Heptavalent symmetric graphs of order $12p$

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Abstract. A graph is symmetric if its automorphism group acts transitively on the set of arcs of the graph. In this paper, we classify connected heptavalent symmetric graphs of order $12p$ for each prime $p$. As a result, there are eleven sporadic such graphs: one for $p = 2$, one for $p = 3$ and nine for $p = 13$.  

Keywords: symmetric graph, $s$-transitive graph, coset graph.

1. Introduction

Throughout this paper graphs are assumed to be finite, simple, connected and undirected. For group-theoretic concepts or graph-theoretic terms not defined here we refer the reader to [24, 27] or [1, 2], respectively. Let $G$ be a permutation group on a set $\Omega$ and $v \in \Omega$. Denote by $G_v$ the stabilizer of $v$ in $G$, that is, the subgroup of $G$ fixing the point $v$. We say that $G$ is semiregular on $\Omega$ if $G_v = 1$ for every $v \in \Omega$ and regular if $G$ is transitive and semiregular.

For a graph $X$, denote by $V(X)$, $E(X)$ and $\text{Aut}(X)$ its vertex set, its edge set and its full automorphism group, respectively. A graph $X$ is said to be $G$-vertex-transitive if $G \leq \text{Aut}(X)$ acts transitively on $V(X)$. $X$ is simply called vertex-transitive if it is $\text{Aut}(X)$-vertex-transitive. An $s$-arc in a graph is an ordered $(s + 1)$-tuple $(v_0, v_1, \ldots, v_{s-1}, v_s)$ of vertices of the graph $X$ such that $v_{i-1}$ is adjacent to $v_i$ for $1 \leq i \leq s$, and $v_{i-1} \neq v_{i+1}$ for $1 \leq i \leq s - 1$. In particular, a 1-arc is just an arc and a 0-arc is a vertex. For a subgroup $G \leq \text{Aut}(X)$, a graph $X$ is said to be $(G, s)$-arc-transitive or $(G, s)$-regular if $G$ is transitive or regular on the set of $s$-arcs in $X$, respectively. A $(G, s)$-arc-transitive graph is said to be $(G, s)$-transitive if it is not $(G, s+1)$-arc-transitive. In particular, a $(G, 1)$-arc-transitive graph is called $G$-symmetric. A graph $X$ is simply called $s$-arc-transitive, $s$-regular or $s$-transitive if it is $(\text{Aut}(X), s)$-arc-transitive, $(\text{Aut}(X), s)$-regular or $(\text{Aut}(X), s)$-transitive, respectively.

As we all known that the structure of the vertex stabilizers of symmetric graphs is very useful to classify such graphs, and this structure of the cubic or tetravalent case was given by Miller [20] and Potočnik [23]. Thus, classifying symmetric graphs with valency 3 or 4 has received considerable attention and a lot of results have been achieved, see [8, 30, 31]. Guo [12] determined the exact
structure of pentavalent case. Following this structure, a series of pentavalent symmetric graphs was classified in [17, 21, 22, 28, 29]. Recently, Guo [13] gave the exact structure of heptavalent case and determined heptavalent symmetric graph of order 6$p$ in [10]. Thus, as an application, we classify connected heptavalent symmetric graphs of order 12$p$ for each prime $p$ in this paper.

2. Preliminary results

Let $X$ be a connected $G$-symmetric graph with $G \leq \text{Aut}(X)$, and let $N$ be a normal subgroup of $G$. The quotient graph $X_N$ of $X$ relative to $N$ is defined as the graph with vertices the orbits of $N$ on $V(X)$ and with two orbits adjacent if there is an edge in $X$ between those two orbits. In view of [18, Theorem 9], we have the following:

**Proposition 2.1.** Let $X$ be a connected heptavalent $G$-symmetric graph with $G \leq \text{Aut}(X)$, and let $N$ be a normal subgroup of $G$. Then one of the following holds:

1. $N$ is transitive on $V(X)$;
2. $X$ is bipartite and $N$ is transitive on each part of the bipartition;
3. $N$ has $r \geq 3$ orbits on $V(X)$, $N$ acts semiregularly on $V(X)$, the quotient graph $X_N$ is a connected heptavalent $G/N$-symmetric graph.

The following proposition characterizes the vertex stabilizers of connected heptavalent $(s)$-transitive graphs (see [13, Theorem 1.1]).

