zadeh [23], several researches were conducted on

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the generalizations of the notion of fuzzy set and application to many logical algebras such as: In 1998, Jun et al. [11] applied the notion of fuzzy sets to BCI-semigroups (it was renamed as an IS-algebra for the convenience of study), and introduced the concept of fuzzy I-ideals. In 2000, Roh et al. [18] considered the fuzzification of an associative I-ideal of an IS-algebra. They proved that every fuzzy associative I-ideal is a fuzzy I-ideal. By giving an appropriate example, they verified that a fuzzy I-ideal may not be a fuzzy associative I-ideal. They gave a condition for a fuzzy I-ideal to be a fuzzy associative I-ideal, and they investigated some related properties. In 2003, Jun and Kondo [12] proved that some concepts of BCK/BCI-algebras expressed by a certain formula can be naturally extended to the fuzzy setting and that many results are obtained immediately with the use of our method. Moreover They proved that these results can be extended to fuzzy IS-algebras. In 2003, Jianming and Dajing [9] introduced the concept of intuitionistic fuzzy associative I-ideals of IS-algebras and they investigated some related properties. In 2007, Prince Williams and Husain [22] studied fuzzy KS-semigroups. In 2016, Endam and Manahon [1] introduced the notion of fuzzy JB-semigroups and they investigated some of its properties.

In this paper, we introduce several types of subsets and of fuzzy sets of fully UP-semigroups, and investigate the algebraic properties of fuzzy sets under the operations of intersection and union. Further, we discuss the relation between t-characteristic fuzzy sets and UP_s-subalgebras (resp., UP_i-subalgebras, UP_s-filters, UP_i-filters, UP_s-ideals, UP_i-ideals, strongly UP_s-ideals and strongly UP_i-ideals).

Before we begin our study, we will introduce the definition of a UP-algebra.

Definition 1.1 ([4])**.** An algebra $A = (A, \cdot, 0)$ of type $(2, 0)$ is called a *UP-algebra* where A is a nonempty set, $\cdot$ is a binary operation on A, and 0 is a fixed element of A (i.e., a nullary operation) if it satisfies the following axioms: for any $x, y, z \in A$,

(UP-1) $(y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = 0$,

(UP-2) $0 \cdot x = x$,

(UP-3) $x \cdot 0 = 0$, and

(UP-4) $x \cdot y = 0$ and $y \cdot x = 0$ imply $x = y$.

From [4], we know that the notion of UP-algebras is a generalization of KU-algebras.

On a UP-algebra $A = (A, \cdot, 0)$, we define a binary relation $\leq$ on A as follows: for all $x, y \in A$,

$$x \leq y \text{ if and only if } x \cdot y = 0.$$

Example 1.2 ([20])**.** Let X be a universal set and let $\Omega \in \mathcal{P}(X)$. Let $\mathcal{P}_\Omega(X) = \{A \in \mathcal{P}(X) \mid \Omega \subseteq A\}$. Define a binary operation $\cdot$ on $\mathcal{P}_\Omega(X)$ by putting $A \cdot B = B \cap (A' \cup \Omega)$ for all $A, B \in \mathcal{P}_\Omega(X)$. Then $(\mathcal{P}_\Omega(X), \cdot, \Omega)$ is a UP-algebra and we shall call it the *generalized power UP-algebra of type 1 with respect to* Ω.

Example 1.3 ([20])**.** Let X be a universal set and let $\Omega \in \mathcal{P}(X)$. Let $\mathcal{P}^\Omega(X) = \{A \in \mathcal{P}(X) \mid A \subseteq \Omega\}$. Define a binary operation $*$ on $\mathcal{P}^\Omega(X)$ by putting $A * B = B \cup (A' \cap \Omega)$ for all $A, B \in \mathcal{P}^\Omega(X)$. Then $(\mathcal{P}^\Omega(X), *, \Omega)$ is a UP-algebra and we shall call it the *generalized power UP-algebra of type 2 with respect to* Ω.

In particular, $(\mathcal{P}(X), \cdot, \emptyset)$ is the power UP-algebra of type 1 and $(\mathcal{P}(X), *, X)$ is the power UP-algebra of type 2.

In a UP-algebra $A = (A, \cdot, 0)$, the following assertions are valid (see [4, 6]).

(1.1) $(\forall x \in A)(x \cdot x = 0)$,

(1.2) $(\forall x, y, z \in A)(x \cdot y = 0, y \cdot z = 0 \Rightarrow x \cdot z = 0)$,

(1.3) $(\forall x, y, z \in A)(x \cdot y = 0 \Rightarrow (z \cdot x) \cdot (z \cdot y) = 0)$,

(1.4) $(\forall x, y, z \in A)(x \cdot y = 0 \Rightarrow (y \cdot z) \cdot (x \cdot z) = 0)$,

(1.5) $(\forall x, y \in A)(x \cdot (y \cdot x) = 0)$,

(1.6) $(\forall x, y \in A)((y \cdot x) \cdot x = 0 \Leftrightarrow x = y \cdot x)$,

(1.7) $(\forall x, y \in A)(x \cdot (y \cdot y) = 0)$,

(1.8) $(\forall a, x, y, z \in A)((x \cdot (y \cdot z)) \cdot (x \cdot ((a \cdot y) \cdot (a \cdot z))) = 0)$,

(1.9) $(\forall a, x, y, z \in A)((((a \cdot x) \cdot (a \cdot y)) \cdot z) \cdot ((x \cdot y) \cdot z) = 0)$,

(1.10) $(\forall x, y, z \in A)(((x \cdot y) \cdot z) \cdot (y \cdot z) = 0)$,

(1.11) $(\forall x, y, z \in A)(x \cdot y = 0 \Rightarrow x \cdot (z \cdot y) = 0)$,

(1.12) $(\forall x, y, z \in A)(((x \cdot y) \cdot z) \cdot (x \cdot (y \cdot z)) = 0)$, and

(1.13) $(\forall a, x, y, z \in A)(((x \cdot y) \cdot z) \cdot (y \cdot (a \cdot z)) = 0)$.

Definition 1.4 ([3, 4, 21])**.** A nonempty subset S of a UP-algebra $(A, \cdot, 0)$ is called

(1) a *UP-subalgebra* of A if for any $x, y \in S$, $x \cdot y \in S$.

(2) a *UP-filter* of A if it satisfies the following properties:

  (i) the constant 0 of A is in S, and

  (ii) for any $x, y \in A$, $x \cdot y \in S$ and $x \in S$ imply $y \in S$.

(3) a *UP-ideal* of A if it satisfies the following properties:

  (i) the constant 0 of A is in S, and

  (ii) for any $x, y, z \in A$, $x \cdot (y \cdot z) \in S$ and $y \in S$ imply $x \cdot z \in S$.

(4) a *strongly UP-ideal* of A if it satisfies the following properties:

(i) the constant 0 of A is in S, and
(ii) for any $x, y, z \in A$, $(z \cdot y) \cdot (z \cdot x) \in S$ and $y \in S$ imply $x \in S$.

Guntasow et al. [3] proved the generalization that the notion of UP-subalgebras is a generalization of UP-filters, the notion of UP-filters is a generalization of UP-ideals, and the notion of UP-ideals is a generalization of strongly UP-ideals. Moreover, they also proved that a UP-algebra A is the only one strongly UP-ideal of itself.

Definition 1.5. A nonempty subset S of a semigroup $(A, *)$ is called

(1) a *subsemigroup* of A if for any $x, y \in S$, $x * y \in S$.

(2) an *ideal* of A if for any $x \in A$ and $s \in S$, $x * s, s * x \in S$.

Clearly, an ideal is a subsemigroup.

Definition 1.6 ([5])**.** Let A be a nonempty set, $\cdot$ and $*$ are binary operations on A, and 0 is a fixed element of A (i.e., a nullary operation). An algebra $A = (A, \cdot, *, 0)$ of type $(2, 2, 0)$ in which $(A, \cdot, 0)$ is a UP-algebra and $(A, *)$ is a semigroup is called a *fully UP-semigroup* (in short, an *f-UP-semigroup*) if the operation "$*$" is distributive (on both sides) over the operation "$\cdot$".

