Certain notions of single-valued neutrosophic $K$-algebras

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Abstract. We apply the notion of single-valued neutrosophic sets to $K$-algebras. We develop the concept of single-valued neutrosophic $K$-subalgebras, and present some of their properties. Moreover, we study the behavior of single-valued neutrosophic $K$-subalgebras under homomorphism. Finally, we discuss $(\varepsilon, \varepsilon \lor \varphi)$-single-valued neutrosophic $K$-algebras.

Keywords: Single-valued neutrosophic sets, $K$-algebras, homomorphism, $(\varepsilon, \varepsilon \lor \varphi)$-single-valued neutrosophic $K$-algebras.

1. Introduction

A new kind of logical algebra, known as $K$-algebra, was introduced by Dar and Akram [9]. A $K$-algebra was built on a group $G$ by adjoining the induced binary operation on $G$. The group $G$ is particularly of the type in which each non-identity element is not of order 2. This algebraic structure is, in general, non-commutative and non-associative with right identity element [5, 10, 11]. Akram et.al [2, 3, 4] introduced fuzzy $K$-algebras. They then developed fuzzy $K$-algebras with other researchers worldwide. The concepts and results of $K$-algebras have been broadened to the fuzzy setting frames by applying Zadeh’s fuzzy set theory and its generalizations, namely, interval- valued fuzzy sets, intuitionistic fuzzy sets, interval-valued intuitionistic fuzzy sets, bipolar fuzzy sets and vague sets.

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In handling information regarding various aspects of uncertainty, non-classical logic (a great extension and development of classical logic) is considered to be a more powerful technique than the classical logic. The non-classical logic has nowadays become a useful tool in computer science. Moreover, non-classical logic deals with fuzzy information and uncertainty. In 1998, Smarandache [15] introduced neutrosophic sets as a generalization of fuzzy sets [19] and intuitionistic fuzzy sets [6]. A neutrosophic set is identified by three functions called truth-membership \((T)\), indeterminacy-membership \((I)\) and falsity-membership \((F)\) whose values are real standard or non-standard subset of unit interval \([-0, 1]^+\), where \(-0 = 0 - \epsilon, 1^+ = 1 + \epsilon, \epsilon\) is an infinitesimal number. To apply neutrosophic set in real-life problems more conveniently, Smarandache [15] and Wang et al. [16] defined single-valued neutrosophic sets which takes the value from the subset of \([0, 1]\). Thus, a single-valued neutrosophic set is an instance of neutrosophic set, and can be used expediently to deal with real-world problems, especially in decision support. Algebraic structures have a vital place with vast applications in various disciplines. Neutrosophic set theory has been applied to algebraic structures [1, 8, 13]. In this research article, we introduce the notion of single-valued neutrosophic \(K\)-subalgebra and investigate some of their properties. We discuss \(K\)-subalgebra in terms of level sets using neutrosophic environment. We study the homomorphisms between the single-valued neutrosophic \(K\)-subalgebras. We discuss characteristic \(K\)-subalgebras and fully invariant \(K\)-subalgebras. Finally, we discuss \((\in, \in \vee q)\)-single-valued neutrosophic \(K\)-algebras.

2. Single-valued neutrosophic \(K\)-algebras

The concept of \(K\)-algebra was first developed by Dar and Akram in [14].

Definition 2.1. Let \((G, \cdot, e)\) be a group in which each non-identity element is not of order 2. Then a \(K\)-algebra is a structure \(K = (G, \cdot, \circ, e)\) on a group \(G\) in which induced binary operation \(\circ : G \times G \rightarrow G\) is defined by \(\circ(x, y) = x \circ y = x.y^{-1}\) and satisfies the following axioms:

(i) \((x \circ y) \circ (x \circ z) = (x \circ ((e \circ z) \circ (e \circ y))) \circ x,\)

(ii) \(x \circ (x \circ y) = (x \circ (e \circ y)) \circ x,\)

(iii) \(x \circ x = e,\)

(iv) \(x \circ e = x,\)

(v) \(e \circ x = x^{-1},\)

for all \(x, y, z \in G.\)

Definition 2.2. [16] Let \(Z\) be a space of objects with a general element \(z \in Z.\) A single-valued neutrosophic set \(\mathcal{A}\) in \(Z\) is characterized by three membership
functions, \(T_A\)-truth membership function, \(I_A\)-indeterminacy membership function and \(F_A\)-falsity membership function, where \(T_A(z), I_A(z), F_A(z) \in [0, 1]\), for all \(z \in Z\).

\(A\) can also be written as \(A = \{< z, T_A(z), I_A(z), F_A(z) > \mid z \in Z\}\).

**Definition 2.3.** A single-valued neutrosophic set \(A = (T_A, I_A, F_A)\) in a \(K\)-algebra \(K\) is called a single-valued neutrosophic \(K\)-subalgebra of \(K\) if it satisfies the following conditions:

\[
\begin{align*}
(a) \quad T_A(s \odot t) & \geq \min\{T_A(s), T_A(t)\}, \\
(b) \quad I_A(s \odot t) & \geq \min\{I_A(s), I_A(t)\}, \\
(c) \quad F_A(s \odot t) & \leq \max\{F_A(s), F_A(t)\},
\end{align*}
\]

for all \(s, t \in G\).

Note that \(T_A(e) \geq T_A(s), I_A(e) \geq I_A(s), F_A(e) \leq F_A(s)\), for all \(s \in G\).

**Example 2.1.** Let \(K = (G, \cdot, \odot, e)\) be a \(K\)-algebra, where \(G = \{e, x, x^2, x^3, x^4, x^5, x^6, x^7, x^8\}\) is the cyclic group of order 9 and Cayley’s table for \(\odot\) is given as:

\[
\begin{array}{c|cccccccc}
\odot & e & x & x^2 & x^3 & x^4 & x^5 & x^6 & x^7 & x^8 \\
\hline
e & e & x^8 & x^4 & x^3 & x^6 & x^7 & x^2 & x & x \\
x & x & e & x^7 & x^6 & x^5 & x^4 & x^3 & x^2 & x \\
x^2 & x^2 & e & x & x^8 & x^7 & x^6 & x^5 & x^4 & x^3 \\
x^3 & x^3 & x^2 & x & e & x^8 & x^7 & x^6 & x^5 & x^4 \\
x^4 & x^4 & x^3 & x^2 & x & e & x^8 & x^7 & x^6 & x^5 \\
x^5 & x^5 & x^4 & x^3 & x^2 & x & e & x^8 & x^7 & x^6 \\
x^6 & x^6 & x^5 & x^4 & x^3 & x^2 & x & e & x^8 & x^7 \\
x^7 & x^7 & x^6 & x^5 & x^4 & x^3 & x^2 & x & e & x^8 \\
x^8 & x^8 & x^7 & x^6 & x^5 & x^4 & x^3 & x^2 & x & e \\
\end{array}
\]

We define a single-valued neutrosophic set \(A = (T_A, I_A, F_A)\) in \(K\)-algebra as follows:

\[
\begin{align*}
T_A(e) &= 0.8, I_A(e) = 0.7, F_A(e) = 0.4, \\
T_A(s) &= 0.2, I_A(s) = 0.3, F_A(s) = 0.6, \text{ for all } s \neq e \in G.
\end{align*}
\]

Clearly, \(A = (T_A, I_A, F_A)\) is a single-valued neutrosophic \(K\)-subalgebra of \(K\).

**Example 2.2.** Let \(K = (G, \cdot, \odot, e)\) be a \(K\)-algebra on dihedral group \(D_4\) given as \(G = \{e, a, b, c, x, y, u, v\}\), where \(c = ab, x = a^2, y = a^3, u = a^2b, v = a^3b\) and
Cayley's table for $\circ$ is given as:

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We define a single-valued neutrosophic set $A = (T_A, I_A, F_A)$ in $K$-algebra as follows: $T_A(e) = 0.9, I_A(e) = 0.3, F_A(e) = 0.3, T_A(s) = 0.6, I_A(s) = 0.2, F_A(s) = 0.4$, for all $s \neq e \in G$. By routine calculations, it can be verified that $A$ is a single-valued neutrosophic $K$-subalgebra of $K$.

**Proposition 2.1.** If $A = (T_A, I_A, F_A)$ is a single-valued neutrosophic $K$-subalgebra of $K$, then

1. $(\forall s, t \in G), (T_A(s \circ t) = T_A(t) \Rightarrow T_A(s) = T_A(e)).$
   $(\forall s, t \in G)(I_A(s \circ t) = I_A(t) \Rightarrow I_A(s) = I_A(e)).$
   $(\forall s, t \in G)(F_A(s \circ t) = F_A(t) \Rightarrow F_A(s) = F_A(e)).$

2. Assume that $T_A(s \circ t) = T_A(t)$, for all $s, t \in G$. Taking $t = e$ and using (iii) of Definition 2.1, we have $T_A(s) = T_A(s \circ e) = T_A(e)$. Let for $s, t \in G$ be such that $T_A(s) = T_A(e)$. Then $T_A(s \circ t) \geq \min\{T_A(s), T_A(t)\} = \min\{T_A(e), T_A(t)\} = T_A(t)$.

3. Assume that $I_A(s \circ t) = I_A(t)$, for all $s, t \in G$. Taking $t = e$ and by (iii) of Definition 2.1, we have $I_A(s) = I_A(s \circ e) = I_A(e)$. Also let $s, t \in G$ be such that $I_A(s) = I_A(e)$. Then $I_A(s \circ t) \geq \min\{I_A(s), I_A(t)\} = \min\{I_A(e), I_A(t)\} = I_A(t)$.