**Proposition 2.2.** Let $X$ be a connected heptavalent $(G, s)$-transitive graph for some $G \leq \text{Aut}(X)$ and $s \geq 1$. Let $v \in V(X)$. Then $s \leq 3$ and one of the following holds:

1. For $s = 1$, $G_v \cong \mathbb{Z}_7$, $D_{14}$, $F_{21}$, $D_{28}$, $F_{21} \times \mathbb{Z}_3$;
2. For $s = 2$, $G_v \cong F_{42}$, $F_{42} \times \mathbb{Z}_2$, $F_{42} \times \mathbb{Z}_3$, $\text{PSL}(3, 2)$, $A_7$, $S_7$, $\mathbb{Z}_2^3 \rtimes \text{SL}(3, 2)$ or $\mathbb{Z}_2^3 \rtimes \text{SL}(3, 2)$;
3. For $s = 3$, $G_v \cong F_{42} \times \mathbb{Z}_6$, $\text{PSL}(3, 2) \times S_4$, $A_7 \times A_6$, $S_7 \times S_6$, $(A_7 \times A_6) \rtimes \mathbb{Z}_2$, $\mathbb{Z}_2^3 \times (\text{SL}(2, 2) \times \text{SL}(3, 2))$ or $[2^{20}] \rtimes (\text{SL}(2, 2) \times \text{SL}(3, 2))$.

From [9, pp.12-14], [26, Theorem 2] and [16, Theorem A], we may obtain the following proposition by checking the orders of non-abelian simple groups:

**Proposition 2.3.** Let $p$ be a prime, and let $G$ be a non-abelian simple group of order $|G| = (2^{26} \cdot 3^{5} \cdot 5^{2} \cdot 7 \cdot p)$. Then $G$ has $3$-prime factor, $4$-prime factor or $5$-prime factor, and is one of the following groups:
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<table>
<thead>
<tr>
<th>$G$</th>
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### 4-prime factor

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### 5-prime factor

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<td>$PSp(8, 2)$</td>
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<td>$PO^+(8, 2)$</td>
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<td>$2^3\cdot3^5$</td>
<td>$G_2(4)$</td>
<td>$2^3\cdot3^5$</td>
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</table>

Table 1: Non-abelian simple $\{2, 3, 5, 7, p\}$—groups

In view of [14, Theorem 3.1], we have the classification of connected heptavalent symmetric graphs of order $4p$ for a prime $p$.

**Proposition 2.4.** Let $X$ be connected heptavalent symmetric graph of order $4p$ with $p$ a prime. Then $X$ is isomorphic to $K_8$.

Next we construct some heptavalent symmetric graphs of order $6p$ with $p$ a prime. In order to construct some heptavalent symmetric graphs, we need to introduce the so called coset graph (see [20, 25]) constructed from a finite group $G$ relative to a subgroup $H$ of $G$ and a union $D$ of some double cosets of $H$ in $G$ such that $D^{-1} = D$. The coset graph $Cos(G, H, D)$ of $G$ with respect to $H$ and $D$ is defined to have vertex set $[G : H]$, the set of right cosets of $H$ in $G$, and edge set $\{ (Hg, Hdg) \mid g \in G, d \in D \}$. The graph $Cos(G, H, D)$ has valency $|D|/|H|$ and is connected if and only if $D$ generates the group $G$. The action of $G$ on $V(Cos(G, H, D))$ by right multiplication induces a vertex-transitive automorphism group, which is arc-transitive if and only if $D$ is a single double coset. Moreover, this action is faithful if and only if $H_G = 1$, where $H_G$ is the largest normal subgroup of $G$ in $H$. Clearly, $Cos(G, H, D) \cong Cos(G, H^\alpha, D^\alpha)$ for every $\alpha \in Aut(G)$. For more details regarding coset graphs, see, for example, [7, 18, 25].

By Atlas [6], $S_8$ has a maximal subgroup $K \cong \mathbb{Z}_4^3 \times S_4$ and $A_8$ has a maximal subgroup $H \cong \mathbb{Z}_2^3 \times PSL(3, 2)$ such that $K \cap H \cong \mathbb{Z}_2^3 \times S_4$. Take an element $g$
of order 2 in $K \setminus (K \cap H)$. Then define the coset graph:

$$C_{30} = \text{Cos}(S_8, H, HgH).$$

Let $G \cong \text{PSL}(2, 13)$ as a permutation group acting naturally on 14 points, and take the following four elements:

- $a = (1, 13, 2, 12, 9, 5, 14)(3, 4, 7, 10, 11, 6, 8),$
- $b = (2, 5)(3, 7)(6, 11)(8, 10)(9, 12)(13, 14),$
- $x = (1, 4)(3, 10)(6, 14)(7, 8)(9, 12)(11, 13),$
- $y = (1, 4)(2, 6)(5, 11)(8, 9)(10, 12)(13, 14).$

Then $H = \langle a, b \rangle \cong D_{14}$. Define the following two coset graph:

$$C_{78}^1 = \text{Cos}(G, H, HxH), \quad C_{78}^2 = \text{Cos}(G, H, HyH).$$

We can get the classification of heptavalent symmetric graphs of order $6p$ with $p$ a prime from [10, Theorem 3.1].

**Proposition 2.5.** Let $X$ be a connected heptavalent symmetric graph of order $6p$ with $p$ a prime. Then $X$ is isomorphic to $C_{30}$, $C_{78}^1$ or $C_{78}^2$.