Definition 1.7 ([23])**.** A *fuzzy set* F in a nonempty set U (or a *fuzzy subset* of U) is described by its membership function f_F. To every point $x \in U$, this function associates a real number $f_F(x)$ in the interval $[0, 1]$. The number $f_F(x)$ is interpreted for the point as a degree of belonging x to the fuzzy set F, that is, $F := \{(u, f_F(u)) \mid u \in U\}$. If $A \subseteq U$ and $t \in (0, 1]$, the *t-characteristic function* [13] χ_A^t of U is a function of U into $\{0, t\}$ defined as follows:

$$\chi_A^t(x) = \begin{cases} t, & \text{if } x \in A, \\ 0, & \text{if } x \notin A. \end{cases}$$

By the definition of t-characteristic function, χ_A^t is a function of U into $\{0, t\} \subset [0, 1]$. We denote the fuzzy set F_A^t in U is described by its membership function χ_A^t, is called the *t-characteristic fuzzy set* of A in U. We say that a fuzzy set F in U is *constant* if its membership function f_F is constant.

Definition 1.8 ([16])**.** Let $\{F_i\}_{i \in I}$ be a nonempty family of fuzzy sets in a nonempty set U where I is an arbitrary index set. The *intersection* of F_i, denoted by $\bigwedge_{i \in I} F_i$, is described by its membership function $f_{\bigwedge_{i \in I} F_i}$ which defined as follows:

$$f_{\bigwedge_{i \in I} F_i}(x) = \inf\{f_{F_i}(x)\}_{i \in I} \text{ for all } x \in U.$$

The *union* of F_i, denoted by $\bigvee_{i \in I} F_i$, is described by its membership function $f_{\bigvee_{i \in I} F_i}$ which defined as follows:

$$\mathrm{f}_{\bigvee_{i\in I}\mathrm{F}_i}(x) = \sup\{\mathrm{f}_{\mathrm{F}_i}(x)\}_{i\in I} \text{ for all } x \in U.$$

Lemma 1.9. *Let S be a nonempty subset of a UP-algebra $(A, \cdot, 0)$ and $t \in (0, 1]$. Then the constant 0 of A is in S if and only if $\chi^t_S(0) \geq \chi^t_S(x)$ for all $x \in A$.*

Proof. Assume that $0 \in S$. Then for all $x \in A$, $\chi^t_S(0) = t \geq \chi^t_S(x)$.

Conversely, assume that $\chi^t_S(0) \geq \chi^t_S(x)$ for all $x \in A$. Since S is a nonempty subset of A, we have an element a in S, that is, $\chi^t_S(a) = t$. Thus $t \geq \chi^t_S(0) \geq \chi^t_S(a) = t$. So $\chi^t_S(0) = t$, that is, $0 \in S$. □

Rosenfeld [19] introduced the notion of fuzzy subsemigroups (resp., fuzzy ideals) of semigroups as follows:

Definition 1.10. A fuzzy set F in a semigroup $A = (A, *)$ is called

(1) a *fuzzy subsemigroup* of A if for any $x, y \in A$,

$$\mathrm{f}_\mathrm{F}(x * y) \geq \min\{\mathrm{f}_\mathrm{F}(x), \mathrm{f}_\mathrm{F}(y)\}.$$

(2) a *fuzzy ideal* of A if for any $x, y \in A$,

$$\mathrm{f}_\mathrm{F}(x * y) \geq \max\{\mathrm{f}_\mathrm{F}(x), \mathrm{f}_\mathrm{F}(y)\}.$$

Clearly, a fuzzy ideal is a fuzzy subsemigroup.

Theorem 1.11. *Let S be a nonempty subset of a semigroup $A = (A, *)$ and $t \in (0, 1]$. Then the following statements hold:*

(1) S is a subsemigroup of A if and only if the t-characteristic fuzzy set F^t_S is a fuzzy subsemigroup of A, and

(2) S is an ideal of A if and only if the t-characteristic fuzzy set F^t_S is a fuzzy ideal of A.

Proof. (1) Assume that S is a subsemigroup of A. Let $x, y \in A$.

Case 1: $x, y \in S$. Then $\chi^t_S(x) = t = \chi^t_S(y)$, so $\min\{\chi^t_S(x), \chi^t_S(y)\} = t$. Since S is a subsemigroup of A, we have $x * y \in S$ and so $\chi^t_S(x * y) = t$. Therefore, $\chi^t_S(x * y) = t \geq t = \min\{\chi^t_S(x), \chi^t_S(y)\}$.

Case 2: $x \notin S$ or $y \notin S$. Then $\chi^t_S(x) = 0$ or $\chi^t_S(y) = 0$, so $\min\{\chi^t_S(x), \chi^t_S(y)\} = 0$. Therefore, $\chi^t_S(x * y) \geq 0 = \min\{\chi^t_S(x), \chi^t_S(y)\}$.

Hence, F^t_S is a fuzzy subsemigroup of A.

Conversely, assume that F^t_S is a fuzzy subsemigroup of A. Let $x, y \in S$. Then $\chi^t_S(y) = t = \chi^t_S(y)$, so $\min\{\chi^t_S(x), \chi^t_S(y)\} = t$. Since F^t_S is a fuzzy subsemigroup of A, we have $t \geq \chi^t_S(x * y) \geq \min\{\chi^t_S(x), \chi^t_S(y)\} = t$. Thus $\chi^t_S(x * y) = t$, that is, $x * y \in S$. Hence, S is a subsemigroup of A.

(2) Assume that S is an ideal of A. Let $x, y \in A$.

Case 1: $x, y \in S$. Then $\chi_S^t(x) = t = \chi_S^t(y)$, so $\max\{\chi_S^t(x), \chi_S^t(y)\} = t$. Since S is an ideal of A, we have $x * y \in S$ and so $\chi_S^t(x * y) = t$. Therefore, $\chi_S^t(x * y) = t \geq t = \max\{\chi_S^t(x), \chi_S^t(y)\}$.

Case 2: $x \notin S$ or $y \notin S$. If $x*y \in S$, then $\chi_S^t(x*y) = t$. Therefore, $\chi_S^t(x*y) = t \geq \max\{\chi_S^t(x), \chi_S^t(y)\}$. If $x * y \notin S$, then $x, y \notin S$. Thus $\chi_S^t(x * y) = 0$ and $\chi_S^t(x) = 0 = \chi_S^t(y)$. Therefore, $\chi_S^t(x * y) = 0 \geq 0 = \max\{\chi_S^t(x), \chi_S^t(y)\}$.

Hence, F_S^t is a fuzzy ideal of A.

Conversely, assume that F_S^t is a fuzzy ideal of A. Let $s \in S$ and $x \in A$. Then $\chi_S^t(s) = t$, so $\max\{\chi_S^t(s), \chi_S^t(x)\} = t$. Since F_S^t is a fuzzy ideal of A, we have $t \geq \chi_S^t(s*x), \chi_S^t(x*s) \geq \max\{\chi_S^t(s), \chi_S^t(x)\} = t$. Thus $\chi_S^t(s*x) = t = \chi_S^t(x*s)$, that is $s * x, x * s \in S$. Hence, S is an ideal of A. □

Somjanta et al. [21] and Guntasow et al. [3] introduced the notion of fuzzy UP-subalgebras (resp., fuzzy UP-filters, fuzzy UP-ideals, fuzzy strongly UP-ideals) of UP-algebras as follows:

Definition 1.12. A fuzzy set F in a UP-algebra $A = (A, \cdot, 0)$ is called

(1) a *fuzzy UP-subalgebra* of A if for any $x, y \in A$,

$$\mathrm{f_F}(x \cdot y) \geq \min\{\mathrm{f_F}(x), \mathrm{f_F}(y)\}.$$

(2) a *fuzzy UP-filter* of A if for any $x, y \in A$,

  (i) $\mathrm{f_F}(0) \geq \mathrm{f_F}(x)$, and
  (ii) $\mathrm{f_F}(y) \geq \min\{\mathrm{f_F}(x \cdot y), \mathrm{f_F}(x)\}$.

(3) a *fuzzy UP-ideal* of A if for any $x, y, z \in A$,

  (i) $\mathrm{f_F}(0) \geq \mathrm{f_F}(x)$, and
  (ii) $\mathrm{f_F}(x \cdot z) \geq \min\{\mathrm{f_F}(x \cdot (y \cdot z)), \mathrm{f_F}(y)\}$.

(4) a *fuzzy strongly UP-ideal* of A if for any $x, y, z \in A$,

  (i) $\mathrm{f_F}(0) \geq \mathrm{f_F}(x)$, and
  (ii) $\mathrm{f_F}(x) \geq \min\{\mathrm{f_F}((z \cdot y) \cdot (z \cdot x)), \mathrm{f_F}(y)\}$.