3. Consider that $F_A(s \circ t) = F_A(t)$, for all $s, t \in G$. Taking $t = e$ and again by (iii) of Definition 2.1, we have $F_A(s) = F_A(s \circ e) = F_A(e)$. Let $s, t \in G$ be such that $F_A(s) = F_A(t)$. Then $F_A(s \circ t) \leq \max\{F_A(s), F_A(t)\} = \max\{F_A(e), F_A(t)\} = F_A(t)$.

This completes the proof.
Definition 2.4. Let $A = (\mathcal{T}_A, \mathcal{I}_A, \mathcal{F}_A)$ be a single-valued neutrosophic set in a $K$-algebra $\mathcal{K}$ and let $(\alpha, \beta, \gamma) \in [0, 1] \times [0, 1] \times [0, 1]$ with $\alpha + \beta + \gamma \leq 3$. Then level subsets of $A$ are defined as:

\[ A_{(\alpha, \beta, \gamma)} = \{ s \in G \mid T_A(s) \geq \alpha, I_A(s) \geq \beta, F_A(s) \leq \gamma \}, \]
\[ A_{(\alpha, \beta, \gamma)} = \{ s \in G \mid T_A(s) \geq \alpha \} \cap \{ s \in G \mid I_A(s) \geq \beta \} \cap \{ s \in G \mid F_A(s) \leq \gamma \}, \]
\[ A_{(\alpha, \beta, \gamma)} = \cup(\mathcal{T}_A, \alpha) \cap \cup'(I_A, \beta) \cap L(F_A, \gamma) \]

are called $(\alpha, \beta, \gamma)$-level subsets of single-valued neutrosophic set $A$.

The set of all $(\alpha, \beta, \gamma) \in \text{Im}(\mathcal{T}_A) \times \text{Im}(\mathcal{I}_A) \times \text{Im}(\mathcal{F}_A)$ is known as image of $A = (\mathcal{T}_A, \mathcal{I}_A, \mathcal{F}_A)$. The set $A_{(\alpha, \beta, \gamma)} = \{ s \in G \mid T_A(s) > \alpha, I_A(s) > \beta, F_A(s) < \gamma \}$ is known as strong $(\alpha, \beta, \gamma)$-level subset of $A$.

**Proposition 2.2.** If $A = (\mathcal{T}_A, \mathcal{I}_A, \mathcal{F}_A)$ is a single-valued neutrosophic $K$-subalgebra of $\mathcal{K}$, then the level subsets $\cup(\mathcal{T}_A, \alpha) = \{ s \in G \mid T_A(s) \geq \alpha \}$, $\cup'(I_A, \beta) = \{ s \in G \mid I_A(s) \geq \beta \}$ and $L(F_A, \gamma) = \{ s \in G \mid F_A(s) \leq \gamma \}$ are level subsets of $\mathcal{K}$, for every $(\alpha, \beta, \gamma) \in \text{Im}(\mathcal{T}_A) \times \text{Im}(\mathcal{I}_A) \times \text{Im}(\mathcal{F}_A) \subseteq [0, 1]$, where $\text{Im}(\mathcal{T}_A)$, $\text{Im}(\mathcal{I}_A)$ and $\text{Im}(\mathcal{F}_A)$ are sets of values of $T_A$, $I_A$ and $F_A$, respectively.

**Proof.** Assume that $A = (\mathcal{T}_A, \mathcal{I}_A, \mathcal{F}_A)$ is a single-valued neutrosophic $K$-subalgebra of $\mathcal{K}$ and let $(\alpha, \beta, \gamma) \in \text{Im}(\mathcal{T}_A) \times \text{Im}(\mathcal{I}_A) \times \text{Im}(\mathcal{F}_A)$ be such that $\cup(\mathcal{T}_A, \alpha) \neq \emptyset, \cup'(I_A, \beta) \neq \emptyset$ and $L(F_A, \gamma) \neq \emptyset$. Now to prove that $\cup, \cup'$ and $L$ are level $K$-subalgebras. Let for $s, t \in \cup(\mathcal{T}_A, \alpha)$, $T_A(s) \geq \alpha$ and $T_A(t) \geq \alpha$. It follows from Definition 2.3 that $\mathcal{T}_A(s \circ t) \geq \min\{\mathcal{T}_A(s), \mathcal{T}_A(t)\} \geq \alpha$. It implies that $s \circ t \in \cup(\mathcal{T}_A, \alpha)$. Hence $\cup(\mathcal{T}_A, \alpha)$ is a level $K$-subalgebra of $\mathcal{K}$. Similar result can be proved for $\cup'(I_A, \beta)$ and $L(F_A, \gamma)$. \(\square\)

**Theorem 2.1.** Let $A = (\mathcal{T}_A, \mathcal{I}_A, \mathcal{F}_A)$ be a single-valued neutrosophic set in $K$-algebra $\mathcal{K}$. Then $A = (\mathcal{T}_A, \mathcal{I}_A, \mathcal{F}_A)$ is a single-valued neutrosophic $K$-subalgebra of $\mathcal{K}$ if and only if $A_{(\alpha, \beta, \gamma)}$ is a $K$-subalgebra of $\mathcal{K}$, for every $(\alpha, \beta, \gamma) \in \text{Im}(\mathcal{T}_A) \times \text{Im}(\mathcal{I}_A) \times \text{Im}(\mathcal{F}_A)$ with $\alpha + \beta + \gamma \leq 3$.

**Proof.** Let $A = (\mathcal{T}_A, \mathcal{I}_A, \mathcal{F}_A)$ be a single-valued neutrosophic set in a $K$-algebra $\mathcal{K}$. Assume that $A = (\mathcal{T}_A, \mathcal{I}_A, \mathcal{F}_A)$ be a single-valued neutrosophic $K$-subalgebra of $\mathcal{K}$.

Let for $(\alpha, \beta, \gamma) \in \text{Im}(\mathcal{T}_A) \times \text{Im}(\mathcal{I}_A) \times \text{Im}(\mathcal{F}_A)$ with $\alpha + \beta + \gamma \leq 3$ be such that $A_{(\alpha, \beta, \gamma)} \neq \emptyset$. Let $s, t \in A_{(\alpha, \beta, \gamma)}$ be such that

\[ T_A(s) \geq \alpha, T_A(t) \geq \alpha', \]
\[ I_A(s) \geq \beta, I_A(t) \geq \beta', \]
\[ F_A(s) \leq \gamma, F_A(t) \leq \gamma'. \]
Without loss of generality we can assume that $\alpha \leq \alpha'$, $\beta \leq \beta'$ and $\gamma \geq \gamma'$. It follows from Definition 2.3 that
\[ T_A(s \circ t) \geq \alpha = \min \{ T_A(s), T_A(t) \}, \]
\[ \mathcal{I}_A(s \circ t) \geq \beta = \min \{ \mathcal{I}_A(s), \mathcal{I}_A(t) \}, \]
\[ \mathcal{F}_A(s \circ t) \leq \gamma = \max \{ \mathcal{F}_A(s), \mathcal{F}_A(t) \}. \]

It implies that $s \circ t \in A_{(\alpha, \beta, \gamma)}$. So, $A_{(\alpha, \beta, \gamma)}$ is a $K$-subalgebra of $K$.

Conversely, we suppose that $A_{(\alpha, \beta, \gamma)}$ is a $K$-subalgebra of $K$. If the condition of the Definition 2.3 is not true, then there exist $u, v \in G$ such that
\[ T_A(u \circ v) < \min \{ T_A(u), T_A(v) \}, \]
\[ \mathcal{I}_A(u \circ v) < \min \{ \mathcal{I}_A(u), \mathcal{I}_A(v) \}, \]
\[ \mathcal{F}_A(u \circ v) > \max \{ \mathcal{F}_A(u), \mathcal{F}_A(v) \}. \]

Taking $\alpha_1 = \frac{1}{2}(T_A(u \circ v) + \min \{ T_A(u), T_A(v) \}), \beta_1 = \frac{1}{2}(\mathcal{I}_A(u \circ v) + \min \{ \mathcal{I}_A(u), \mathcal{I}_A(v) \}), \gamma_1 = \frac{1}{2}(\mathcal{F}_A(u \circ v) + \min \{ \mathcal{F}_A(u), \mathcal{F}_A(v) \})$, we have $T_A(u \circ v) < \alpha_1 < \min \{ T_A(u), T_A(v) \}, \mathcal{I}_A(u \circ v) < \beta_1 < \min \{ \mathcal{I}_A(u), \mathcal{I}_A(v) \}$ and $\mathcal{F}_A(u \circ v) > \gamma_1 > \max \{ \mathcal{F}_A(u), \mathcal{F}_A(v) \}$. It implies that $u, v \in A_{(\alpha, \beta, \gamma)}$ and $u \circ v \notin A_{(\alpha, \beta, \gamma)}$, a contradiction. Therefore, the condition of Definition 2.3 is true. Hence $A = (T_A, \mathcal{I}_A, \mathcal{F}_A)$ is a single-valued neutrosophic $k$-subalgebra of $K$. \hfill \Box

**Theorem 2.2.** Let $A = (T_A, \mathcal{I}_A, \mathcal{F}_A)$ be a single-valued neutrosophic $k$-subalgebra and $(\alpha_1, \beta_1, \gamma_1), (\alpha_2, \beta_2, \gamma_2) \in \text{Im}(T_A) \times \text{Im}(\mathcal{I}_A) \times \text{Im}(\mathcal{F}_A)$ with $\alpha_j + \beta_j + \gamma_j \leq 3$ for $j = 1, 2$. Then $A_{(\alpha_1, \beta_1, \gamma_1)} = A_{(\alpha_2, \beta_2, \gamma_2)}$ if $(\alpha_1, \beta_1, \gamma_1) = (\alpha_2, \beta_2, \gamma_2)$.