Let $G \cong \text{PSL}(2, 8)$. Then by Atlas [6], $G$ has a maximal subgroup $H \cong D_{14}$ and a Sylow 2-subgroup $P \cong \mathbb{Z}_2^3$ such that $P \cap H \cong \mathbb{Z}_2$. Take $g \in P \setminus H$. Then $g^2 = 1$, $H \cap H^g \cong \mathbb{Z}_2$ and $\langle H, g \rangle = G$. Define the coset graph:

$$C_{36} = \text{Cos}(G, H, HgH).$$

From [11, Theorem 3.1], we have the classification of connected heptavalent symmetric graphs of order 36.

**Proposition 2.6.** There is only one connected heptavalent symmetric graph of order 36, that is, $C_{36}$.

Now we construct some heptavalent symmetric graphs of order $12p$. If $p = 2$, then by [5], there is only one such graph of order 24, and we define this graph as follows:

**Construction 2.7.** Set $a = (3, 5)(4, 7)(6, 8)$, $b = (1, 2)(4, 8)(6, 7)$ and $c = (1, 4, 8, 3, 5, 6, 7)$. Then $G = \langle a, b, c \rangle \cong \text{PGL}(2, 7)$. Let $H = \langle a, c \rangle$. Clearly, $H \cong D_{14}$ and $b$ centralizes $a$. Define the coset graph:

$$C_{24} = \text{Cos}(G, H, HbH).$$

Then $C_{24}$ is a connected heptavalent symmetric graph of order 24. By Magma [3], $\text{Aut}(C_{24}) \cong \text{PGL}(2, 7)$ and any connected heptavalent symmetric graph of order 24 admitting $\text{PGL}(2, 7)$ as an arc-transitive automorphism group is isomorphic to $C_{24}$. 
By using the isomorphisms of coset graph and calculation of Magma [3], we have that there are nine non-isomorphic heptavalent symmetric graphs of order 156 with PSL(2, 13) as an automorphism group.

**Construction 2.8.** Take the following six elements in $S_{16}$:

\[
\begin{align*}
a &= (3, 13, 11, 9, 7, 5)(4, 14, 12, 10, 8, 6), \\
b &= (1, 2, 9)(3, 8, 10)(4, 5, 12)(6, 13, 14), \\
c &= (15, 16), \\
f_1 &= (2, 10)(3, 13)(4, 11)(5, 14)(6, 7)(8, 12), \\
f_2 &= (2, 10)(3, 13)(4, 11)(5, 14)(6, 7)(8, 12)(15, 16), \\
f_3 &= (1, 10, 8, 5, 14, 12, 2)(3, 6, 11, 4, 7, 13, 9).
\end{align*}
\]

Then $G = \langle a, b, c \rangle \cong PSL(2, 13) \times \mathbb{Z}_2$, $H_1 = \langle f_1, f_3 \rangle \cong D_{14}$ and $H_2 = \langle f_2, f_3 \rangle \cong D_{14}$. Take five elements in $G$:

\[
\begin{align*}
x_1 &= (3, 9)(4, 10)(5, 11)(6, 12)(7, 13)(8, 14)(15, 16), \\
x_2 &= (2, 6)(3, 14)(4, 8)(7, 12)(9, 11)(10, 13)(15, 16), \\
x_3 &= (2, 9)(3, 10)(4, 13)(5, 6)(7, 11)(12, 14)(15, 16), \\
x_4 &= (1, 9)(2, 6)(3, 13)(5, 8)(7, 10)(12, 14)(15, 16), \\
x_5 &= (1, 9)(2, 4)(3, 14)(5, 13)(8, 12)(10, 11)(15, 16).
\end{align*}
\]

Define the following five coset graphs:

\[
\begin{align*}
SG_{156}^1 &= \text{Cos}(G, H_1, H_1x_iH_1), i = 1, 2, \\
SG_{156}^i &= \text{Cos}(G, H_2, H_2x_iH_2), i = 3, 4, 5.
\end{align*}
\]

By Magma [3], Aut($SG_{156}^i$) $\cong PSL(2, 13) \times \mathbb{Z}_2$, Aut($SG_{156}^2$) $\cong PGL(2, 13) \times \mathbb{Z}_2$ and Aut($SG_{156}^3$) $\cong PSL(2, 13) \times \mathbb{Z}_2$ for $i = 3, 4, 5$. In particular, $SG_{156}^1$ and $SG_{156}^2$ are bipartite.