Guntasow et al. [3] also proved that the notion of fuzzy UP-subalgebras is a generalization of fuzzy UP-filters, the notion of fuzzy UP-filters is a generalization of fuzzy UP-ideals, and the notion of fuzzy UP-ideals is a generalization of fuzzy strongly UP-ideals.

Theorem 1.13 ([3]). *Fuzzy strongly UP-ideals and constant fuzzy sets coincide in UP-algebras.*

Lemma 1.14. *Let* F *be a fuzzy UP-filter of a UP-algebra* $A = (A, \cdot, 0)$. *Then for any* $x, y \in A$,

$$x \leq y \textit{ implies } \mathrm{f_F}(x) \leq \mathrm{f_F}(y) \leq \mathrm{f_F}(x \cdot y).$$

Proof. Let $x, y \in A$ be such that $x \leq y$. Then $x \cdot y = 0$, so

$$\mathrm{f_F}(y) \geq \min\{\mathrm{f_F}(x \cdot y), \mathrm{f_F}(x)\} = \min\{\mathrm{f_F}(0), \mathrm{f_F}(x)\} = \mathrm{f_F}(x).$$

By (1.5), we have $y \leq x \cdot y$ and thus $\mathrm{f_F}(y) \leq \mathrm{f_F}(x \cdot y)$. □

Theorem 1.15. *Let S be a nonempty subset of a UP-algebra $A = (A, \cdot, 0)$ and $t \in (0, 1]$. Then the following statements hold:*

(1) S is a UP-subalgebra of A if and only if the t-characteristic fuzzy set F^t_S is a fuzzy UP-subalgebra of A,

(2) S is a UP-filter of A if and only if the t-characteristic fuzzy set F^t_S is a fuzzy UP-filter of A,

(3) S is a UP-ideal of A if and only if the t-characteristic fuzzy set F^t_S is a fuzzy UP-ideal of A, and

(4) S is a strongly UP-ideal of A if and only if the t-characteristic fuzzy set F^t_S is a fuzzy strongly UP-ideal of A.

Proof. (1) Assume that S is a UP-subalgebra of A. Let $x, y \in A$.

Case 1: $x, y \in S$. Then $\chi^t_S(x) = t = \chi^t_S(y)$, so $\min\{\chi^t_S(x), \chi^t_S(y)\} = t$. Since S is a UP-subalgebra of A, we have $x \cdot y \in S$ and so $\chi^t_S(x \cdot y) = t$. Therefore, $\chi^t_S(x \cdot y) = t \geq t = \min\{\chi^t_S(x), \chi^t_S(y)\}$.

Case 2: $x \notin S$ or $y \notin S$. Then $\chi^t_S(x) = 0$ or $\chi^t_S(y) = 0$, so $\min\{\chi^t_S(x), \chi^t_S(y)\} = 0$. Therefore, $\chi^t_S(x \cdot y) \geq 0 = \min\{\chi^t_S(x), \chi^t_S(y)\}$.

Hence, F^t_S is a fuzzy UP-subalgebra of A.

Conversely, assume that F^t_S is a fuzzy UP-subalgebra of A. Let $x, y \in S$. Then $\chi^t_S(y) = t = \chi^t_S(y)$, so $\min\{\chi^t_S(x), \chi^t_S(y)\} = t$. Since F^t_S is a fuzzy UP-subalgebra of A, we have $t \geq \chi^t_S(x \cdot y) \geq \min\{\chi^t_S(x), \chi^t_S(y)\} = t$. Thus $\chi^t_S(x \cdot y) = t$, that is, $x \cdot y \in S$. Hence, S is a UP-subalgebra of A.

(2) Assume that S is a UP-filter of A. Since $0 \in S$, it follows from Lemma 1.9 that $\chi^t_S(0) \geq \chi^t_S(x)$ for all $x \in A$. Next, let $x, y \in A$.

Case 1: $x, y \in S$. Then $\chi^t_S(x) = t = \chi^t_S(y)$. Thus $\chi^t_S(y) = t \geq \chi^t_S(x \cdot y) = \min\{\chi^t_S(x \cdot y), \chi^t_S(x)\}$.

Case 2: $x \notin S$ or $y \notin S$. If $x \notin S$, then $\chi^t_S(x) = 0$. Thus $\chi^t_S(y) \geq 0 = \min\{\chi^t_S(x \cdot y), \chi^t_S(x)\}$. If $y \notin S$, then $\chi^t_S(y) = 0$. Since S is a UP-filter of A, we have $x \cdot y \notin S$ or $x \notin S$ and so $\chi^t_S(x \cdot y) = 0$ or $\chi^t_S(x) = 0$. Thus $\chi^t_S(y) = 0 \geq 0 = \min\{\chi^t_S(x \cdot y), \chi^t_S(x)\}$.

Hence, F^t_S is a fuzzy UP-filter of A.

Conversely, assume that F^t_S is a fuzzy UP-filter of A. Since $\chi^t_S(0) \geq \chi^t_S(x)$ for all $x \in A$, it follows from Lemma 1.9 that $0 \in S$. Next, let $x, y \in A$ be such that $x \cdot y \in S$ and $x \in S$. Then $\chi^t_S(x \cdot y) = t = \chi^t_S(x)$, so $\min\{\chi^t_S(x \cdot y), \chi^t_S(x)\} = t$.

Since F_S^t is a fuzzy UP-filter of A, we have $t \geq \chi_S^t(y) \geq \min\{\chi_S^t(x\cdot y), \chi_S^t(x)\} = t$. Thus $\chi_S^t(y) = t$, that is, $y \in S$. Hence, S is a UP-filter of A.

(3) Assume that S is a UP-ideal of A. Since $0 \in S$, it follows from Lemma 1.9 that $\chi_S^t(0) \geq \chi_S^t(x)$ for all $x \in A$. Next, let $x, y, z \in A$.

Case 1: $x \cdot (y \cdot z), y \in S$. Then $\chi_S^t(x \cdot (y \cdot z)) = t = \chi_S^t(y)$, so $\min\{\chi_S^t(x \cdot (y \cdot z)), \chi_S^t(y)\} = t$. Since S is a UP-ideal of A, we have $x\cdot z \in S$ and so $\chi_S^t(x\cdot z) = t$. Thus $\chi_S^t(x \cdot z) = t \geq t = \min\{\chi_S^t(x \cdot (y \cdot z)), \chi_S^t(y)\}$.

Case 2: $x \cdot (y \cdot z) \notin S$ or $y \notin S$. Then $\chi_S^t(x \cdot (y \cdot z)) = 0$ or $\chi_S^t(y) = 0$, so $\min\{\chi_S^t(x \cdot (y \cdot z)), \chi_S^t(y)\} = 0$. Thus $\chi_S^t(x \cdot z) \geq 0 = \min\{\chi_S^t(x \cdot (y \cdot z)), \chi_S^t(y)\}$.

Hence, F_S^t is a fuzzy UP-ideal of A.

Conversely, assume that F_S^t is a fuzzy UP-ideal of A. Since $\chi_S^t(0) \geq \chi_S^t(x)$ for all $x \in A$, it follows from Lemma 1.9 that $0 \in S$. Next, let $x, y, z \in A$ such that $x \cdot (y \cdot z) \in S$ and $y \in S$. Then $\chi_S^t(x \cdot (y \cdot z)) = t = \chi_S^t(y)$, so $\min\{\chi_S^t(x \cdot (y \cdot z)), \chi_S^t(y)\} = t$. Since F_S^t is a fuzzy UP-ideal of A, we have $t \geq \chi_S^t(x \cdot z) \geq \min\{\chi_S^t(x \cdot (y \cdot z)), \chi_S^t(y)\} = t$. Thus $\chi_S^t(x \cdot z) = t$, that is, $x \cdot z \in S$. Hence, S is a UP-ideal of A.

(4) It is straightforward by Theorem 1.13, and A is the only one strongly UP-ideal of itself. □

2. Special subsets of fully UP-semigroups

In this section, we introduce the notions of $\mathrm{UP_s}$-subalgebras, $\mathrm{UP_i}$-subalgebras, $\mathrm{UP_s}$-filters, $\mathrm{UP_i}$-filters, $\mathrm{UP_s}$-ideals, $\mathrm{UP_i}$-ideals, strongly $\mathrm{UP_s}$-ideals, and strongly $\mathrm{UP_i}$-ideals of fully UP-semigroups, provide the necessary examples and prove its generalizations.