**Proof.** If $(\alpha_1, \beta_1, \gamma_1) = (\alpha_2, \beta_2, \gamma_2)$, then clearly $A_{(\alpha_1, \beta_1, \gamma_1)} = A_{(\alpha_2, \beta_2, \gamma_2)}$.

Assume that $A_{(\alpha_1, \beta_1, \gamma_1)} = A_{(\alpha_2, \beta_2, \gamma_2)}$. Since $(\alpha_1, \beta_1, \gamma_1) \in \text{Im}(T_A) \times \text{Im}(\mathcal{I}_A) \times \text{Im}(\mathcal{F}_A)$, there exists $s \in G$ such that $T_A(s) = \alpha_1, \mathcal{I}_A(s) = \beta_1$ and $\mathcal{F}_A(s) = \gamma_1$. It follows that $s \in A_{(\alpha_1, \beta_1, \gamma_1)} = A_{(\alpha_2, \beta_2, \gamma_2)}$. So that $\alpha_1 = T_A(s) \geq \alpha_2, \beta_1 = \mathcal{I}_A(s) \geq \beta_2$ and $\gamma_1 = \mathcal{F}_A(s) \leq \gamma_2$. Also $(\alpha_2, \beta_2, \gamma_2) \in \text{Im}(T_A) \times \text{Im}(\mathcal{I}_A) \times \text{Im}(\mathcal{F}_A)$, there exists $t \in G$ such that $T_A(t) = \alpha_2, \mathcal{I}_A(t) = \beta_2$ and $\mathcal{F}_A(t) = \gamma_2$. It follows that $t \in A_{(\alpha_2, \beta_2, \gamma_2)} = A_{(\alpha_1, \beta_1, \gamma_1)}$. So that $\alpha_2 = T_A(t) \geq \alpha_1, \beta_2 = \mathcal{I}_A(t) \geq \beta_1$ and $\gamma_2 = \mathcal{F}_A(t) \leq \gamma_1$. Hence $(\alpha_1, \beta_1, \gamma_1) = (\alpha_2, \beta_2, \gamma_2)$. \hfill \Box

**Theorem 2.3.** Let $H$ be a $K$-subalgebra of $K$-algebra $K$. Then there exists a single-valued neutrosophic $K$-subalgebra $A = (T_A, \mathcal{I}_A, \mathcal{F}_A)$ of $K$-algebra $K$ such that $A = (T_A, \mathcal{I}_A, \mathcal{F}_A) = H$, for some $\alpha, \beta \in (0, 1], \gamma \in [0, 1]$.

**Proof.** Let $A = (T_A, \mathcal{I}_A, \mathcal{F}_A)$ be a single-valued neutrosophic set in $K$-algebra $K$ given by
\[ T_A(s) = \begin{cases} \alpha \in (0, 1], & \text{if } s \in H, \\ 0, & \text{otherwise.} \end{cases} \]
Let \( A \subset K \) be a level \( K \)-subalgebra of \( K \). Then there exists a single-valued neutrosophic \( K \)-subalgebra whose level \( K \)-subalgebras are exactly the \( K \)-subalgebras in this chain.

**Proof.** Let \( \{\alpha_k \mid k = 0, 1, ..., n\}, \{\beta_k \mid k = 0, 1, ..., n\} \) be finite decreasing sequences and \( \{\gamma_k \mid k = 0, 1, ..., n\} \) be finite increasing sequence in \([0, 1]\) such that \( \alpha_i + \beta_i + \gamma_i \leq 3 \), for \( i = 0, 1, 2, ..., n \). Let \( A = (\mathcal{T}_A, \mathcal{I}_A, \mathcal{F}_A) \) be a single-valued neutrosophic set in \( K \) defined by \( \mathcal{T}_A(A_0) = \alpha_0, \mathcal{I}_A(A_0) = \beta_0, \mathcal{F}_A(A_0) = \gamma_0 \), \( \mathcal{T}_A(A_k \setminus A_{k-1}) = \alpha_k, \mathcal{I}_A(A_k \setminus A_{k-1}) = \beta_k \) and \( \mathcal{F}_A(A_k \setminus A_{k-1}) = \gamma_k \), for \( 0 < k \leq n \). We claim that \( A = (\mathcal{T}_A, \mathcal{I}_A, \mathcal{F}_A) \) is a single-valued neutrosophic \( K \)-subalgebra of \( K \). Let \( s, t \in G \). If \( s, t \in A_k \setminus A_{k-1} \), then it implies that \( \mathcal{T}_A(s) = \alpha_k = \mathcal{T}_A(t), \mathcal{I}_A(s) = \beta_k = \mathcal{I}_A(t) \) and \( \mathcal{F}_A(s) = \gamma_k = \mathcal{F}_A(t) \). Since each \( A_k \) is a \( K \)-subalgebra, it follows that \( s \circ t \in A_k \). So that either \( s \circ t \in A_k \setminus A_{k-1} \) or \( s \circ t \in A_{k-1} \). In any case, we conclude that

\[
\mathcal{T}_A(s \circ t) = \alpha_k = \min\{\mathcal{T}_A(s), \mathcal{T}_A(t)\},
\mathcal{I}_A(s \circ t) = \beta_k = \min\{\mathcal{I}_A(s), \mathcal{I}_A(t)\},
\mathcal{F}_A(s \circ t) = \gamma_k = \max\{\mathcal{F}_A(s), \mathcal{F}_A(t)\}.
\]

For \( i > j \), if \( s \in A_i \setminus A_{i-1} \) and \( t \in A_j \setminus A_{j-1} \), then \( \mathcal{T}_A(s) = \alpha_i, \mathcal{T}_A(t) = \alpha_j, \mathcal{I}_A(s) = \beta_i, \mathcal{I}_A(t) = \beta_j \), and \( \mathcal{F}_A(s) = \gamma_i, \mathcal{F}_A(t) = \gamma_j \) and \( s \circ t \in A_i \). Because \( A_i \) is a \( K \)-subalgebra and \( A_j \subset A_i \). It follows that

\[
\mathcal{T}_A(s \circ t) = \alpha_i = \min\{\mathcal{T}_A(s), \mathcal{T}_A(t)\},
\mathcal{I}_A(s \circ t) = \beta_i = \min\{\mathcal{I}_A(s), \mathcal{I}_A(t)\},
\mathcal{F}_A(s \circ t) = \gamma_i = \max\{\mathcal{F}_A(s), \mathcal{F}_A(t)\}.
\]

Thus, \( A = (\mathcal{T}_A, \mathcal{I}_A, \mathcal{F}_A) \) is a single-valued neutrosophic \( K \)-subalgebra of \( K \) and all its non empty level subsets are level \( K \)-subalgebras of \( K \). Since \( \text{Im}(\mathcal{T}_A) = \{\alpha_0, \alpha_1, ..., \alpha_n\}, \text{Im}(\mathcal{I}_A) = \{\beta_0, \beta_1, ..., \beta_n\}, \text{Im}(\mathcal{F}_A) = \{\gamma_0, \gamma_1, ..., \gamma_n\} \). Therefore,
the level $K$-subalgebras of $A = (\mathcal{T}_A, \mathcal{I}_A, \mathcal{F}_A)$ are given by the chain of $K$-subalgebras:

\[
\bigcup(\mathcal{T}_A, \alpha_0) \subset \bigcup(\mathcal{T}_A, \alpha_1) \subset \ldots \subset \bigcup(\mathcal{T}_A, \alpha_n) = G,
\]
\[
\bigcup'(\mathcal{I}_A, \beta_0) \subset \bigcup'(\mathcal{I}_A, \beta_1) \subset \ldots \subset \bigcup'(\mathcal{I}_A, \beta_n) = G,
\]
\[
L(\mathcal{F}_A, \gamma_0) \subset L(\mathcal{F}_A, \gamma_1) \subset \ldots \subset L(\mathcal{F}_A, \gamma_n) = G,
\]

respectively. Indeed,

\[
\bigcup(\mathcal{T}_A, \alpha_0) = \{s \in G \mid \mathcal{T}_A(s) \geq \alpha_0\} = A_0,
\]
\[
\bigcup'(\mathcal{I}_A, \beta_0) = \{s \in G \mid \mathcal{I}_A(s) \geq \beta_0\} = A_0,
\]
\[
L(\mathcal{F}_A, \gamma_0) = \{s \in G \mid \mathcal{F}_A(s) \leq \gamma_0\} = A_0.
\]

Now we prove that $\bigcup(\mathcal{T}_A, \alpha_k) = A_k$, $\bigcup'(\mathcal{I}_A, \beta_k) = A_k$ and $L(\mathcal{F}_A, \gamma_k) = A_k$, for $0 < k \leq n$. Clearly, $A_k \subseteq \bigcup(\mathcal{T}_A, \alpha_k)$, $A_k \subseteq \bigcup'(\mathcal{I}_A, \beta_k)$ and $A_k \subseteq L(\mathcal{F}_A, \gamma_k)$. If $s \in \bigcup(\mathcal{T}_A, \alpha_k)$, then $\mathcal{T}_A(s) \geq \alpha_k$ and so $s \notin A_i$, for $i > k$.