Take the following eight elements:

\[
\begin{align*}
d &= (1, 2)(3, 6)(4, 5)(7, 14)(8, 13)(9, 12)(10, 11), \\
e &= (1, 4)(2, 5)(3, 8)(6, 13)(7, 10)(11, 14), \\
f &= (1, 9, 4, 14, 3, 8, 11)(2, 10, 13, 6, 7, 5, 12), \\
y_1 &= (3, 9)(4, 10)(5, 11)(6, 12)(7, 13)(8, 14), \\
y_2 &= (2, 11)(3, 6)(4, 14)(5, 12)(7, 8)(9, 13), \\
y_3 &= (2, 13)(3, 11)(4, 6)(5, 8)(7, 14)(9, 10), \\
y_4 &= (2, 9, 5, 12)(3, 14, 7, 13)(6, 10, 11, 8), \\
y_5 &= (2, 13, 6, 10)(3, 4, 14, 8)(7, 9, 12, 11).
\end{align*}
\]

Set $T = \langle a, b, d \rangle \cong PGL(2, 13)$, $K_1 = \langle d, f \rangle \cong D_{14}$ and $K_2 = \langle e, f \rangle \cong D_{14}$.

Define the following coset graphs:

\[
\begin{align*}
CG_{156}^i &= \text{Cos}(T, K_1, K_1y_iK_1), i = 1, 2, 3, \\
CG_{156}^i &= \text{Cos}(T, K_2, K_2y_iK_2), i = 4, 5.
\end{align*}
\]
By Magma [3], $CG_{156}^2 \cong SG_{156}^2$, and $\text{Aut}(CG_{156}^i) \cong \text{PGL}(2, 13)$ for $i = 1, 2, 3, 4$. In particular, $CG_{156}^4$ is bipartite.

We use the same notations as above to state the following lemma.

**Lemma 2.9.** Let $X$ be a connected heptavalent symmetric graph of order 156 and $A = \text{Aut}(X)$. Then

1. If $A$ has an arc-transitive subgroup isomorphic to $G$, then $X \cong SG_{156}^i$ for $i = 1, 2, 3, 4, 5$;

2. If $A$ has an arc-transitive subgroup isomorphic to $T$, then $X \cong CG_{156}^i$ for $i = 1, 2, 3, 4, 5$;

**Proof.** Assume that $A$ has an arc-transitive subgroup $G$. Then $X \cong \text{Cos}(G, G_v, G_v g G_v)$. Let $G_v \cong D_{14}$, $(G_v, g) = G$ and $|G_v : G_v^g \cap G_v| = 7$. In particular, $g$ can be chosen as a 2-element. By Magma [3], $G$ has two conjugacy classes of subgroups isomorphic to $D_{14}$ with $H_1$ and $H_2$ as their representatives.

Suppose that $G_v = H_1$. Then by Magma [3], $g$ has 42 choices, denoted this set by $U$. Note that $N_G(G_v) \cong D_{28}$ by Atlas [6]. Thus, again by Magma [3], $N_G(G_v)$ acting on $U$ has 6 orbits. Clearly, the coset graphs formed by the elements in the same orbit are isomorphic each other. Thus, we obtain six coset graphs. By Magma [3], these six representatives of 6 orbits form two coset graphs, and $X \cong SG_{156}^i$ with $i = 1, 2$.

Suppose that $G_v = H_2$. Then by Magma [3], $g$ has 84 choices, denoted this set by $V$. Since $N_G(G_v) \cong D_{28}$, we have that $N_G(G_v)$ acting on $V$ has 12 orbits by Magma [3]. All the representatives of 12 orbits form three coset graphs, and $X \cong SG_{156}^i$ with $i = 3, 4, 5$.

Assume that $A$ has an arc-transitive subgroup $T$. Then $X \cong \text{Cos}(T, T_v, T_v t T_v)$ with $T_v \cong D_{14}$, $(T_v, t) = T$ and $|T_v : T_v^t \cap T_v| = 7$. In particular, $t$ can be chosen as a 2-element. By Atlas [6], $T$ has two conjugacy classes of subgroups isomorphic to $D_{14}$ with $K_1$ and $K_2$ as their representatives.

Suppose that $T_v = K_1$. Then by Magma [3], $t$ has 112 choices, and $N_T(T_v) \cong D_{28}$ acting on these elements has seven orbits. These seven orbits form seven coset graphs and by Magma [3], there are 3 non-isomorphic graphs, that is, $X \cong CG_{156}^i$ with $i = 1, 2, 3$.

Suppose that $T_v = K_2$. Then by Magma [3], $t$ has 98 choices, and $N_T(T_v) \cong D_{28}$ acting on these elements has five orbits. These five orbits form five coset graphs and by Magma [3], there are 2 non-isomorphic graphs, that is, $X \cong CG_{156}^i$ with $i = 4, 5$.

**3. Classification**

This section is devoted to classify connected heptavalent symmetric graphs of order $12p$ for each prime $p$.
Theorem 3.1. Any connected heptavalent symmetric graph of order $12p$ with $p$ a prime is isomorphic to $C_{24}$, $C_{36}$, $S_{156}^i$ with $i = 1, 2, 3, 4, 5$, or $C_{156}^j$ with $j = 1, 2, 3, 4, 5$.