From now on, we shall let A be an f-UP-semigroup $A = (A, \cdot, *, 0)$ unless otherwise specified.

Definition 2.1. A subset S of an f-UP-semigroup A is called

(1) a *UP_{s}-subalgebra* of A if S is a UP-subalgebra of $(A, \cdot, 0)$, and S is a subsemigroup of $(A, *)$.

(2) a *UP_{i}-subalgebra* of A if S is a UP-subalgebra of $(A, \cdot, 0)$, and S is an ideal of $(A, *)$.

We have Theorem 2.2, 2.8, and 2.13 directly from Definition 1.5.

Theorem 2.2. *Every UP_{i}-subalgebra of A is a UP_{s}-subalgebra of A.*

Example 2.3. Let $A = \{0, 1, 2, 3\}$ be a set with two binary operations $\cdot$ and $*$ defined by the following Cayley tables:

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	2	3
2	0	1	0	3
3	0	1	2	0

$*$	0	1	2	3
0	0	0	0	0
1	0	1	0	0
2	0	0	2	0
3	0	3	0	0

Then $A = (A, \cdot, *, 0)$ is an f-UP-semigroup. Let $S = \{0, 1, 2\}$. Then S is a UP_s-subalgebra of A. Since $1 \in S$ and $3 \in A$ but $3 * 1 = 3 \notin S$, we have S is not an ideal of $(A, *)$. Thus S is not a UP_i-subalgebra of A.

Definition 2.4. A subset S of an f-UP-semigroup $A = (A, \cdot, *, 0)$ is called

(1) a *UP_s-filter* of A if S is a UP-filter of $(A, \cdot, 0)$, and S is a subsemigroup of $(A, *)$.

(2) a *UP_i-filter* of A if S is a UP-filter of $(A, \cdot, 0)$, and S is an ideal of $(A, *)$.

We have Theorem 2.5, 2.7, 2.10, 2.12, 2.15, and 2.17 directly from a result quoted in Definition 1.4.

Theorem 2.5. *Every UP_s-filter of A is a UP_s-subalgebra of A.*

Example 2.6. Let $A = \{0, 1, 2, 3\}$ be a set with two binary operations $\cdot$ and $*$ defined by the following Cayley tables:

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	2	3
2	0	0	0	3
3	0	0	0	0

$*$	0	1	2	3
0	0	0	0	0
1	0	0	0	0
2	0	0	0	0
3	0	0	0	1

Then $A = (A, \cdot, *, 0)$ is an f-UP-semigroup. Let $S = \{0, 2\}$. Then S is a UP_s-subalgebra of A. Since $2 \cdot 1 = 0 \in S$ and $2 \in S$ but $1 \notin S$, we have S is not a UP-filter of $(A, \cdot, 0)$. Thus S is not a UP_s-filter of A.

Theorem 2.7. *Every UP_i-filter of A is a UP_i-subalgebra of A.*

In Example 2.6, we have S is a UP_i-subalgebra of A. Since S is not a UP-filter of $(A, \cdot, 0)$, we have S is not a UP_i-filter of A.

Theorem 2.8. *Every UP_i-filter of A is a UP_s-filter of A.*

In Example 2.3, we have S is a UP_s-filter of A. Since S is not an ideal of $(A, *)$, we have S is not a UP_i-filter of A.

Definition 2.9. A subset S of an f-UP-semigroup A is called

(1) a *UP_s-ideal* of A if S is a UP-ideal of $(A, \cdot, 0)$, and S is a subsemigroup of $(A, *)$.

(2) a *UP_i-ideal* of A if S is a UP-ideal of $(A, \cdot, 0)$, and S is an ideal of $(A, *)$.

Theorem 2.10. *Every UP_s-ideal of A is a UP_s-filter of A.*

Example 2.11. Let $A = \{0, 1, 2, 3\}$ be a set with two binary operations $\cdot$ and $*$ defined by the following Cayley tables:

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	2	2
2	0	1	0	2
3	0	1	0	0

$*$	0	1	2	3
0	0	0	0	0
1	0	0	0	0
2	0	0	0	0
3	0	0	0	0

Then $A = (A, \cdot, *, 0)$ is an f-UP-semigroup. Let $S = \{0, 1\}$. Then S is a UP_s-filter of A. Since $2 \cdot (1 \cdot 3) = 0 \in S$ and $1 \in S$ but $2 \cdot 3 = 2 \notin S$, we have S is not a UP-ideal of $(A, \cdot, 0)$. Thus S is not a UP_s-ideal of A.

Theorem 2.12. *Every UP_i-ideal of A is a UP_i-filter of A.*

In Example 2.11, we have S is a UP_i-filter of A. Since S is not a UP-ideal of $(A, \cdot, 0)$, we have S is not a UP_i-ideal of A.

Theorem 2.13. *Every UP_i-ideal of A is a UP_s-ideal of A.*

In Example 2.3, we have S is a UP_s-ideal of A. Since S is not an ideal of $(A, *)$, we have S is not a UP_i-ideal of A.

Definition 2.14. A subset S of an f-UP-semigroup A is called

(1) a *strongly UP_s-ideal* of A if S is a strongly UP-ideal of $(A, \cdot, 0)$, and S is a subsemigroup of $(A, *)$.

(2) a *strongly UP_i-ideal* of A if S is a strongly UP-ideal of $(A, \cdot, 0)$, and S is an ideal of $(A, *)$.

Theorem 2.15. *Every strongly UP_s-ideal of A is a UP_s-ideal of A.*

Example 2.16. Let $A = \{0, 1, 2, 3\}$ be a set with two binary operations $\cdot$ and $*$ defined by the following Cayley tables:

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	2	3
2	0	1	0	3
3	0	1	2	0

$*$	0	1	2	3
0	0	0	0	0
1	0	0	0	0
2	0	0	0	1
3	0	0	1	0

Then $A = (A, \cdot, *, 0)$ is an f-UP-semigroup. Let $S = \{0, 1, 2\}$. Then S is a UP_s-ideal of A. Since $S \neq A$, we have S is not a strongly UP-ideal of $(A, \cdot, 0)$. Thus S is not a strongly UP_s-ideal of A.

Theorem 2.17. *Every strongly* $UP_{\rm i}$*-ideal of* A *is a* $UP_{\rm i}$*-ideal of* A.

In Example 2.16, we have S is a $\mathrm{UP_i}$-ideal of A. Since S is not a strongly UP-ideal of $(A, \cdot, 0)$, we have S is not a strongly $\mathrm{UP_i}$-ideal of A.

Theorem 2.18. *Strongly* $UP_{\rm s}$*-ideals and strongly* $UP_{\rm i}$*-ideals coincide in* A *and it is only* A.

Proof. It is straightforward by A is the only one strongly UP-ideal of itself. □

3. Fuzzy sets in fully UP-semigroups

In this section, we introduce the notions of fuzzy $\mathrm{UP_s}$-subalgebras, fuzzy $\mathrm{UP_i}$-subalgebras, fuzzy $\mathrm{UP_s}$-filters, fuzzy $\mathrm{UP_i}$-filters, fuzzy $\mathrm{UP_s}$-ideals, fuzzy $\mathrm{UP_i}$-ideals, fuzzy strongly $\mathrm{UP_s}$-ideals, and fuzzy strongly $\mathrm{UP_i}$-ideals of fully UP-semigroups, provide the necessary examples, prove its generalizations and investigate the algebraic properties of fuzzy sets under the operations of intersection and union.

Definition 3.1. A fuzzy set F in an f-UP-semigroup A is called

(1) a *fuzzy* $UP_{\rm s}$*-subalgebra* of A if F is a fuzzy UP-subalgebra of $(A, \cdot, 0)$ and a fuzzy subsemigroup of $(A, *)$.

(2) a *fuzzy* $UP_{\rm i}$*-subalgebra* of A if F is a fuzzy UP-subalgebra of $(A, \cdot, 0)$ and a fuzzy ideal of $(A, *)$.

Clearly, a fuzzy $\mathrm{UP_i}$-subalgebra is a fuzzy $\mathrm{UP_s}$-subalgebra.

In Example 2.16, we define a membership function $\mathrm{f_F}$ as follows:

$$\mathrm{f_F}(0) = 1,\ \mathrm{f_F}(1) = 0.4,\ \mathrm{f_F}(2) = 0.5,\ \text{and } \mathrm{f_F}(3) = 0.2.$$

Then F is a fuzzy $\mathrm{UP_s}$-subalgebra of A. Since $\mathrm{f_F}(2 * 3) = \mathrm{f_F}(1) = 0.4 \ngeq 0.5 = \max\{0.5, 0.2\} = \max\{\mathrm{f_F}(2), \mathrm{f_F}(3)\}$, we have F is not a fuzzy $\mathrm{UP_i}$-subalgebra of A.