Hence $\mathcal{T}_A(s) \in \{\alpha_0, \alpha_1, \ldots, \alpha_k\}$ which implies that $s \in A_i$, for some $i \leq k$ since $A_i \subseteq A_k$. It follows that $s \in A_k$. Consequently, $\bigcup(\mathcal{T}_A, \alpha_k) = A_k$ for some $0 < k \leq n$. Similar case can be proved for $\bigcup'(\mathcal{I}_A, \beta_k) = A_k$. Now if $t \in L(\mathcal{F}_A, \gamma_k)$, then $\mathcal{F}_A(s) \leq \gamma_k$ and so $t \notin A_i$, for some $i \leq k$. Thus, $\mathcal{F}_A(s) \in \{\gamma_0, \gamma_1, \ldots, \gamma_k\}$ which implies that $s \in A_j$, for some $j \leq k$. Since $A_j \subseteq A_k$. It follows that $t \in A_k$. Consequently, $L(\mathcal{F}_A, \gamma_k) = A_k$, for some $0 < k \leq n$. Hence the proof.

\[\square\]

### 2.1 Homomorphism of single-valued neutrosophic $K$-algebras

**Definition 2.5.** Let $\mathcal{K}_1 = (G_1, \cdot, \circ, e_1)$ and $\mathcal{K}_2 = (G_2, \cdot, \circ, e_2)$ be two $K$-algebras and let $\phi$ be a function from $\mathcal{K}_1$ into $\mathcal{K}_2$. If $B = (\mathcal{T}_B, \mathcal{I}_B, \mathcal{F}_B)$ is a single-valued neutrosophic $K$-subalgebra of $\mathcal{K}_2$, then the preimage of $B = (\mathcal{T}_B, \mathcal{I}_B, \mathcal{F}_B)$ under $\phi$ is a single-valued neutrosophic $K$-subalgebra of $\mathcal{K}_1$ defined by $\phi^{-1}(\mathcal{T}_B)(s) = \mathcal{T}_B(\phi(s))$, $\phi^{-1}(\mathcal{I}_B)(s) = \mathcal{I}_B(\phi(s))$ and $\phi^{-1}(\mathcal{F}_B)(s) = \mathcal{F}_B(\phi(s))$, for all $s \in G_1$.

**Theorem 2.5.** Let $\phi : \mathcal{K}_1 \rightarrow \mathcal{K}_2$ be an epimorphism of $K$-algebras. If $B = (\mathcal{T}_B, \mathcal{I}_B, \mathcal{F}_B)$ is a single-valued neutrosophic $K$-subalgebra of $\mathcal{K}_2$, then $\phi^{-1}(B)$ is a single-valued neutrosophic $K$-subalgebra of $\mathcal{K}_1$.

**Proof.** It is easy to see that $\phi^{-1}(\mathcal{T}_B)(e) \geq \phi^{-1}(\mathcal{T}_B)(s)$, $\phi^{-1}(\mathcal{I}_B)(e) \geq \phi^{-1}(\mathcal{I}_B)(s)$ and $\phi^{-1}(\mathcal{F}_B)(e) \leq \phi^{-1}(\mathcal{F}_B)(s)$ for all $s \in G_1$. Let $s, t \in G_1$, then

\[
\phi^{-1}(\mathcal{T}_B)(s \circ t) = \mathcal{T}_B(\phi(s \circ t))
\]
\[
\phi^{-1}(\mathcal{T}_B)(s \circ t) = \mathcal{T}_B(\phi(s) \circ \phi(t))
\]
\[
\phi^{-1}(\mathcal{T}_B)(s \circ t) \geq \min\{\mathcal{T}_B(\phi(s)), \mathcal{T}_B(\phi(t))\}
\]
\[
\phi^{-1}(\mathcal{T}_B)(s \circ t) \geq \min\{\phi^{-1}(\mathcal{T}_B)(s), \phi^{-1}(\mathcal{T}_B)(t)\},
\]
Let a mapping 

\[ \phi^{-1}(I_B)(s \circ t) = I_B(\phi(s \circ t)) \]

\[ \phi^{-1}(I_B)(s \circ t) = I_B(\phi(s) \circ \phi(t)) \]

\[ \phi^{-1}(I_B)(s \circ t) \geq \min\{I_B(\phi(s)), I_B(\phi(t))\} \]

\[ \phi^{-1}(I_B)(s \circ t) \geq \min\{\phi^{-1}(I_B)(s), \phi^{-1}(I_B)(t)\}; \]

\[ \phi^{-1}(F_B)(s \circ t) = F_B(\phi(s) \circ \phi(t)) \]

\[ \phi^{-1}(F_B)(s \circ t) = F_B(\phi(s) \circ \phi(t)) \]

\[ \phi^{-1}(F_B)(s \circ t) \leq \max\{F_B(\phi(s)), F_B(\phi(t))\} \]

\[ \phi^{-1}(F_B)(s \circ t) \leq \max\{\phi^{-1}(F_B)(s), \phi^{-1}(F_B)(t)\}. \]

Hence \( \phi^{-1}(B) \) is a single-valued neutrosophic \( K \)-subalgebra of \( K_1 \). \( \square \)

**Theorem 2.6.** Let \( \phi : K_1 \rightarrow K_2 \) be an epimorphism of \( K \)-algebras. If \( B = (T_B, I_B, F_B) \) is a single-valued neutrosophic \( K \)-subalgebra of \( K_2 \) and \( A = (T_A, I_A, F_A) \) is the preimage of \( B \) under \( \phi \). Then \( A \) is a single-valued neutrosophic \( K \)-subalgebra of \( K_1 \).

**Proof.** It is easy to see that \( T_A(e) \geq T_A(s), I_A(e) \geq I_A(s) \) and \( F_A(e) \leq F_A(s) \), for all \( s \in G_1 \). Now for any \( s, t \in G_1 \),

\[ T_A(s \circ t) = T_B(\phi(s \circ t)) \]

\[ T_A(s \circ t) = T_B(\phi(s) \circ \phi(t)) \]

\[ T_A(s \circ t) \geq \min\{T_B(\phi(s)), T_B(\phi(t))\} \]

\[ T_A(s \circ t) \geq \min\{T_A(s), T_A(t)\}; \]

\[ I_A(s \circ t) = I_B(\phi(s \circ t)) \]

\[ I_A(s \circ t) = I_B(\phi(s) \circ \phi(t)) \]

\[ I_A(s \circ t) \geq \min\{I_B(\phi(s)), I_B(\phi(t))\} \]

\[ I_A(s \circ t) \geq \min\{I_A(s), I_A(t)\}; \]

\[ F_A(s \circ t) = F_B(\phi(s \circ t)) \]

\[ F_A(s \circ t) = F_B(\phi(s) \circ \phi(t)) \]

\[ F_A(s \circ t) \leq \max\{F_B(\phi(s)), F_B(\phi(t))\} \]

\[ F_A(s \circ t) \leq \max\{F_A(s), F_A(t)\}. \]

Hence \( A \) is a single-valued neutrosophic \( K \)-subalgebra of \( K_1 \). \( \square \)

**Definition 2.6.** Let a mapping \( \phi : K_1 \rightarrow K_2 \) from \( K_1 \) into \( K_2 \) of \( K \)-algebras and let \( A = (T_A, I_A, F_A) \) be a single-valued neutrosophic set of \( K_2 \). The map \( A = (T_A, I_A, F_A) \) is called the preimage of \( A \) under \( \phi \), if \( T_A^\phi(s) = T_A(\phi(s)), I_A^\phi(s) = I_A(\phi(s)) \) and \( F_A^\phi(s) = F_A(\phi(s)) \) for all \( s \in G_1 \).
Proposition 2.3. Let $\phi : K_1 \to K_2$ be an epimorphism of $K$-algebras. If $\mathcal{A} = (\mathcal{T}_A, \mathcal{I}_A, \mathcal{F}_A)$ is a single-valued neutrosophic $K$-subalgebra of $K_2$, then $\mathcal{A}^\phi = (\mathcal{T}_A^\phi, \mathcal{I}_A^\phi, \mathcal{F}_A^\phi)$ is a single-valued neutrosophic $K$-subalgebra of $K_1$.