Proof. Let $X$ be a connected heptavalent symmetric graph of order $12p$ and $A = \text{Aut}(X)$. If $p = 2$, then by [5] and Construction 2.7, $X \cong C_{24}$ and $A \cong \text{PGL}(2, 7)$. If $p = 3$, then by Proposition 2.6, $X \cong C_{36}$. Thus, in the following we assume that $p \geq 5$. Take $v \in V(X)$. Then by Proposition 2.2, $|A_v| = 2^{24}3^45^27$ and hence $|A| = 2^{26}3^55^27p$. We separate the proof into two cases: $A$ has a solvable minimal normal subgroup; $A$ has no solvable minimal normal subgroup.

Case 1: $A$ has a solvable minimal normal subgroup.

Let $N$ be a solvable minimal normal subgroup of $A$. Then $|N| = 2^{26}3^55^27p$, and $N$ is elementary abelian. Thus, $N \cong \mathbb{Z}_q^k$ with $q = 2, 3, 5, 7$ or $p$ and $k$ a positive integer. By Proposition 2.1, $N$ is semiregular and $X_N$ is also a connected heptavalent $A/N$-symmetric graph. It follows that $|N| | 12p$ and $N \cong \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_5$ or $\mathbb{Z}_p$. Note that there is no connected heptavalent regular graph of odd order. Thus, $N \not\cong \mathbb{Z}_q^2$.

Assume that $X_N \cong C_{30}$. Then $p = 5$. Since $\text{Aut}(C_{30}) \cong S_8$ by Proposition 2.5, we have that $A/N \cong S_8$. By Magma [3], $\text{Aut}(C_{30})$ has a minimal arc-transitive subgroup isomorphic to $S_7$. It follows that $A/N$ has an arc-transitive subgroup $M/N \cong S_7$. Note that $C_{30}$ is bipartite. Set $H/N \cong A_7$. Then $H/N$ has two orbits on $V(X_N)$ and hence $H \cong \text{PSL}(2, 7)$. Since $N \leq C_H(N)$ and $H/N$ is simple, we have that $C_H(N) = H$. It forces that $H \cong \mathbb{Z}_2 \times A_7$ or $\mathbb{Z}_2A_7$. If $H \cong \mathbb{Z}_2 \times A_7$, then $H$ is a characteristic subgroup $K \cong A_7$. The normality of $H$ in $M$ implies that $K$ is also normal in $M$. Thus, the block graph $X_K$ has order 4. By Proposition 2.1, $K$ is semiregular and $|K| | 12p$. This is impossible because $K \cong A_7$. If $H \cong \mathbb{Z}_2A_7$, then $|H_v| = 168$. However, by Magma [3], $\mathbb{Z}_2A_7$ has no subgroups of order 168, a contradiction.

Assume that $X_N \cong C_{78}^1$ or $C_{78}^2$. Then $p = 13$. By Atlas [6], $C_{78}^1$ or $C_{78}^2$ has a minimal arc-transitive subgroup isomorphic to $\text{PSL}(2, 13)$. Thus, $A/N$ has an arc-transitive subgroup $M/N \cong \text{PSL}(2, 13)$. Since $N \cong \mathbb{Z}_2$, we have that $C_M(N) = M$. By Atlas [6], $\text{Mult}(\text{PSL}(2, 13)) = 2$. It follows that $M \cong \text{SL}(2, 13)$ or $\text{PSL}(2, 13) \times \mathbb{Z}_2$. If $M \cong \text{SL}(2, 13)$, then $M_v \cong \mathbb{Z}_{14}$ by Magma [3]. This is impossible by Proposition 2.2. It follows that $M \cong \text{PSL}(2, 13) \times \mathbb{Z}_2$. Note that $M$ is arc-transitive. Thus, by Construction 2.8 and Lemma 2.9, $X \cong S_{156}^i$ with $i = 1, 2, 3, 4, 5$.

Case 2: $A$ has no solvable minimal normal subgroup.

For convenience, we still use $N$ to denote a minimal normal subgroup of $A$. Since $N$ is non-solvable and minimal normal, we have that $N = T^k$ with $T$ a
non-abelian simple group and $k$ an positive integer. It follows that $T$ has at least 3-prime factors. Note that $|N| = 2^{26} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot p$. Thus, $T$ is one of the simple groups listed in Proposition 2.3. By Proposition 2.1, $N$ has at most two orbits on $V(X)$, and hence $|N| = 12p|N_v|$ or $6p|N_v|$.

Assume that $k \geq 2$. Since $T$ is a non-abelian simple group, we have that $2^{k} | |T|$ and $|T_v| \neq 1$. If $p > 7$, then $p \nmid |T|$ because $p^2 \nmid |N| = |T^k|$. It follows that $p$ divides the order of $X_v$. By Proposition 2.1, $N = T^k$ is semiregular and hence $T_v = 1$, a contradiction. Thus, $p \leq 7$. Recall that $|T^k| = 2^{26} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot p$ and $6 \cdot p | |T^k|$.

