# DERIVED NUMBERS OF ONE VARIABLE CONVEX FUNCTIONS

**S. Kadry***
*Department of Mathematics and Computer Science*
*Faculty of Science*
*Beirut Arab University*
*Lebanon*
*s.kadry@bau.edu.lb*

**G. Alferov**
**G. Ivanov**
**A. Sharlay**
*Sain-Petersburg State University*
*Russia, 198504*
*Saint-Petersburg, Peterhof*
*Universitetskiy prospekt, 35*
*alferovgv@gmail.com*

**Abstract.** The Fermat, Roll and Lagrange theorems are generalized into the class of nondifferentiable functions, the necessary and sufficient conditions for convexity of one variable functions are given.

**Keywords:** derived number, periodic solution, almost periodic solution, nonsmooth analysis, Dini-Hölder derived number.

## 1. Introduction

In this paper, a method of periodic and almost periodic ordinary differential equations development is considered. It is based on the ideas of functional analysis. I.P. Natanson briefly outlined the theory of derived numbers [1]. Developing this theory, several theorems of mathematical analysis are proved. Implementation of this theory let reducing the restrictions on smoothness degree of the right-hand sides of the equations considered, which made it possible to extend the scope of the results obtained [2-11]. In many problems of classical and celestial mechanics, robotics and mechatronics, there are processes which the time dependence is not periodic in [12-21]. In this connection, the interest in derived theory implementation to the study of periodic and almost periodic solutions of differential equations and differential equations with almost periodic coefficients has arisen [22-26].

*. Corresponding author

## 2. Basic definitions

Let $f$ be a function defined on an open interval $(a,b)$, taking values in the set of real numbers $R$, i.e. $f : (a,b) \to R$, $a,b \in R$, $a < b$. Consider an arbitrary point $x_0$ in $(a,b)$.

Let a number $\lambda$ be a derived number of function $f$ at $x_0$ if there exists a sequence $\{x_k\}$, such that $x_k \to x_0$ as $k \to \infty$ and

$$\lim_{k\to\infty} \frac{f(x_k) - f(x_0)}{x_k - x_0} = \lambda.$$

The fact that $\lambda$ is the derived number of function $f$ at $x_0$ is represented as $\lambda = \lambda[f](x_0)$.

The set of all derived numbers of function $f$ at $x_0$ is denoted by $\Lambda[f](x_0)$

If in the definition of a derived number it is required the sequence $\{x_k\}$ to satisfy one more additional condition, which means that for all $k$ the inequality $x_k - x_0 > 0$ is fulfilled, then such derived number is determined as the right derived number and denoted by $\lambda^+[f](x_0)$. If $x_k - x_0 < 0$ for all $k$, then such derived number is determined as the left derived number of function $f$ at $x_0$ and denoted by $\lambda^-[f](x_0)$

Let the set of right derived number of function $f$ at $x_0$ be denoted by $\Lambda^+[f](x_0)$, and the set of left derived number be denoted by $\Lambda^-[f](x_0)$.

It is clear that $\sup_{\lambda\in\Lambda^+[f](x_0)} \lambda$ determines $D^+f(x_0)$ that is the right upper derived number of a Dini function at a point $x_0$. Similarly, the remaining three derived number of Dini function at a point $x_0$ can be introduced.

Suppose

$$\lambda^\alpha = \lim_{k\to\infty} \frac{f(x_k) - f(x_0)}{(x_k - x_0)^\alpha}.$$

In this relation consider $\alpha$ such that for any $\varepsilon > 0$ the equalities $\lambda^{\alpha-\varepsilon} = 0$ and $\lambda^{\alpha+\varepsilon} = \infty$ are realised. If the function $f$ is defined in some neighborhood of the point $x_0$, then such $\alpha$ obviously exists. The magnitude can depend only on the choice of convergence to $x_0$ of the subsequence $\{x_k\}$.

Let the number $\lambda$ be called the derived number of a Hölder function at $x_0$ if there exist $\alpha \leq 0$ and a sequence $\{x_k\}$ converging to $x_0$, such that

$$\lambda = \lim_{k\to\infty} \frac{f(x_k) - f(x_0)}{(x_k - x_0)^\alpha},$$

and for any $\varepsilon > 0$

$$\lim_{k\to\infty} \frac{f(x_k) - f(x_0)}{(x_k - x_0)^{\alpha-\varepsilon}} = 0,$$

and

$$\lim_{k\to\infty} \frac{f(x_k) - f(x_0)}{(x_k - x_0)^{\alpha+\varepsilon}} = \infty.$$

Let the number $\alpha$ appearing in the definition of the Hölder derived number be called the exponent of this derived number.

The fact that $\lambda$ is a Hölder derived number of function $f$ at $x_0$ can be represented as following:

$$\lambda = \lambda_H[f](x_0).$$

The set of Hölder derived number of function $f$ at $x_0$ is denoted by $\Lambda_H[f](x_0)$.

If in the definition of the Hölder derived number it is required that $x_k - x_0 > 0$ for all $k$, then such a derived number is determined as the right Hölder derived number and denoted by $\lambda_H^+[f](x_0)$. If $x_k - x_0 < 0$ for all $k$, then such a derived number is determined as the left Hölder derived number and denoted by $\lambda_H^-[f](x_0)$.

Let the set of all right Hölder derived numbers of function $f$ at $x_0$ be denoted by $\Lambda_H^+[f](x_0)$, and the set of all left Hölder derived numbers at the same point be denoted by $\Lambda_H^-[f](x_0)$.