Proof. For any $s \in G_1$, we have
\[
\mathcal{T}_A^\phi(e_1) = \mathcal{T}_A(\phi(e_1)) = \mathcal{T}_A(e_2) \geq \mathcal{T}_A(\phi(s)) = \mathcal{T}_A^\phi(s),
\]
\[
\mathcal{T}_A^\phi(e_1) = \mathcal{I}_A(\phi(e_1)) = \mathcal{I}_A(e_2) \geq \mathcal{I}_A(\phi(s)) = \mathcal{I}_A^\phi(s),
\]
\[
\mathcal{F}_A^\phi(e_1) = \mathcal{F}_A(\phi(e_1)) = \mathcal{F}_A(e_2) \leq \mathcal{F}_A(\phi(s)) = \mathcal{F}_A^\phi(s).
\]

For any $s, t \in G_1$, since $\mathcal{A}$ is a single-valued neutrosophic $K$-subalgebra of $K_2$

\[
\mathcal{T}_A^\phi(s \circ t) = \mathcal{T}_A(\phi(s \circ t))
\]
\[
\mathcal{T}_A^\phi(s \circ t) = \mathcal{T}_A(\phi(s) \circ \phi(t))
\]
\[
\mathcal{T}_A^\phi(s \circ t) \geq \min\{\mathcal{T}_A(\phi(s)), \mathcal{T}_A(\phi(t))\}
\]
\[
\mathcal{T}_A^\phi(s \circ t) \geq \min\{\mathcal{T}_A^\phi(s), \mathcal{T}_A^\phi(t)\},
\]
\[
\mathcal{I}_A^\phi(s \circ t) = \mathcal{I}_A(\phi(s \circ t))
\]
\[
\mathcal{I}_A^\phi(s \circ t) = \mathcal{I}_A(\phi(s) \circ \phi(t))
\]
\[
\mathcal{I}_A^\phi(s \circ t) \geq \min\{\mathcal{I}_A(\phi(s)), \mathcal{I}_A(\phi(t))\}
\]
\[
\mathcal{I}_A^\phi(s \circ t) \geq \min\{\mathcal{I}_A^\phi(s), \mathcal{I}_A^\phi(t)\},
\]
\[
\mathcal{F}_A^\phi(s \circ t) = \mathcal{F}_A(\phi(s \circ t))
\]
\[
\mathcal{F}_A^\phi(s \circ t) = \mathcal{F}_A(\phi(s) \circ \phi(t))
\]
\[
\mathcal{F}_A^\phi(s \circ t) \leq \max\{\mathcal{F}_A(\phi(s)), \mathcal{F}_A(\phi(t))\}
\]
\[
\mathcal{F}_A^\phi(s \circ t) \leq \max\{\mathcal{F}_A^\phi(s), \mathcal{F}_A^\phi(t)\}.
\]

Hence $\mathcal{A}^\phi = (\mathcal{T}_A, \mathcal{I}_A, \mathcal{F}_A)$ is a single-valued neutrosophic $K$-subalgebra of $K_1$.

Proposition 2.4. Let $\phi : K_1 \to K_2$ be an epimorphism of $K$-algebras. If $\mathcal{A}^\phi = (\mathcal{T}_A^\phi, \mathcal{I}_A^\phi, \mathcal{F}_A^\phi)$ is a single-valued neutrosophic $K$-subalgebra of $K_2$, then $\mathcal{A} = (\mathcal{T}_A, \mathcal{I}_A, \mathcal{F}_A)$ is a single-valued neutrosophic $K$-subalgebra of $K_1$.

Proof. Since there exists $s \in G_1$ such that $t = \phi(s)$, for any $t \in G_2$

\[
\mathcal{T}_A(t) = \mathcal{T}_A(\phi(s)) = \mathcal{T}_A^\phi(e_1) = \mathcal{T}_A(e_2),
\]
\[
\mathcal{I}_A(t) = \mathcal{I}_A(\phi(s)) = \mathcal{I}_A^\phi(e_1) = \mathcal{I}_A(e_2),
\]
\[
\mathcal{F}_A(t) = \mathcal{F}_A(\phi(s)) = \mathcal{F}_A^\phi(e_1) = \mathcal{F}_A(e_2).
\]
for any $s, t \in G_2$, $u, v \in G_1$ such that $s = \phi(u)$ and $t = \phi(v)$. It follows that

\[ T_A(s \circ t) = T_A(\phi(u \circ v)) \]
\[ T_A(s \circ t) = T_A^\phi(u \circ v) \]
\[ T_A(s \circ t) \geq \min\{T_A^\phi(u), T_A^\phi(v)\} \]
\[ T_A(s \circ t) \geq \min\{T_A(\phi(u)), T_A(\phi(v))\} \]
\[ T_A(s \circ t) \geq \min\{T_A(s), T_A(t)\} \]

\[ I_A(s \circ t) = I_A(\phi(u \circ v)) \]
\[ I_A(s \circ t) = I_A^\phi(u \circ v) \]
\[ I_A(s \circ t) \geq \min\{I_A^\phi(u), I_A^\phi(v)\} \]
\[ I_A(s \circ t) \geq \min\{I_A(\phi(u)), I_A(\phi(v))\} \]
\[ I_A(s \circ t) \geq \min\{I_A(s), I_A(t)\} \]

\[ F_A(s \circ t) = F_A(\phi(u \circ v)) \]
\[ F_A(s \circ t) = F_A^\phi(u \circ v) \]
\[ F_A(s \circ t) \leq \max\{F_A^\phi(u), F_A^\phi(v)\} \]
\[ F_A(s \circ t) \leq \max\{F_A(\phi(u)), F_A(\phi(v))\} \]
\[ F_A(s \circ t) \leq \max\{F_A(s), F_A(t)\} \]

Hence $A = (T_A, I_A, F_A)$ is a single-valued neutrosophic $K$-subalgebra of $K_2$. \qed

From above two propositions, we obtain the following theorem.

**Theorem 2.7.** Let $\phi : K_1 \to K_2$ be an epimorphism of $K$-algebras. Then $A^\phi = (T_A^\phi, I_A^\phi, F_A^\phi)$ is a single-valued neutrosophic $K$-subalgebra of $K_1$ if and only if $A = (T_A, I_A, F_A)$ is a single-valued neutrosophic $K$-subalgebra of $K_2$.

**Definition 2.7.** A single-valued neutrosophic $K$-subalgebra $A = (T_A, I_A, F_A)$ of a $K$-algebra $K$ is called characteristic if $T_A(\phi(s)) = T_A(s)$, $I_A(\phi(s)) = I_A(s)$ and $F_A(\phi(s)) = F_A(s)$, for all $s \in G$ and $\phi \in \text{Aut}(K)$.

**Definition 2.8.** A $K$-subalgebra $S$ of a $K$-algebra $K$ is said to be fully invariant if $\phi(S) \subseteq S$, for all $\phi \in \text{End}(K)$, where $\text{End}(K)$ is the set of all endomorphisms of a $K$-algebra $K$. A single-valued neutrosophic $K$-subalgebra $A = (T_A, I_A, F_A)$ of a $K$-algebra $K$ is called fully invariant if $T_A(\phi(s)) \leq T_A(s)$, $I_A(\phi(s)) \leq I_A(s)$ and $F_A(\phi(s)) \leq F_A(s)$, for all $s \in G$ and $\phi \in \text{End}(K)$.

**Definition 2.9.** Let $A_1 = (T_{A_1}, I_{A_1}, F_{A_1})$ and $A_2 = (T_{A_2}, I_{A_2}, F_{A_2})$ be single-valued neutrosophic $K$-subalgebras of $K$. Then $A_1 = (T_{A_1}, I_{A_1}, A_1)$ is said to be the same type of $A_2 = (T_{A_2}, I_{A_2}, F_{A_2})$ if there exists $\phi \in \text{Aut}(K)$ such that $A_1 = A_2 \circ \phi$, i.e., $T_{A_1}(s) = T_{A_2}(\phi(s))$, $I_{A_1}(s) = I_{A_2}(\phi(s))$ and $F_{A_1}(s) = F_{A_2}(\phi(s))$, for all $s \in G$. 
Theorem 2.8. Let \( A_1 = (T_{A_1}, I_{A_1}, F_{A_1}) \) and \( A_2 = (T_{A_2}, I_{A_2}, F_{A_2}) \) be single-valued neutrosophic \( K \)-subalgebras of \( K \). Then \( A_1 = (T_{A_1}, I_{A_1}, F_{A_1}) \) is a single-valued neutrosophic \( K \)-subalgebra if and only if \( A_1 \) is isomorphic to \( A_2 \).

Proof. Sufficient condition holds trivially so we only need to prove the necessary condition. Let \( A_1 = (T_{A_1}, I_{A_1}, F_{A_1}) \) be a single-valued neutrosophic \( K \)-subalgebra having same type of \( A_2 = (T_{A_2}, I_{A_2}, F_{A_2}) \). Then there exists \( \phi \in \text{Aut}(K) \) such that \( T_{A_1}(s) = T_{A_2}(\phi(s)) \), \( I_{A_1}(s) = I_{A_2}(\phi(s)) \) and \( F_{A_1} = F_{A_2}(\phi(s)) \), for all \( s \in G \).

Let \( f : A_1(K) \rightarrow A_2(K) \) be a mapping defined by \( f(A_1(s)) = A_2(\phi(s)) \), for all \( s \in G \), that is, \( f(T_{A_1}(s)) = T_{A_2}(\phi(s)) \), \( f(I_{A_1}(s)) = I_{A_2}(\phi(s)) \) and \( f(F_{A_1}(s)) = F_{A_2}(\phi(s)) \), for all \( s \in G \). Clearly, \( f \) is surjective. Also, \( f \) is injective because if \( f(T_{A_1}(s)) = f(T_{A_1}(t)) \), for all \( s, t \in G \), then \( T_{A_2}(\phi(s)) = T_{A_2}(\phi(t)) \) and we have \( T_{A_1}(s) = T_{A_1}(t) \). Similarly, \( I_{A_1}(s) = I_{A_1}(t) \), \( F_{A_1}(s) = F_{A_1}(t) \).