Let $p = 5$. Then $7^2 | |T^k|$ and $T$ is a simple $\{2, 3, 5\}$-group. By Proposition 2.3, $T \cong A_5$ or $A_6$. The normality of $N$ in $A$ implies that $N_v \leq A_v$. If $k \geq 3$, then $5^2 | |N_v|$ and hence $N_v$ has a subgroup isomorphic to $A_7$ by Proposition 2.2. It follows that $7 | |T|$, a contradiction. Thus, $k = 2$. If $T \cong A_5$, then $N \cong A_5^2$ and $|N_v| = 60$ or 120. By Atlas [6], for $|N_v| = 60$, we have that $N_v \cong D_{10} \times S_3$, $A_4 \times Z_5$ or $A_5$, and for $|N_v| = 120$, we have that $N_v \cong A_5 \times Z_2$ or $A_4 \times D_{10}$. However, by Proposition 2.2, $A_v$ has no such normal subgroups, a contradiction. If $T \cong A_6$, then $N \cong A_6^2$, and $|N_v| = 2^3 \cdot 3^5 \cdot 5$ or $2^5 \cdot 3^3 \cdot 5$. By Magma [3], for $|N_v| = 2^3 \cdot 3^5 \cdot 5$, we have that $N_v \cong A_6 \times S_3$ or $A_5 \times F_{36}$, and for $|N_v| = 2^5 \cdot 3^3 \cdot 5$, we have that $N_v \cong A_6 \times A_4$. Similarly, $A_v$ has no such normal subgroups, a contradiction.

Let $p = 7$. Then $|T^k| = 2^{26} \cdot 3^5 \cdot 5^2 \cdot 7^2$ and $6 \cdot 7 | |T^k|$. It follows that $7 | |T|$ and $k = 2$. By Proposition 2.3, $T$ is isomorphic to one of the following groups: $\text{PSL}(2, 7)$, $\text{PSL}(2, 8)$, $A_7$, $A_8$, $\text{PSL}(3, 4)$.

If $T \cong \text{PSL}(2, 7)$, then $|N_v| = 2^5 \cdot 3 \cdot 7$ or $2^4 \cdot 3 \cdot 7$. By Magma [3], for $|N_v| = 2^5 \cdot 3 \cdot 7$, $N_v \cong \text{PSL}(2, 7) \times Z_2$ or $\text{PSL}(2, 7) \times Z_4$; for $|N_v| = 2^4 \cdot 3 \cdot 7$, $N_v \cong \text{PSL}(2, 7) \times Z_2$. The normality of $N_v$ in $A_v$ implies that $A_v \cong \text{PSL}(2, 7) \times S_3$ and $N_v \cong \text{PSL}(2, 7) \times Z_2$ by Proposition 2.2. It follows that $N$ has two orbits on $V(X)$ and $|A| = 2|A_v, N| = 6|N|$. Clearly, $C_A(N) \cap N = 1$. Thus, $|C_A(N)| | 6$ and $C_A(N)$ is solvable. Since $C_A(N) \leq A$, we have that $C_A(N) = 1$ by our assumption. By “N/C-Theorem” (see [15, Chapter I, Theorem 4.5]), $A \cong A/C_A(N) \subset \text{Aut}(N)$. However, by Magma [3], $|\text{Aut}(N)| = 8|N|$, which is contrary to the fact that $|A| = 6|N|$. If $T \cong \text{PSL}(2, 8)$, then $|N_v| = 2^5 \cdot 3 \cdot 7$ or $2^4 \cdot 3 \cdot 7$. By Magma [3], the only possible is that $N_v \cong \text{PSL}(2, 8) \times S_3$. However, $A_v$ has no such normal subgroup by Proposition 2.2, a contradiction. If $T \cong A_7$, then by Magma [3], the only possible is that $|N_v| = 2^5 \cdot 3 \cdot 5 \cdot 7$ and $N_v \cong A_7 \times A_5$. Similarly, $A_v$ has no such normal subgroup, a contradiction. If $T \cong A_8$ or $\text{PSL}(3, 4)$, then $|N_v| = |N|/6p$ or $|N|/12p$. By Magma [3], $A_5^2$ or $\text{PSL}(3, 4)^2$ has no subgroups of such orders, a contradiction.

Thus, $k = 1$ and $N = T$ is a non-abelian simple group. Since $|N| = 2^{26} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot p$, we have that $N$ is isomorphic to one of the simple group listed in Proposition 2.3. By Proposition 2.1, we have that $N$ has at most two orbits on $V(X)$ and $|N_v| = |N|/6p$ or $|N|/12p$. 