Theorem 3.2. *The intersection of any nonempty family of fuzzy* $UP_{\rm s}$*-subalgebra of* A *is also a fuzzy* $UP_{\rm s}$*-subalgebra of* A.

Proof. Let F_i be a fuzzy $\mathrm{UP_s}$-subalgebra of A for all $i \in I$. Then

$$\begin{aligned}
\mathrm{f}_{\bigwedge_{i\in I}\mathrm{F}_i}(x \cdot y) &= \inf\{\mathrm{f}_{\mathrm{F}_i}(x \cdot y)\}_{i\in I} \\
&\geq \inf\{\min\{\mathrm{f}_{\mathrm{F}_i}(x), \mathrm{f}_{\mathrm{F}_i}(y)\}\}_{i\in I} \\
&= \min\{\inf\{\mathrm{f}_{\mathrm{F}_i}(x)\}_{i\in I}, \inf\{\mathrm{f}_{\mathrm{F}_i}(y)\}_{i\in I}\} \\
&= \min\{\mathrm{f}_{\bigwedge_{i\in I}\mathrm{F}_i}(x), \mathrm{f}_{\bigwedge_{i\in I}\mathrm{F}_i}(y)\} \text{ and} \\
\mathrm{f}_{\bigwedge_{i\in I}\mathrm{F}_i}(x * y) &= \inf\{\mathrm{f}_{\mathrm{F}_i}(x * y)\}_{i\in I} \\
&\geq \inf\{\min\{\mathrm{f}_{\mathrm{F}_i}(x), \mathrm{f}_{\mathrm{F}_i}(y)\}\}_{i\in I} \\
&= \min\{\inf\{\mathrm{f}_{\mathrm{F}_i}(x)\}_{i\in I}, \inf\{\mathrm{f}_{\mathrm{F}_i}(y)\}_{i\in I}\} \\
&= \min\{\mathrm{f}_{\bigwedge_{i\in I}\mathrm{F}_i}(x), \mathrm{f}_{\bigwedge_{i\in I}\mathrm{F}_i}(y)\}.
\end{aligned}$$

Hence, $\bigwedge_{i\in I} \mathrm{F}_i$ is a fuzzy $\mathrm{UP_s}$-subalgebra of A. □

In Example 2.16, we define two membership functions $\mathrm{f}_{\mathrm{F}1}$ and $\mathrm{f}_{\mathrm{F}2}$ as follow:

A	0	1	2	3
$\mathrm{f}_{\mathrm{F}1}$	0.7	0.5	0.7	0.3
$\mathrm{f}_{\mathrm{F}2}$	0.7	0.3	0.2	0.6

Then F_1 and F_2 are fuzzy $\mathrm{UP_s}$-subalgebras of A. Since $\mathrm{f}_{\mathrm{F}_1\vee\mathrm{F}_2}(3*2) = \mathrm{f}_{\mathrm{F}_1\vee\mathrm{F}_2}(1) = 0.5 \ngeq 0.6 = \min\{0.6, 0.7\} = \min\{\mathrm{f}_{\mathrm{F}_1\vee\mathrm{F}_2}(3), \mathrm{f}_{\mathrm{F}_1\vee\mathrm{F}_2}(2)\}$, we have $\mathrm{F}_1 \vee \mathrm{F}_2$ is not a fuzzy $\mathrm{UP_s}$-subalgebra of A.

Theorem 3.3. *A nonempty subset S of A is a UP_s-subalgebra of A if and only if the t-characteristic fuzzy set F_S^t is a fuzzy UP_s-subalgebra of A.*

Proof. It is straightforward by Theorem 1.11 (1) and Theorem 1.15 (1). □

Theorem 3.4. *The intersection of any nonempty family of fuzzy UP_i-subalgebra of A is also a fuzzy UP_i-subalgebra of A.*

Proof. Let F_i be a fuzzy $\mathrm{UP_i}$-subalgebra of A for all $i \in I$. Then

$$\begin{aligned}
\mathrm{f}_{\bigwedge_{i\in I}\mathrm{F}_i}(x\cdot y) &= \inf\{\mathrm{f}_{\mathrm{F}i}(x\cdot y)\}_{i\in I}\\
&\geq \inf\{\min\{\mathrm{f}_{\mathrm{F}i}(x), \mathrm{f}_{\mathrm{F}i}(y)\}\}_{i\in I}\\
&= \min\{\inf\{\mathrm{f}_{\mathrm{F}i}(x)\}_{i\in I}, \inf\{\mathrm{f}_{\mathrm{F}i}(y)\}_{i\in I}\}\\
&= \min\{\mathrm{f}_{\bigwedge_{i\in I}\mathrm{F}_i}(x), \mathrm{f}_{\bigwedge_{i\in I}\mathrm{F}_i}(y)\} \text{ and}\\
\mathrm{f}_{\bigwedge_{i\in I}\mathrm{F}_i}(x * y) &= \inf\{\mathrm{f}_{\mathrm{F}i}(x * y)\}_{i\in I}\\
&\geq \inf\{\max\{\mathrm{f}_{\mathrm{F}i}(x), \mathrm{f}_{\mathrm{F}i}(y)\}\}_{i\in I}\\
&\geq \max\{\inf\{\mathrm{f}_{\mathrm{F}i}(x)\}_{i\in I}, \inf\{\mathrm{f}_{\mathrm{F}i}(y)\}_{i\in I}\}\\
&= \max\{\mathrm{f}_{\bigwedge_{i\in I}\mathrm{F}_i}(x), \mathrm{f}_{\bigwedge_{i\in I}\mathrm{F}_i}(y)\}.
\end{aligned}$$

Hence, $\bigwedge_{i\in I} \mathrm{F}_i$ is a fuzzy $\mathrm{UP_i}$-subalgebra of A. □

In Example 2.11, we define two membership functions $\mathrm{f}_{\mathrm{F}1}$ and $\mathrm{f}_{\mathrm{F}2}$ as follow:

A	0	1	2	3
$\mathrm{f}_{\mathrm{F}_1}$	0.9	0.7	0.1	0.1
$\mathrm{f}_{\mathrm{F}_2}$	0.8	0.4	0.5	0.6

Then F_1 and F_2 are fuzzy $\mathrm{UP_i}$-subalgebras of A. Since $\mathrm{f}_{\mathrm{F}_1\vee\mathrm{F}_2}(1\cdot 3) = \mathrm{f}_{\mathrm{F}_1\vee\mathrm{F}_2}(2) = 0.5 \ngeq 0.6 = \min\{0.7, 0.6\} = \min\{\mathrm{f}_{\mathrm{F}_1\vee\mathrm{F}_2}(1), \mathrm{f}_{\mathrm{F}_1\vee\mathrm{F}_2}(3)\}$, we have $\mathrm{F}_1 \vee \mathrm{F}_2$ is not a fuzzy $\mathrm{UP_i}$-subalgebra of A.

Theorem 3.5. *A nonempty subset S of A is a UP_i-subalgebra of A if and only if the t-characteristic fuzzy set F_S^t is a fuzzy UP_i-subalgebra of A.*

Proof. It is straightforward by Theorem 1.11 (2) and Theorem 1.15 (1). □

Definition 3.6. A fuzzy set F in an f-UP-semigroup A is called

(1) a *fuzzy* UP_{s}*-filter* of A if F is a fuzzy UP-filter of $(A,\cdot,0)$ and a fuzzy subsemigroup of $(A,*)$.

(2) a *fuzzy* UP_{i}*-filter* of A if F is a fuzzy UP-filter of $(A,\cdot,0)$ and a fuzzy ideal of $(A,*)$.

Clearly, a fuzzy $\mathrm{UP_i}$-filter is a fuzzy $\mathrm{UP_s}$-filter.

In Example 2.16, we define a membership function $\mathrm{f_F}$ as follows:

$$\mathrm{f_F}(0)=1,\ \mathrm{f_F}(1)=0.4,\ \mathrm{f_F}(2)=0.5,\ \text{and } \mathrm{f_F}(3)=0.2.$$

Then F is a fuzzy $\mathrm{UP_s}$-filter of A. Since$\mathrm{f_F}(2*3)=\mathrm{f_F}(1)=0.4\ngeq 0.5=\max\{0.5,0.2\}=\max\{\mathrm{f_F}(2),\mathrm{f_F}(3)\}$, we have F is not a fuzzy $\mathrm{UP_i}$-filter of A.