Therefore, \( f \) is a homomorphism, such that for \( s, t \in G \), we have

\[
\begin{align*}
f(T_{A_1}(s \circ t)) &= T_{A_2}(\phi(s \circ t)), \\
f(I_{A_1}(s \circ t)) &= I_{A_2}(\phi(s \circ t)), \\
f(F_{A_1}(s \circ t)) &= F_{A_2}(\phi(s \circ t)).
\end{align*}
\]

Hence \( A_1 = (T_{A_1}, I_{A_1}, F_{A_1}) \) is isomorphic to \( A_2 = (T_{A_2}, I_{A_2}, F_{A_2}) \). Hence the proof.

3. \((\tilde{a}, \tilde{b})\)-single-valued neutrosophic \( K \)-algebras

Definition 3.1. A single-valued neutrosophic set \( A = (T_A, I_A, F_A) \) in a set \( G \) is called an \((\tilde{a}, \tilde{b})\)-single-valued neutrosophic \( K \)-subalgebra of \( K \) if it satisfies the following conditions:

- \( u_{(\alpha_1, \beta_1, \gamma_1)} \tilde{a} A, v_{(\alpha_2, \beta_2, \gamma_2)} \tilde{a} A \Rightarrow (u \circ v)_{(\min(\alpha_1, \alpha_2), \min(\beta_1, \beta_2), \max(\gamma_1, \gamma_2))} \tilde{b} A \), for all \( u, v \in G, \alpha_1, \beta_1, \alpha_2, \beta_2 \in (0, 1], \gamma_1, \gamma_2 \in [0, 1) \).

Twelve different types of single-valued neutrosophic \( K \)-subalgebras can be obtained by replacing the values of \( \tilde{a}(\neq v \land q) \) and \( \tilde{b} \) by any two values in the set \( \{ e, q, \in, \lor \} \) in Definition 3.1.

Remark 3.1. Every \((e, \in)\)-single-valued neutrosophic \( K \)-subalgebra is in fact, a single-valued neutrosophic \( K \)-subalgebra.

Proposition 3.1. Every \((e, \in)\)-single-valued neutrosophic \( K \)-subalgebra is an \((e, \in) \in \lor \)-single-valued neutrosophic \( K \)-subalgebra.

Proof. Let \( A = (T_A, I_A, F_A) \) be a single-valued neutrosophic \( K \)-subalgebra of \( K \). Let \( u, v \in G \) and \( \alpha_1, \alpha_2 \in (0, 1], \beta_1, \beta_2 \in (0, 1], \gamma_1, \gamma_2 \in [0, 1) \) be such that \( u_{(\alpha_1, \beta_1, \gamma_1)} \in A, v_{(\alpha_2, \beta_2, \gamma_2)} \in A \). Then \( u_{(\alpha_1, \beta_1, \gamma_1)} \in A, v_{(\alpha_2, \beta_2, \gamma_2)} \in A \Rightarrow (u \circ v)_{(\min(\alpha_1, \alpha_2), \min(\beta_1, \beta_2), \max(\gamma_1, \gamma_2))} \in \lor A \). Hence \( A \) is an \((e, \in) \in \lor \)-single-valued neutrosophic \( K \)-subalgebra of \( K \).
**Proposition 3.2.** Every \((\in, \in)\)-single-valued neutrosophic K-subalgebra is an \((\in, \in)\)-single-valued neutrosophic K-subalgebra of \(\mathcal{K}\).

**Definition 3.2.** Let \(\mathcal{A} = (T_\mathcal{A}, I_\mathcal{A}, F_\mathcal{A})\) be a single-valued neutrosophic set in \(\mathcal{G}\). The set \(\underline{A} = \{u \in G \mid T_\mathcal{A}(u) \neq 0, I_\mathcal{A}(u) \neq 0, F_\mathcal{A}(u) \neq 0\}\) is called the support of \(\mathcal{A}\).

**Lemma 3.1.** If \(\mathcal{A} = (T_\mathcal{A}, I_\mathcal{A}, F_\mathcal{A})\) is a non-zero \((\in, \in)\)-single-valued neutrosophic K-subalgebra of \(\mathcal{K}\), then \(\underline{A}\) is a K-subalgebra of \(\mathcal{K}\).

**Proof.** Let \(\mathcal{A} = (T_\mathcal{A}, I_\mathcal{A}, F_\mathcal{A})\) be a non-zero \((\in, \in)\)-single-valued neutrosophic K-subalgebra of \(\mathcal{K}\) and let \(u, v \in \underline{A}\). Then \(T_\mathcal{A}(u) \neq 0\) and \(T_\mathcal{A}(v) \neq 0\) and \(I_\mathcal{A}(u) \neq 0\) and \(I_\mathcal{A}(v) \neq 0\) and \(F_\mathcal{A}(u) \neq 0\) and \(F_\mathcal{A}(v) \neq 0\). Let \(T_\mathcal{A}(u \circ v) = 0, I_\mathcal{A}(u \circ v) = 0\) and \(F_\mathcal{A}(u \circ v) = 0\). Since \(u T_\mathcal{A}(u) \in \mathcal{A}\) and \(v T_\mathcal{A}(v) \in \mathcal{A}\), \(u T_\mathcal{A}(u) \in \mathcal{A}\) and \(v T_\mathcal{A}(v) \in \mathcal{A}\) but \((u \circ v)_{\min(T_\mathcal{A}(u), T_\mathcal{A}(v)), \min(I_\mathcal{A}(u), I_\mathcal{A}(v)), \max(F_\mathcal{A}(u), F_\mathcal{A}(v))) \notin \mathcal{A}\).

Since \(T_\mathcal{A}(u \circ v) = 0, I_\mathcal{A}(u \circ v) = 0\) and \(F_\mathcal{A}(u \circ v) = 0\). A contradiction. Hence \(T_\mathcal{A}(u \circ v) \neq 0, I_\mathcal{A}(u \circ v) \neq 0\) and \(F_\mathcal{A}(u \circ v) \neq 0\) which shows that \((u \circ v) \in \underline{A}, \) consequently \(\underline{A}\) is a K-subalgebra of \(\mathcal{A}\). \(\Box\)

**Lemma 3.2.** If \(\mathcal{A} = (T_\mathcal{A}, I_\mathcal{A}, F_\mathcal{A})\) is a non-zero \((\in, q)\)-single-valued neutrosophic K-subalgebra of \(\mathcal{K}\), then \(\underline{A}\) is a K-subalgebra of \(\mathcal{K}\).

**Lemma 3.3.** If \(\mathcal{A} = (T_\mathcal{A}, I_\mathcal{A}, F_\mathcal{A})\) is a non-zero \((q, \in)\)-single-valued neutrosophic K-subalgebra of \(\mathcal{K}\), then \(\underline{A}\) is a K-subalgebra of \(\mathcal{K}\).

**Lemma 3.4.** If \(\mathcal{A} = (T_\mathcal{A}, I_\mathcal{A}, F_\mathcal{A})\) is a non-zero \((q, q)\)-single-valued neutrosophic K-subalgebra of \(\mathcal{K}\), then \(\underline{A}\) is a K-subalgebra of \(\mathcal{K}\).

**Theorem 3.1.** If \(\mathcal{A} = (T_\mathcal{A}, I_\mathcal{A}, F_\mathcal{A})\) is a non-zero \((\bar{a}, \bar{b})\)-single-valued neutrosophic K-subalgebra of \(\mathcal{K}\), then \(\underline{A}\) is a K-subalgebra of \(\mathcal{K}\).

**Definition 3.3.** A neutrosophic set \(\mathcal{A} = (T_\mathcal{A}, I_\mathcal{A}, F_\mathcal{A})\) in a K-algebra \(\mathcal{K}\) is called an \((\in, \in \vee q)\)-single-valued neutrosophic K-subalgebra of \(\mathcal{K}\) if it satisfies the following conditions:

(a) \(e_{(\alpha, \beta, \gamma)} \in \mathcal{A} \Rightarrow (u)_{(\alpha, \beta, \gamma)} \in \mathcal{A}\),

(b) \(u_{(\alpha_1, \beta_1, \gamma_1)} \in \mathcal{A}, v_{(\alpha_2, \beta_2, \gamma_2)} \in \mathcal{A} \Rightarrow (u \circ v)_{\min(\alpha_1, \alpha_2), \min(\beta_1, \beta_2), \max(\gamma_1, \gamma_2)} \in \mathcal{A}\),

For all \(u, v \in G, \alpha, \alpha_1, \alpha_2 \in (0, 1], \beta, \beta_1, \beta_2 \in (0, 1], \gamma, \gamma_1, \gamma_2 \in [0, 1)\).
Example 3.1. Consider a K-algebra $\mathcal{K} = (G, \cdot, \circ, e)$, where $G = \{e, x, x^2, x^3, x^4, x^5, x^6\}$ is the cyclic group of order 7 and Cayley’s table for $\circ$ is given as:

$$
\begin{array}{ccccccc}
\circ & e & x & x^2 & x^3 & x^4 & x^5 \\
\hline
e & e & x & x^2 & x^3 & x^4 & x^5 \\
x & x & e & x^6 & x^4 & x^3 & x^2 \\
x^2 & x^2 & x & e & x^6 & x^4 & x^3 \\
x^3 & x^3 & x^2 & x & e & x^6 & x^4 \\
x^4 & x^4 & x^3 & x^2 & x & e & x^6 \\
x^5 & x^5 & x^4 & x^3 & x^2 & x & e \\
x^6 & x^6 & x^5 & x^4 & x^3 & x^2 & x
\end{array}
$$

We define a single-valued neutrosophic set $\mathcal{A} = (\mathcal{T}_A, \mathcal{I}_A, \mathcal{F}_A)$ in $\mathcal{K}$ as follows:

$$
\mathcal{T}_A(u) = \begin{cases} 
1, & \text{if } u = e, \\
0.7, & \text{otherwise}
\end{cases}
$$

$$
\mathcal{I}_A(u) = \begin{cases} 
1, & \text{if } u = e, \\
0.6, & \text{otherwise}
\end{cases}
$$

$$
\mathcal{F}_A(u) = \begin{cases} 
0, & \text{if } u = e, \\
0.5, & \text{otherwise}
\end{cases}
$$

Now take $\alpha = 0.4, \alpha_1 = 0.5, \alpha_2 = 0.3, \beta = 0.5, \beta_1 = 0.6, \beta_2 = 0.3, \gamma = 0.6, \gamma_1 = 0.6, \gamma_2 = 0.5$, where $\alpha, \alpha_1, \alpha_2 \in (0, 1], \beta, \beta_1, \beta_2 \in (0, 1], \gamma, \gamma_1, \gamma_2 \in [0, 1)$.