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**Subcase 2.1:** Suppose that \( p = 5 \). Then \(|N| = 2^{26} \cdot 3^5 \cdot 5^3 \cdot 7^2\). By Proposition 2.3, \( N \) is isomorphic to one of the following simple groups:

\[
A_5, \ A_6, \ \text{PSU}(4, 2), \ A_7, \ A_8, \ A_9, \ A_{10} \\
\text{PSL}(3, 4), \ \text{PSp}(6, 2), \ J_2, \ \text{PΩ}^+(8, 2).
\]

Note that \(|N_v| = |N|/6 \cdot 5 \) or \(|N|/12 \cdot 5\).

Let \( N \cong A_5 \). Then \( N_v \cong \mathbb{Z}_2 \) or \( 1 \). If \( N_v \cong \mathbb{Z}_2 \), then since \( N_v \leq A_v \), we have that \( A_v \cong D_{28}, \ F_{42} \times \mathbb{Z}_2 \) or \( F_{12} \times \mathbb{Z}_6 \) by Proposition 2.2. Note that \( C_A(N) \leq \mathbb{Z}_6 \). It follows that \( A \) is non-solvable and has no solvable characteristic subgroup by our assumption. Clearly, \( C_A(N) \) is regular on \( V(X) \). By “N/C-Theorem” (see [15, Chapter I, Theorem 4.5]), \( A/C_A(N) \leq \text{Aut}(N) \). Since \( \text{Aut}(A_5) \leq S_5 \), we have that \(|C_A(N)| \leq 7 \) and since \( C_A(N) \not\leq A \), we have that \( C_A(N) \) is non-solvable. Clearly, \( C_A(N) \) is a transitive subgroup of \( V(X) \). By Proposition 2.1, \( C_A(N) \) is regular and hence solvable, a contradiction.

Let \( N \cong A_6 \). Then \(|N_v| = 12 \) or \( 6 \). By Atlas [6], \( N_v \cong A_4 \), or \( S_3 \). Since \( N \not\leq A_6 \), we have that \( N_v \leq A_v \). By Proposition 2.2, \( N_v \cong A_4 \) and \( A_v \cong \text{PSL}(2, 7) \times S_4 \). It follows that \(|A| = 2^{4} \cdot 3 \cdot 7 \cdot |N| \). Similar as above, \( C_A(N) \not\cong A_6 \) and \( C_A(N) \) is non-solvable. Clearly, \( C_A(N) \cap N = 1 \). Thus, \(|C_A(N)| = 2^4 \cdot 3 \cdot 7 \). This implies that \( C_A(N) \) acting on \( V(X) \) has at least five orbits. By Proposition 2.1, \( C_A(N) \) is regular and hence solvable, a contradiction.

**Subcase 2.2:** Suppose that \( p = 7 \). Then \(|N| = 2^{26} \cdot 3^5 \cdot 5^3 \cdot 7^2 \) and \( 6 \cdot 7 \) or \( 12 \cdot 7 \) or \( |N| \). By Proposition 2.3, \( N \) is isomorphic to one of the following simple groups:

\[
\text{PSL}(2, 7), \ \text{PSL}(2, 8), \ \text{PSU}(3, 3), \ A_7, \ A_8, \ A_9, \ A_{10} \\
\text{PSL}(2, 19), \ \text{PSL}(3, 4), \ \text{PSp}(6, 2), \ J_2, \ \text{PΩ}^+(8, 2).
\]

Let \( N \cong \text{PSL}(2, 7) \). Then \( N_v \cong \mathbb{Z}_2 \) or \( \mathbb{Z}_2 \). Set \( C = C_A(N) \). If \( C = 1 \), then by “N/C-Theorem”, \( A \cong A/C \leq \text{Aut}(N) \cong \text{PGL}(2, 7) \). However, since \( p = 7 \) and \( X \) is arc-transitive, we have that \(|A| = 7^2 |C| \), a contradiction. Thus, \( C \neq 1 \).

Since \( C \leq A \), we have that \( C \) is non-solvable by our assumption. Note that \( C \cong C \cap N \leq A/N \) and \( A_v/N_v \cong A_v/N \leq A/N \) with \(|A : A_v/N_v| \leq 2 \). Thus, \( A_v \) is non-solvable. Since \( N_v \leq A_v \), we have that \( N_v \cong \mathbb{Z}_2 \) and \( A_v \cong \text{PSL}(2, 7) \times S_4 \) by Proposition 2.2. It follows that \( A_v/N_v \cong \text{PSL}(2, 7) \times S_3 \) and \( C \cong \text{PSL}(2, 7) \) because \( C \) is non-solvable and has no solvable characteristic subgroup. Thus, \(|A/C : NC/C| = 12 \) and \( N \not\leq A/C \leq \text{Aut}(N) \cong \text{PGL}(2, 7) \), this is impossible.
Let $N \cong \text{PSL}(2,8)$. Then $|N_v| = 6$ or 12. By Atlas [6], $N_v \cong S_3$. Since $N \leq A_4$ we have that $N_v \leq A_v$. However, by Proposition 2.2, $A_v$ has no normal subgroup isomorphic to $S_3$, a contradiction.