Theorem 3.7. *The intersection of any nonempty family of fuzzy* UP_{s}*-filter of* A *is also a fuzzy* UP_{s}*-filter of* A*.*

Proof. Let F_i be a fuzzy $\mathrm{UP_s}$-filter of A for all $i\in I$. Then

$$\begin{aligned}
\mathrm{f}_{\bigwedge_{i\in I}\mathrm{F}_i}(0)&=\inf\{\mathrm{f}_{\mathrm{F}i}(0)\}_{i\in I}\\
&\geq\inf\{\mathrm{f}_{\mathrm{F}i}(x)\}_{i\in I}\\
&=\mathrm{f}_{\bigwedge_{i\in I}\mathrm{F}_i}(x),\\
\mathrm{f}_{\bigwedge_{i\in I}\mathrm{F}_i}(y)&=\inf\{\mathrm{f}_{\mathrm{F}i}(y)\}_{i\in I}\\
&\geq\inf\{\min\{\mathrm{f}_{\mathrm{F}i}(x\cdot y),\mathrm{f}_{\mathrm{F}i}(x)\}\}_{i\in I}\\
&=\min\{\inf\{\mathrm{f}_{\mathrm{F}i}(x\cdot y)\}_{i\in I},\inf\{\mathrm{f}_{\mathrm{F}i}(x)\}_{i\in I}\}\\
&=\min\{\mathrm{f}_{\bigwedge_{i\in I}\mathrm{F}_i}(x\cdot y),\mathrm{f}_{\bigwedge_{i\in I}\mathrm{F}_i}(x)\},\ \text{and}\\
\mathrm{f}_{\bigwedge_{i\in I}\mathrm{F}_i}(x*y)&=\inf\{\mathrm{f}_{\mathrm{F}i}(x*y)\}_{i\in I}\\
&\geq\inf\{\min\{\mathrm{f}_{\mathrm{F}i}(x),\mathrm{f}_{\mathrm{F}i}(y)\}\}_{i\in I}\\
&=\min\{\inf\{\mathrm{f}_{\mathrm{F}i}(x)\}_{i\in I},\inf\{\mathrm{f}_{\mathrm{F}i}(y)\}_{i\in I}\}\\
&=\min\{\mathrm{f}_{\bigwedge_{i\in I}\mathrm{F}_i}(x),\mathrm{f}_{\bigwedge_{i\in I}\mathrm{F}_i}(y)\}.
\end{aligned}$$

Hence, $\bigwedge_{i\in I}\mathrm{F}_i$ is a fuzzy $\mathrm{UP_s}$-filter of A. □

In Example 2.16, we define two membership functions $\mathrm{f_{F1}}$ and $\mathrm{f_{F2}}$ as follow:

A	0	1	2	3
$\mathrm{f_{F_1}}$	0.7	0.5	0.7	0.3
$\mathrm{f_{F_2}}$	0.7	0.3	0.2	0.6

Then $\mathrm{F_1}$ and $\mathrm{F_2}$ are fuzzy $\mathrm{UP_s}$-filters of A. Since $\mathrm{f_{F_1\vee F_2}}(2*3)=\mathrm{f_{F_1\vee F_2}}(1)=0.5\ngeq 0.6=\min\{0.7,0.6\}=\min\{\mathrm{f_{F_1\vee F_2}}(2),\mathrm{f_{F_1\vee F_2}}(3)\}$, we have $\mathrm{F_1}\vee\mathrm{F_2}$ is not a fuzzy $\mathrm{UP_s}$-filter of A.

Theorem 3.8. *A nonempty subset S of A is a UP$_{\mathrm{s}}$-filter of A if and only if the t-characteristic fuzzy set F_S^t is a fuzzy UP$_{\mathrm{s}}$-filter of A.*

Proof. It is straightforward by Theorem 1.11 (1) and Theorem 1.15 (2). □

Theorem 3.9. *The intersection of any nonempty family of fuzzy UP$_{\mathrm{i}}$-filter of A is also a fuzzy UP$_{\mathrm{i}}$-filter of A.*

Proof. Let F_i be a fuzzy UP$_{\mathrm{i}}$-filter of A for all $i \in I$. Then

$$\begin{aligned}
\mathrm{f}_{\bigwedge_{i\in I}\mathrm{F}_i}(0) &= \inf\{\mathrm{f}_{\mathrm{F}_i}(0)\}_{i\in I} \\
&\geq \inf\{\mathrm{f}_{\mathrm{F}_i}(x)\}_{i\in I} \\
&= \mathrm{f}_{\bigwedge_{i\in I}\mathrm{F}_i}(x), \\
\mathrm{f}_{\bigwedge_{i\in I}\mathrm{F}_i}(y) &= \inf\{\mathrm{f}_{\mathrm{F}_i}(y)\}_{i\in I} \\
&\geq \inf\{\min\{\mathrm{f}_{\mathrm{F}_i}(x\cdot y), \mathrm{f}_{\mathrm{F}_i}(x)\}\}_{i\in I} \\
&= \min\{\inf\{\mathrm{f}_{\mathrm{F}_i}(x\cdot y)\}_{i\in I}, \inf\{\mathrm{f}_{\mathrm{F}_i}(x)\}_{i\in I}\} \\
&= \min\{\mathrm{f}_{\bigwedge_{i\in I}\mathrm{F}_i}(x\cdot y), \mathrm{f}_{\bigwedge_{i\in I}\mathrm{F}_i}(x)\}, \text{ and} \\
\mathrm{f}_{\bigwedge_{i\in I}\mathrm{F}_i}(x*y) &= \inf\{\mathrm{f}_{\mathrm{F}_i}(x*y)\}_{i\in I} \\
&\geq \inf\{\max\{\mathrm{f}_{\mathrm{F}_i}(x), \mathrm{f}_{\mathrm{F}_i}(y)\}\}_{i\in I} \\
&\geq \max\{\inf\{\mathrm{f}_{\mathrm{F}_i}(x)\}_{i\in I}, \inf\{\mathrm{f}_{\mathrm{F}_i}(y)\}_{i\in I}\} \\
&= \max\{\mathrm{f}_{\bigwedge_{i\in I}\mathrm{F}_i}(x), \mathrm{f}_{\bigwedge_{i\in I}\mathrm{F}_i}(y)\}.
\end{aligned}$$

Hence, $\bigwedge_{i\in I}\mathrm{F}_i$ is a fuzzy UP$_{\mathrm{i}}$-filter of A. □

Example 3.10. Let $A = \{0, 1, 2, 3\}$ be a set with two binary operations $\cdot$ and $*$ defined by the following Cayley tables:

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	2	2
2	0	1	0	1
3	0	0	0	0

$*$	0	1	2	3
0	0	0	0	0
1	0	0	0	0
2	0	0	0	0
3	0	0	0	0

We define two membership functions $\mathrm{f}_{\mathrm{F}_1}$ and $\mathrm{f}_{\mathrm{F}_2}$ as follow:

A	0	1	2	3
$\mathrm{f}_{\mathrm{F}_1}$	0.9	0.9	0.5	0.5
$\mathrm{f}_{\mathrm{F}_2}$	1	0.5	0.6	0.5

Then F_1 and F_2 are fuzzy UP$_{\mathrm{i}}$-filters of A. Since $\mathrm{f}_{\mathrm{F}_1\vee\mathrm{F}_2}(3) = 0.5 \ngeq 0.6 = \min\{0.9, 0.6\} = \min\{\mathrm{f}_{\mathrm{F}_1\vee\mathrm{F}_2}(1), \mathrm{f}_{\mathrm{F}_1\vee\mathrm{F}_2}(2)\} = \min\{\mathrm{f}_{\mathrm{F}_1\vee\mathrm{F}_2}(2\cdot 3), \mathrm{f}_{\mathrm{F}_1\vee\mathrm{F}_2}(2)\}$, we have $\mathrm{F}_1 \vee \mathrm{F}_2$ is not a fuzzy UP$_{\mathrm{i}}$-filter of A.

Theorem 3.11. *A nonempty subset S of A is a UP$_{\mathrm{i}}$-filter of A if and only if the t-characteristic fuzzy set F_S^t is a fuzzy UP$_{\mathrm{i}}$-filter of A.*

Proof. It is straightforward by Theorem 1.11 (2) and Theorem 1.15 (2). □

We have Theorem 3.12, 3.13, 3.19, 3.20, and 3.27 directly from a result quoted in Definition 1.12.