By direct calculations, it is easy to see that $\mathcal{A}$ is an $(\varepsilon, \epsilon, \gamma)$-single-valued neutrosophic K-subalgebra of $\mathcal{K}$.

Theorem 3.2. Let $\mathcal{A} = (\mathcal{T}_A, \mathcal{I}_A, \mathcal{F}_A)$ be a single-valued neutrosophic set in $\mathcal{K}$. Then $\mathcal{A}$ is an $(\varepsilon, \epsilon, \gamma)$-single-valued neutrosophic K-subalgebra of $\mathcal{K}$ if and only if

(i) $\mathcal{T}_A(u) \geq \min(\mathcal{T}_A(e), 0.5),$  
$\mathcal{I}_A(u) \geq \min(\mathcal{I}_A(e), 0.5),$  
$\mathcal{F}_A(u) \leq \max(\mathcal{F}_A(e), 0.5).$

(ii) $\mathcal{T}_A(u \circ v) \geq \min(\mathcal{T}_A(u), \mathcal{T}_A(v), 0.5),$  
$\mathcal{I}_A(u \circ v) \geq \min(\mathcal{I}_A(u), \mathcal{I}_A(v), 0.5),$  
$\mathcal{F}_A(u \circ v) \leq \max(\mathcal{F}_A(u), \mathcal{F}_A(v), 0.5),$ for all $u, v \in G$.

Proof. Assume that $\mathcal{A}$ is an $(\varepsilon, \epsilon, \gamma)$-single-valued neutrosophic K-subalgebra.

Let for $u, v \in G$. Assume that $\mathcal{T}_A(u \circ v) < \min(\mathcal{T}_A(u), \mathcal{T}_A(v), 0.5)$, $\mathcal{I}_A(u \circ v) < \min(\mathcal{I}_A(u), \mathcal{I}_A(v), 0.5)$, $\mathcal{F}_A(u \circ v) > \max(\mathcal{F}_A(u), \mathcal{F}_A(v), 0.5)$. Then $\mathcal{T}_A(u \circ v) < \min(\mathcal{T}_A(u), \mathcal{T}_A(v))$, $\mathcal{I}_A(u \circ v) < \min(\mathcal{I}_A(u), \mathcal{I}_A(v))$ and $\mathcal{F}_A(u \circ v) > \max(\mathcal{F}_A(u), \mathcal{F}_A(v))$. Take $\alpha, \beta, \gamma$ such that $\mathcal{T}_A(u \circ v) < \alpha < \min(\mathcal{T}_A(u), \mathcal{T}_A(v)$,
Let \( I_A(u \circ v) < \beta < \min(I_A(u), I_A(v), F_A(u \circ v) > \gamma > \max(F_A(u), F_A(v)) \). Then \( u_\alpha, v_\alpha \in T_A, u_\beta, v_\beta \in I_A \) and \( u_\gamma, v_\gamma \in F_A \) but \((u \circ v)_{(\min(\alpha_1, \alpha_2), \min(\beta_1, \beta_2), \max(\gamma_1, \gamma_2))} \in \sqrt{q} A \), contradicting the assumption that \( A \) is a K-subalgebra of \( K \).

Now if \( T_A(u \circ v) < 0.5 \), \( I_A(u \circ v) < 0.5 \), \( F_A(u \circ v) > 0.5 \). Then \((u \circ v)_{(0.5, 0.5, 0.5)} \in A \), but \((u \circ v)_{(0.5, 0.5, 0.5)} \notin \sqrt{q} A \) which is also a contradiction. Hence (i) holds.

Let \( u(\alpha_1, \beta_1, \gamma_1), v(\alpha_2, \beta_2, \gamma_2) \in A \) which means that \( T_A(u) \geq \alpha_1, T_A(v) \geq \alpha_2 \), \( I_A(u) \geq \beta_1, I_A(v) \geq \beta_2 \), \( F_A(u) \leq \gamma_1, F_A(v) \leq \gamma_2 \). We have \( T_A(u \circ v) \geq \min(T_A(u), T_A(v), 0.5) \), \( I_A(u \circ v) \geq \min(I_A(u), I_A(v), 0.5) \), \( F_A(u \circ v) \leq \max(F_A(u), F_A(v), 0.5) \). If \( \min(\alpha_1, \alpha_2) > 0.5, \min(\beta_1, \beta_2) > 0.5, \min(\gamma_1, \gamma_2) < 0.5 \), then \( T_A(u \circ v) > 0.5 \Rightarrow T_A(u \circ v) + \min(\alpha_1, \alpha_2) > 1, I_A(u \circ v) \geq 0.5 \Rightarrow I_A(u \circ v) + \min(\beta_1, \beta_2) > 1, F_A(u \circ v) \leq 0.5 \Rightarrow F_A(u \circ v) + \max(\gamma_1, \gamma_2) < 1 \).

But if \( \min(\alpha_1, \alpha_2) \leq 0.5, \min(\beta_1, \beta_2) \leq 0.5, \min(\gamma_1, \gamma_2) \leq 0.5 \), then \( T_A(u \circ v) \geq \min(\alpha_1, \alpha_2), I_A(u \circ v) \geq \min(\beta_1, \beta_2), F_A(u \circ v) \leq \max(\gamma_1, \gamma_2) \). Hence \((u \circ v)_{(\min(\alpha_1, \alpha_2), \min(\beta_1, \beta_2), \max(\gamma_1, \gamma_2))} \in \sqrt{q} A \). Which completes the proof.

**Theorem 3.3.** Let \( A = (T_A, I_A, F_A) \) be a single-valued neutrosophic set in \( K \). Then \( A \) is a neutrosophic K-subalgebra of \( K \) if and only if each non-empty \( A_{(\alpha, \beta, \gamma)} \) is a K-subalgebra of \( K \), for \( \alpha, \beta \in (0.5, 1], \gamma \in [0.5, 1) \).

**Proof.** Assume that \( A = (T_A, I_A, F_A) \) is a single-valued neutrosophic K-subalgebra of \( K \) and let \( \alpha, \beta \in (0.5, 1], \gamma \in [0.5, 1) \). To prove that \( A_{(\alpha, \beta, \gamma)} = \{ u \in G \mid T_A(u) \geq \alpha, I_A(u) \geq \beta, F_A(u) \leq \gamma \} \) is a K-subalgebra of \( K \). If \( u, v \in A_{(\alpha, \beta, \gamma)} \), then \( T_A(u) \geq \alpha, T_A(v) \geq \alpha, I_A(u) \geq \beta, I_A(v) \geq \beta, F_A(u) \leq \gamma, F_A(v) \leq \gamma \). Thus, \( T_A(u \circ v) \geq 0.5 \), \( I_A(u \circ v) \geq 0.5 \), \( F_A(u \circ v) \leq 0.5 \). Thus, \( u \circ v \in A_{(\alpha, \beta, \gamma)} \). Hence \( A_{(\alpha, \beta, \gamma)} \) is a K-subalgebra of \( K \). Converse part is obvious.

**Theorem 3.4.** Let \( A = (T_A, I_A, F_A) \) be a single-valued neutrosophic set in \( K \). Then \( A_{(\alpha, \beta, \gamma)} \) is a K-subalgebra of \( K \) if and only if

(a) \( \max(T_A(u \circ v), 0.5) \geq \min(T_A(u), T_A(v)) \),
\( \min(I_A(u \circ v), 0.5) \geq \min(I_A(u), I_A(v)) \),
\( \min(F_A(u \circ v), 0.5) \leq \max(F_A(u), F_A(v)) \),

(b) \( \max(T_A(e), 0.5) \geq (T_A(u)) \),
\( \max(I_A(e), 0.5) \geq (I_A(u)) \),
\( \min(F_A(e), 0.5) \leq (F_A(u)) \), for all \( u, v \in G \).
Suppose that $A_{(\alpha, \beta, \gamma)}$ is a $K$-subalgebra of $K$ and let $\min(T_A(u \odot v), 0.5) < \min(T_A(u), T_A(v)) = \alpha$, $\max(I_A(u \odot v), 0.5) < \min(I_A(u), I_A(v)) = \beta$, $\min(F_A(u \odot v), 0.5) > \max(F_A(u), F_A(v)) = \gamma$. Then for $\alpha, \beta \in [0.5, 1]$ and $\gamma \in [0.5, 1]$ and $u, v \in A_{(\alpha, \beta, \gamma)}$, $T_A(u \odot v) < \alpha, I_A(u \odot v) < \beta, F_A(u \odot v) > \gamma$. Since $u, v \in A_{(\alpha, \beta, \gamma)}$ and $A_{(\alpha, \beta, \gamma)}$ is a $K$-subalgebra of $K$, so $u, v \in A_{(\alpha, \beta, \gamma)}$ or $T_A(u \odot v) \geq \alpha, I_A(u \odot v) \geq \beta, F_A(u \odot v) \leq \gamma$, a contradiction.