Let $N \cong \text{PSU}(3,3)$, $A_8$, $A_{10}$, $\text{PSL}(2,49)$, $\text{PSL}(3,4)$, $\text{PSp}(6,2)$, $J_2$ or $\Omega^+(8,2)$. Then $|N_v| = |N|/42$ or $|N|/84$. By Magma [3], $N$ has no subgroups of such orders, a contradiction.

Let $N \cong A_7$. Then $|N_v| = 30$ or 60. By Atlas [6], $N_v \cong A_5$. The normality of $N$ in $A$ implies that $A_v$ has a normal subgroup isomorphic to $A_5$. This is impossible by Proposition 2.2.

Let $N \cong A_9$. Then by Atlas [6], $N_v \cong (A_6 \times \mathbb{Z}_3) \rtimes Z_2$. This is also impossible because $A_v$ has no normal subgroup isomorphic to $(A_6 \times \mathbb{Z}_3) \rtimes Z_2$.

**Subcase 2.3:** Suppose that $p > 7$. Then $|N| \mid 2^{56} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot p$ and $6 \cdot p \mid |N|$ or $12 \cdot p \mid |N|$. By Proposition 2.3, $N$ is isomorphic to one of the following simple groups:

$$\text{PSL}(2,17), \text{PSL}(3,3), \text{PSL}(2,11), \text{PSL}(2,13), \text{PSL}(2,16), \text{PSL}(2,19), \text{PSL}(2,25), \text{PSL}(2,27), \text{PSL}(2,31), \text{PSL}(2,81), \text{PSL}(2,127), \text{PSU}(3,4), \text{PSU}(5,2), \text{PSp}(4,4), M_{11}, M_{12}, \text{PSL}(2,29), \text{PSL}(2,41), \text{PSL}(2,71), \text{PSL}(2,449), \text{PSL}(2,2^{26}), \text{PSL}(4,4), \text{PSL}(5,2), \text{PSp}(8,2), M_{22}, \text{PSL}(3,7)$$

Let $N \cong \text{PSL}(2,17)$. Then by Atlas [6], $N_v \cong A_4$ or $S_4$. Since $N_v \leq A_v$, we have that $A_v \cong \text{PSL}(2,7) \rtimes S_4$ by Proposition 2.2. Similar as above, $C_A(N) \cong \text{PSL}(2,7)$. Since $p \nmid |C_A(N)|$, we have that $C_A(N)$ has at least $p$ orbits on $V(X)$, and by Proposition 2.1, $C_A(N)$ is semiregular and hence solvable, a contradiction.

Let $N \cong \text{PSL}(3,3)$. Then $|N_v| = 2^2 \cdot 3^2$ or $2^3 \cdot 3^2$. By Atlas [6], $N_v$ has a characteristic subgroup isomorphic to $Z_3^2$. The normality of $N_v$ in $A_v$ implies that $A_v$ has a normal subgroup isomorphic to $Z_3^2$. This is impossible by Proposition 2.2.

Let $N \cong \text{PSL}(2,11)$, $\text{PSL}(2,25)$, $M_{11}$, $M_{12}$ or $A_{12}$. Then by Atlas [6], $N_v \cong Z_5$ or $D_{10}$ for $N \cong \text{PSL}(2,11)$; $N_v \cong Z_2^2 \times Z_2$ or $Z_2^2 \times Z_4$ for $N \cong \text{PSL}(2,15)$; $N_v \cong A_5$ or $S_5$ for $N \cong M_{11}$; $N_v \cong A_6 \rtimes Z_2^2$ or $A_6 \times Z_2$ for $N \cong M_{12}$; $N_v \cong A_{10}$ or $S_{10}$ for $N \cong A_{12}$. However, By Proposition 2.2, $A_v$ has no such normal subgroups, a contradiction.

Let $N \cong \text{PSL}(2,13)$. Then by Atlas [6], $N_v \cong Z_7$ or $D_{14}$. Since $N_v \leq A_v$, we have that $A_v$ is solvable by Proposition 2.2. Note that $C_A(N) \cong C_A(N).N/N \leq A/N$ and $|A/N : A_v.N/N| \leq 2$. Thus, $C_A(N)$ is solvable. By our assumption, $C_A(N) = 1$ and by “N/C-Theorem”, $A \cong A/C \leq \text{Aut}(N) \cong \text{PGL}(2,13)$. By Construction 2.8 and Lemma 2.9, $X \cong C_{G_1^5}$ with $i = 1, 2, 3, 4$.

For the remaining simple groups listed in the above, with the calculation of Magma [3] or check the information of maximal subgroups in Atlas [6], we can deduce that all the groups do not have subgroups of order $|N_v| = |N|/6p$ or $|N|/12p$, a contradiction. \hfill \Box
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