Theorem 3.12. *Every fuzzy* $UP_{\rm s}$*-filter of* A *is a fuzzy* $UP_{\rm s}$*-subalgebra of* A*.*

In Example 2.6, we define a membership function $\mathrm{f_F}$ as follows:

$$\mathrm{f_F}(0) = 1,\ \mathrm{f_F}(1) = 0.2,\ \mathrm{f_F}(2) = 0.9,\ \text{and}\ \mathrm{f_F}(3) = 0.1.$$

Then F is a fuzzy $\mathrm{UP_s}$-subalgebra of A. Since $\mathrm{f_F}(1) = 0.2 \ngeq 0.9 = \min\{1, 0.9\} = \min\{\mathrm{f_F}(0), \mathrm{f_F}(2)\} = \min\{\mathrm{f_F}(2 \cdot 1), \mathrm{f_F}(2)\}$, we have F is not a fuzzy $\mathrm{UP_s}$-filter of A.

Theorem 3.13. *Every fuzzy* $UP_{\rm i}$*-filter of* A *is a fuzzy* $UP_{\rm i}$*-subalgebra of* A*.*

In Example 2.6, we define a membership function $\mathrm{f_F}$ as follows:

$$\mathrm{f_F}(0) = 1,\ \mathrm{f_F}(1) = 0.2,\ \mathrm{f_F}(2) = 0.9,\ \text{and}\ \mathrm{f_F}(3) = 0.1.$$

Then F is a fuzzy $\mathrm{UP_i}$-subalgebra of A. Since $\mathrm{f_F}(1) = 0.2 \ngeq 0.9 = \min\{1, 0.9\} = \min\{\mathrm{f_F}(0), \mathrm{f_F}(2)\} = \min\{\mathrm{f_F}(2 \cdot 1), \mathrm{f_F}(2)\}$, we have F is not a fuzzy $\mathrm{UP_i}$-filter of A.

Definition 3.14. A fuzzy set F in an f-UP-semigroup A is called

(1) a *fuzzy* $UP_{\rm s}$*-ideal* of A if F is a fuzzy UP-ideal of $(A, \cdot, 0)$ and a fuzzy subsemigroup of $(A, *)$.

(2) a *fuzzy* $UP_{\rm i}$*-ideal* of A if F is a fuzzy UP-ideal of $(A, \cdot, 0)$ and a fuzzy ideal of $(A, *)$.

Clearly, a fuzzy $\mathrm{UP_i}$-ideal is a fuzzy $\mathrm{UP_s}$-ideal.

In Example 2.16, we define a membership function $\mathrm{f_F}$ as follows:

$$\mathrm{f_F}(0) = 1,\ \mathrm{f_F}(1) = 0.4,\ \mathrm{f_F}(2) = 0.5,\ \text{and}\ \mathrm{f_F}(3) = 0.2.$$

Then F is a fuzzy $\mathrm{UP_s}$-ideal of A. Since $\mathrm{f_F}(3 * 2) = \mathrm{f_F}(1) = 0.4 \ngeq 0.5 = \max\{0.2, 0.5\} = \max\{\mathrm{f_F}(3), \mathrm{f_F}(2)\}$, we have F is not a fuzzy $\mathrm{UP_i}$-ideal of A.

Theorem 3.15. *The intersection of any nonempty family of fuzzy* $UP_{\rm s}$*-ideal of* A *is also a fuzzy* $UP_{\rm s}$*-ideal of* A*.*

Proof. Let F_i be a fuzzy $\mathrm{UP_s}$-ideal of A for all $i \in I$. Then

$$\begin{aligned}
\mathrm{f}_{\bigwedge_{i\in I}\mathrm{F}_i}(0) &= \inf\{\mathrm{f}_{\mathrm{F}i}(0)\}_{i\in I} \\
&\geq \inf\{\mathrm{f}_{\mathrm{F}i}(x)\}_{i\in I} \\
&= \mathrm{f}_{\bigwedge_{i\in I}\mathrm{F}_i}(x), \\
\mathrm{f}_{\bigwedge_{i\in I}\mathrm{F}_i}(x\cdot z) &= \inf\{\mathrm{f}_{\mathrm{F}i}(x\cdot z)\}_{i\in I} \\
&\geq \inf\{\min\{\mathrm{f}_{\mathrm{F}i}(x\cdot(y\cdot z)), \mathrm{f}_{\mathrm{F}i}(y)\}\}_{i\in I} \\
&= \min\{\inf\{\mathrm{f}_{\mathrm{F}i}(x\cdot(y\cdot z))\}_{i\in I}, \inf\{\mathrm{f}_{\mathrm{F}i}(y)\}_{i\in I}\} \\
&= \min\{\mathrm{f}_{\bigwedge_{i\in I}\mathrm{F}_i}(x\cdot(y\cdot z)), \mathrm{f}_{\bigwedge_{i\in I}\mathrm{F}_i}(y)\}, \text{ and} \\
\mathrm{f}_{\bigwedge_{i\in I}\mathrm{F}_i}(x * y) &= \inf\{\mathrm{f}_{\mathrm{F}i}(x * y)\}_{i\in I} \\
&\geq \inf\{\min\{\mathrm{f}_{\mathrm{F}i}(x), \mathrm{f}_{\mathrm{F}i}(y)\}\}_{i\in I} \\
&= \min\{\inf\{\mathrm{f}_{\mathrm{F}i}(x)\}_{i\in I}, \inf\{\mathrm{f}_{\mathrm{F}i}(y)\}_{i\in I}\} \\
&= \min\{\mathrm{f}_{\bigwedge_{i\in I}\mathrm{F}_i}(x), \mathrm{f}_{\bigwedge_{i\in I}\mathrm{F}_i}(y)\}.
\end{aligned}$$

Hence, $\bigwedge_{i\in I}\mathrm{F}_i$ is a fuzzy $\mathrm{UP_s}$-ideal of A. □

In Example 2.16, we define two membership functions $\mathrm{f}_{\mathrm{F}1}$ and $\mathrm{f}_{\mathrm{F}2}$ as follow:

A	0	1	2	3
$\mathrm{f}_{\mathrm{F}_1}$	0.7	0.5	0.7	0.3
$\mathrm{f}_{\mathrm{F}_2}$	0.7	0.3	0.2	0.6

Then F_1 and F_2 are fuzzy $\mathrm{UP_s}$-ideals of A. Since $\mathrm{f}_{\mathrm{F}_1\vee\mathrm{F}_2}(3*2) = \mathrm{f}_{\mathrm{F}_1\vee\mathrm{F}_2}(1) = 0.5 \ngeq 0.6 = \min\{0.6, 0.7\} = \min\{\mathrm{f}_{\mathrm{F}_1\vee\mathrm{F}_2}(3), \mathrm{f}_{\mathrm{F}_1\vee\mathrm{F}_2}(2)\}$, we have $\mathrm{F}_1 \vee \mathrm{F}_2$ is not a fuzzy $\mathrm{UP_s}$-ideal of A.

Theorem 3.16. *A nonempty subset S of A is a UP_{s}-ideal of A if and only if the t-characteristic fuzzy set F_S^t is a fuzzy UP_{s}-ideal of A.*

Proof. It is straightforward by Theorem 1.11 (1) and Theorem 1.15 (3). □

Theorem 3.17. *The intersection of any nonempty family of fuzzy UP_{i}-ideal of A is also a fuzzy UP_{i}-ideal of A.*