Conversely, suppose that conditions (a) and (b) holds. Let for $u, v \in A_{(\alpha, \beta, \gamma)}$, we have $0.5 < \alpha \leq \min(T_A(u), T_A(v)) \leq \max(T_A(u \odot v), 0.5) \Rightarrow T_A(u \odot v) \geq \alpha$, $0.5 < \beta \leq \min(I_A(u), I_A(v)) \leq \max(I_A(u \odot v), 0.5) \Rightarrow I_A(u \odot v) \geq \beta$, $0.5 > \gamma \geq \max(F_A(u), F_A(v)) \geq \min(F_A(u \odot v), 0.5) \Rightarrow F_A(u \odot v) \leq \gamma$. $0.5 < \alpha \leq T_A(u) \leq \max(T_A(e), 0.5) \Rightarrow T_A(mu) \geq \alpha$, $0.5 < \beta \leq I_A(u) \leq \max(I_A(e), 0.5) \Rightarrow I_A(mu) \geq \beta$, $0.5 > \gamma \geq F_A(u) \geq \min(F_A(e), 0.5) \Rightarrow F_A(mu) \leq \gamma$, for some $m \in G u \odot v \in A_{(\alpha, \beta, \gamma)}$. Hence $A_{(\alpha, \beta, \gamma)}$ is a $K$-subalgebra of $K$.

**Theorem 3.5.** The intersection of any family of $(\in, \in \lor q)$-single-valued neutrosophic $K$-subalgebra of $K$ is an $(\in, \in \lor q)$-single-valued neutrosophic $K$-subalgebra of $K$.

**Proof.** Let $\{A_j : j \in I\}$ be a family of $(\in, \in \lor q)$-single-valued neutrosophic $K$-subalgebras of $K$.

Let $A = \bigcap_{j \in I} A_j = (\sup_{j \in I} T_{A_j}, \sup_{j \in I} I_{A_j}, \inf_{j \in I} F_{A_j})$, for $u, v \in G$, we have

$$T_A(u \odot v) \geq \min(T_A(u), T_A(v), 0.5),$$

$$I_A(u \odot v) \geq \min(I_A(u), I_A(v), 0.5),$$

$$F_A(u \odot v) \leq \max(F_A(u), F_A(v), 0.5),$$

$$T_A(u \odot v) = \sup_{j \in I} T_{A_j}(u \odot v),$$

$$T_A(u \odot v) \geq \sup_{j \in I} \min(T_{A_j}(u), T_{A_j}(v), 0.5),$$

$$T_A(u \odot v) = \min(\sup_{j \in I} T_{A_j}(u), \sup_{j \in I} T_{A_j}(v), 0.5),$$

$$T_A(u \odot v) = \min(\bigcap_{j \in I} T_{A_j}(u), \bigcap_{j \in I} T_{A_j}(v), 0.5),$$

$$T_A(u \odot v) = \min(T_A(u), T_A(v), 0.5),$$

$$I_A(u \odot v) = \sup_{j \in I} I_{A_j}(u \odot v),$$

$$I_A(u \odot v) \geq \sup \min(I_{A_j}(u), I_{A_j}(v), 0.5),$$

$$I_A(u \odot v) = \min(\sup I_{A_j}(u), \sup I_{A_j}(v), 0.5),$$

$$I_A(u \odot v) = \min(\bigcap_{j \in I} I_{A_j}(u), \bigcap_{j \in I} I_{A_j}(v), 0.5),$$

$$I_A(u \odot v) = \min(I_A(u), I_A(v), 0.5),$$

$$I_A(u \odot v) = \sup(I_A(u \odot v), 0.5).$$
\[ I_A(u \odot v) = \min(I_A(u), I_A(v), 0.5), \]
\[ F_A(u \odot v) = \inf_{j \in I} F_A(u \odot v), \]
\[ F_A(u \odot v) \leq \inf_{j \in I} \max(F_A(u), F_A(v), 0.5), \]
\[ F_A(u \odot v) = \max(\inf_{j \in I} F_A(u), \inf_{j \in I} F_A(v), 0.5), \]
\[ F_A(u \odot v) = \max(\bigcap_{j \in I} F_A(u), \bigcap_{j \in I} F_A(v), 0.5), \]
\[ F_A(u \odot v) = \max(F_A(u), F_A(v), 0.5). \]

It follows that \( A \) is an \((\varepsilon, \varepsilon \lor q)\)-single-valued neutrosophic \( K \)-subalgebra of \( K \). \( \Box \)

**Definition 3.4.** Let \( \epsilon_1, \epsilon_2 \in [0,1] \) and \( \epsilon_1 < \epsilon_2 \). Let \( A = (T_A, I_A, F_A) \) be a single-valued neutrosophic \( K \)-subalgebra of \( K \). Then \( A \) is called a single-valued neutrosophic \( K \)-subalgebra with thresholds \((\epsilon_1, \epsilon_2)\) of \( K \) if

\[
\begin{align*}
\max(T_A(u \odot v), \epsilon_1) &\geq \min(T_A(u), T_A(v), \epsilon_2), \\
\max(I_A(u \odot v), \epsilon_1) &\geq \min(I_A(u), I_A(v), \epsilon_2), \\
\min(F_A(u \odot v), \epsilon_1) &\leq \max(F_A(u), F_A(v), \epsilon_2), \quad \text{for all } u, v \in G.
\end{align*}
\]

**Example 3.2.** Consider a \( K \)-algebra on a cyclic group of order 9 and ... table for \( \odot \) is given in example 2.1. It is easy to see that \( A = (T_A, I_A, F_A) \) is a single-valued neutrosophic \( K \)-subalgebra with thresholds \((\epsilon_1 = 0.3, \epsilon_2 = 0.56)\) and for \((\epsilon_1 = 0.55, \epsilon_2 = 0.78)\).

**Remark 3.2.** Let for \( \epsilon_1, \epsilon_2 \in [0,1] \) and \( \epsilon_1 < \epsilon_2 \) unless otherwise specified.

(i) When \( \epsilon_1 = 0 \) and \( \epsilon_2 = 1 \) in single-valued neutrosophic \( K \)-subalgebra with thresholds \((\epsilon_1, \epsilon_2)\), \( A \) is an ordinary single-valued neutrosophic \( K \)-subalgebra.

(ii) When \( \epsilon_1 = 0 \) and \( \epsilon_2 = 0.5 \) in single-valued neutrosophic \( K \)-subalgebra with thresholds \((\epsilon_1, \epsilon_2)\), \( A \) is an \((\varepsilon, \varepsilon \lor q)\)-single-valued neutrosophic \( K \)-subalgebra.

**Theorem 3.6.** A single-valued neutrosophic set \( A \) in \( K \) is a single-valued neutrosophic \( K \)-subalgebra with thresholds \((\epsilon_1, \epsilon_2)\) if and only if \( \cup(T_A, \alpha), \cup(I_A, \beta), L(F_A, \gamma)(\neq \emptyset) \), \( \alpha, \beta, \gamma \in (\epsilon_1, \epsilon_2) \) is a \( K \)-subalgebra of \( K \).

**Proof.** Assume that \( A \) is a single-valued neutrosophic \( K \)-subalgebra with thresholds \((\epsilon_1, \epsilon_2)\). Let for \( u, v \in \cup(T_A, \alpha) \) and \( \alpha \in (\epsilon_1, \epsilon_2) \), \( T_A(u) \geq \alpha \) and \( T_A(v) \geq \alpha \). Since \( A \) is a single-valued neutrosophic \( K \)-subalgebra, it follows that \( \max(T_A(u \odot v), \epsilon_1) \geq \min(T_A(u), T_A(v), \epsilon_2) = \alpha \), so that \( u \odot v \in \cup(T_A, \alpha) \). Hence \( \cup(T_A, \alpha) \) is a \( K \)-subalgebra of \( K \). Similarly, we can prove for \( \cup(I_A, \beta) \) and \( L(F_A, \gamma) \). Hence \( A(\alpha, \beta, \gamma) \) is a \( K \)-subalgebra of \( K \).

Conversely, suppose that for \((\epsilon_1, \epsilon_2) \in [0,1] \) and \( \epsilon_1 < \epsilon_2 \), \( A \) be a ...... \( K \)-subalgebra of \( K \) such that \( \max(T_A(u \odot v), \epsilon_1) < \min(T_A(u), T_A(v), \epsilon_2) = \alpha \), then \( T_A(u \odot v) < \alpha \), where \( u \in \cup(T_A, \alpha), v \in \cup(T_A, \alpha), \alpha \in (\epsilon_1, \epsilon_2) \). Since \( u, v \in \cup(T_A, \alpha) \) and \( \cup(T_A, \alpha) \) is a \( K \)-subalgebra, \( u \odot v \in \cup(T_A, \alpha) \), i.e., \( T_A(u \odot v) \geq \alpha \), a contradiction. Similar results can be obtained for \( \cup(I_A, \beta) \) and \( L(F_A, \gamma) \). \( \Box \)
References


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