Proof. Let F_i be a fuzzy $\mathrm{UP_i}$-ideal of A for all $i \in I$. Then

$$\begin{aligned}
\mathrm{f}_{\bigwedge_{i\in I}\mathrm{F}_i}(0) &= \inf\{\mathrm{f}_{\mathrm{F}i}(0)\}_{i\in I} \\
&\geq \inf\{\mathrm{f}_{\mathrm{F}i}(x)\}_{i\in I} \\
&= \mathrm{f}_{\bigwedge_{i\in I}\mathrm{F}_i}(x), \\
\mathrm{f}_{\bigwedge_{i\in I}\mathrm{F}_i}(x\cdot z) &= \inf\{\mathrm{f}_{\mathrm{F}i}(x\cdot z)\}_{i\in I}
\end{aligned}$$

$$
\begin{aligned}
&\geq \inf\{\min\{f_{F_i}(x \cdot (y \cdot z)), f_{F_i}(y)\}\}_{i \in I} \\
&= \min\{\inf\{f_{F_i}(x \cdot (y \cdot z))\}_{i \in I}, \inf\{f_{F_i}(y)\}_{i \in I}\} \\
&= \min\{f_{\bigwedge_{i \in I} F_i}(x \cdot (y \cdot z)), f_{\bigwedge_{i \in I} F_i}(y)\}, \text{ and} \\
f_{\bigwedge_{i \in I} F_i}(x * y) &= \inf\{f_{F_i}(x * y)\}_{i \in I} \\
&\geq \inf\{\max\{f_{F_i}(x), f_{F_i}(y)\}\}_{i \in I} \\
&\geq \max\{\inf\{f_{F_i}(x)\}_{i \in I}, \inf\{f_{F_i}(y)\}_{i \in I}\} \\
&= \max\{f_{\bigwedge_{i \in I} F_i}(x), f_{\bigwedge_{i \in I} F_i}(y)\}.
\end{aligned}
$$

Hence, $\bigwedge_{i \in I} F_i$ is a fuzzy UP_i-ideal of A. □

In Example 3.10, we define two membership functions f_{F_1} and f_{F_2} as follow:

A	0	1	2	3
f_{F_1}	0.7	0.3	0.4	0.3
f_{F_2}	0.8	0.5	0.2	0.2

Then F_1 and F_2 are fuzzy UP_i-ideals of A. Since $f_{F_1 \vee F_2}(0 \cdot 3) = f_{F_1 \vee F_2}(3) = 0.3 \not\geq 0.4 = \min\{0.4, 0.5\} = \min\{f_{F_1 \vee F_2}(2), f_{F_1 \vee F_2}(1)\} = \min\{f_{F_1 \vee F_2}(0 \cdot (1 \cdot 3)), f_{F_1 \vee F_2}(1)\}$, we have $F_1 \vee F_2$ is not a fuzzy UP_i-ideal of A.

Theorem 3.18. *A nonempty subset S of A is a UP_i-ideal of A if and only if the t-characteristic fuzzy set F_S^t is a fuzzy UP_i-ideal of A.*

Proof. It is straightforward by Theorem 1.11 (2) and Theorem 1.15 (3). □

Theorem 3.19. *Every fuzzy UP_s-ideal of A is a fuzzy UP_s-filter of A.*

In Example 2.11, we define a membership function f_F as follows:

$$f_F(0) = 0.8,\ f_F(1) = 0.6,\ f_F(2) = 0.3, \text{ and } f_F(3) = 0.3.$$

Then F is a fuzzy UP_s-filter of A. Since $f_F(2 \cdot 3) = f_F(2) = 0.3 \not\geq 0.6 = \min\{0.8, 0.6\} = \min\{f_F(0), f_F(1)\} = \min\{f_F(2 \cdot (1 \cdot 3)), f_F(1)\}$, we have F is not a fuzzy UP_s-ideal of A.

Theorem 3.20. *Every fuzzy UP_i-ideal of A is a fuzzy UP_i-filter of A.*

In Example 2.11, we define a membership function f_F as follows:

$$f_F(0) = 0.8,\ f_F(1) = 0.6,\ f_F(2) = 0.3, \text{ and } f_F(3) = 0.3.$$

Then F is a fuzzy UP_i-filter of A. Since $f_F(2 \cdot 3) = f_F(2) = 0.3 \not\geq 0.6 = \max\{0.8, 0.6\} = \max\{f_F(0), f_F(1)\} = \max\{f_F(2 \cdot (1 \cdot 3)), f_F(1)\}$, we have F is not a fuzzy UP_i-ideal of A.

Definition 3.21. A fuzzy set F in an f-UP-semigroup A is called

(1) a *fuzzy strongly* UP_{s}*-ideal* of A if F is a fuzzy strongly UP-ideal of $(A, \cdot, 0)$ and a fuzzy subsemigroup of $(A, *)$.

(2) a *fuzzy strongly* UP_{i}*-ideal* of A if F is a fuzzy strongly UP-ideal of $(A, \cdot, 0)$ and a fuzzy ideal of $(A, *)$.

Theorem 3.22. *Fuzzy strongly* UP_{s}*-ideals, fuzzy strongly* UP_{i}*-ideals, and constant fuzzy sets coincide in* A.

Proof. It is straightforward by Theorem 1.13. □

If a fuzzy set F_i is constant for all $i \in I$, then we see that the fuzzy sets $\bigwedge_{i\in I} \mathrm{F}_i$ and $\bigvee_{i\in I} \mathrm{F}_i$ are constant. From this, we have Theorem 3.23 and 3.24.

Theorem 3.23. *The intersection and union of any nonempty family of fuzzy strongly* UP_{s}*-ideal of* A *are also a fuzzy strongly* UP_{s}*-ideal of* A.

Theorem 3.24. *The intersection and union of any nonempty family of fuzzy strongly* UP_{i}*-ideal of* A *are also a fuzzy strongly* UP_{i}*-ideal of* A.

Theorem 3.25. *A nonempty subset* S *of* A *is a strongly* UP_{s}*-ideal of* A *if and only if the t-characteristic fuzzy set* F_S^t *is a fuzzy strongly* UP_{s}*-ideal of* A.

Proof. It is straightforward by Theorem 1.11 (1) and Theorem 1.15 (4). □

Theorem 3.26. *A nonempty subset* S *of* A *is a strongly* UP_{i}*-ideal of* A *if and only if the t-characteristic fuzzy set* F_S^t *is a fuzzy strongly* UP_{i}*-ideal of* A.

Proof. It is straightforward by Theorem 1.11 (2) and Theorem 1.15 (4). □

Theorem 3.27. *Every fuzzy strongly* UP_{s}*-ideal (fuzzy strongly* UP_{i}*-ideal) of* A *is a fuzzy* UP_{s}*-ideal and a fuzzy* UP_{i}*-ideal of* A.

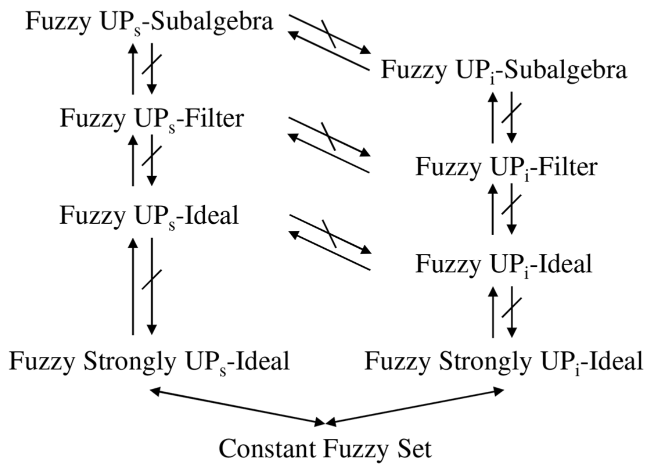
In Example 2.3, we define a membership function $\mathrm{f_F}$ as follows:

$$\mathrm{f_F}(0) = 0.7,\ \mathrm{f_F}(1) = 0.5,\ \mathrm{f_F}(2) = 0.3,\ \text{and } \mathrm{f_F}(3) = 0.6.$$

Then F is a fuzzy $\mathrm{UP_i}$-ideal of A. Since F is not constant, we have F is not a fuzzy strongly $\mathrm{UP_s}$-ideal and a fuzzy strongly $\mathrm{UP_i}$-ideal of A.

4. Conclusions and future works

In this paper, we have introduced the notions of fuzzy $\mathrm{UP_s}$-subalgebras, fuzzy $\mathrm{UP_i}$-subalgebras, fuzzy $\mathrm{UP_s}$-filters, fuzzy $\mathrm{UP_i}$-filters, fuzzy $\mathrm{UP_s}$-ideals, fuzzy $\mathrm{UP_i}$-ideals, fuzzy strongly $\mathrm{UP_s}$-ideals, and fuzzy strongly $\mathrm{UP_i}$-ideals of fully UP-semigroups, proved its generalizations and investigated some of its important properties. Then we have the generalization diagram of fuzzy sets in fully UP-semigroups below.



In our future study of fully UP-semigroups, may be the following topics should be considered:

- To get more results in fuzzy sets in fully UP-semigroups.
- To define fuzzy translations in fully UP-semigroups.
- To define fuzzy soft sets over fully UP-semigroups.

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