Let $\alpha^+$ denote the minimal of the exponents of the derived numbers being into $\Lambda_H^+[f](x_0)$, and $\Lambda_H^{\alpha^+}[f](x_0)$ denote a set of derived numbers belonging to the set $\Lambda_H^+[f](x_0)$ and having the exponent $\alpha^+$. Similarly, for a set $\Lambda_H^-[f](x_0)$, a number $\alpha^-$ and a set $\Lambda_H^{\alpha^-}[f](x_0)$ are introduced.

Let the number

$$\lambda = \sup_{\mu \in \Lambda_H^{\alpha^+}[f](x_0)} \mu$$

be called the right upper derivative of Dini-Hölder function $f$ at $x_0$ and denoted by $DH^+[f](x_0)$.

Let the number

$$\lambda = \inf_{\mu \in \Lambda_H^{\alpha^+}[f](x_0)} \mu$$

be called the right lower derivative of Dini-Hölder function $f$ at $x_0$.

Analogously, the notions of the left upper and left lower Dini-Hölder derivatives of function $f$ at $x_0$ are introduced. These derivatives are denoted by $DH^-[f](x_0)$ and $DH_-[f](x_0)$, respectively. Let $DH^*f$ denote any of the four Dini-Hölder derivatives of the function $f$.

**Theorem 1.** *For the function $f$ to be continuous from the right at $x_0$, it is necessary and sufficient that either the two right Dini-Hölder derivatives $DH^+[f](x_0)$ and $DH_+[f](x_0)$ to be equal to zero or the exponent $\alpha^+$ from the definition of Dini-Hölder derivative is greater than zero.*

**Proof.** *Necessity.* Let the function $f$ be continuous from the right at $x_0$. Consider the right Dini-Hölder derivatives at this point, and let at least one of them, for example $DH^+[f](x_0)$, be non-zero. This means that there exist a sequence $\{x_k\}$ converging to $x_0$ and a number $\alpha^+$, such that

$$DH^+[f](x_0) = \lim_{k\to\infty} \frac{f(x_k) - f(x_0)}{(x_k - x_0)^{\alpha^+}}.$$

If it turned out that in this expression $\alpha^+ = 0$, then the function $f$ would obviously discontinue at the point $x_0$ which the assertion of this part of the theorem follows from.

*Sufficiency.* Let the conditions of the theorem be satisfied, and let the function $f$ have discontinuity at a point $x_0$ contrary to our assertion. Then there exist $\varepsilon > 0$ and a sequence $\{x_k\}$ converging to $x_0$, such that inequality

$$\lim_{k\to\infty} |f(x_k) - f(x_0)| \geq \varepsilon.$$

It follows from this relation that the derived Hölder number realized on this sequence has an exponent equal to zero, and that at least one of the right Dini-Hölder derivatives at this point is nonzero. This contradiction proves the continuity on the right of the function $f$ at the point $x_0$, and also the validity of the theorem assertion.

Similarly, the continuity of the function $f$ at the point $x_0$ on the left is proved.

Obviously, in order for the function $f$ to be continuous at the point $x_0$, it is necessary and sufficient that it be continuous both on the right and on the left.

Comparing the definition of the Dini-Holder derivative with the definition of the derived number of the function $f$ or the derivative of a function $f$, it is obvious that Theorem 1 implies the validity of the following assertion.

The function $f$ is continuous at a point $x_0$ if one of the following conditions is satisfied:

1. The set $\Lambda[f](x_0)$ is bounded;
2. Each of the Dini derivatives $D^* f(x_0)$ is bounded;
3. The derivative $f^{'}(x_0)$ is bounded.

**Theorem 2.** *Let the function $f$ be defined in a neighborhood of the point $x_0$, and the function $g$ be defined in a neighborhood of a point $f(x_0)$. Then, if sets $\Lambda_H[g](f(x_0))$ and $\Lambda_H[f](x_0)$ are bounded, then every derived Hölder number of a function $h = g \circ f$ at the point $x_0$ can be represented in the form*

$$\lambda_H^\gamma[h](x_0) = \lambda_H^\alpha[g](f(x_0)) \cdot \lambda_H^{\beta\ \alpha}[f](x_0),$$

*where the exponent $\gamma$ is equal to the product of exponents $\alpha$ and $\beta$, that is $\gamma = \alpha\beta$, $\lambda_H^\alpha[g](f(x_0))$ and $\lambda_H^\beta[f](x_0)$ are some derived Hölder numbers from sets $\Lambda_H[g](f(x_0))$ and $\Lambda_H[f](x_0)$, respectively.*

**Proof.** Since the function $f$ is defined in a neighborhood of the point $x_0$ and $g$ in a neighborhood of the point $f(x_0)$, a function $h$ is also defined in some neighborhood of the point $x_0$. Let $\{x_k\}$ be a sequence from the range of definition of the function $h$, such that a certain Hölder number $\lambda_H^\gamma[h](x_0)$ with exponent $\gamma$ is realized on it. Without loss of generality, it can assumed that the derived Hölder number $\lambda_H^\beta[f](x_0)$ of the function $f$ with exponent $\beta$ is realized on the sequence $\{x_k\}$, and the derived Hölder number $\lambda_H^\alpha[g](f(x_0))$ with exponent $\alpha$ of the function $g$ is realized on the sequence $\{f(x_k)\}$. Since $h = g \circ f$, then

$$\lim_{k\to\infty} \frac{h(x_k) - h(x_0)}{(x_k - x_0)^{\alpha\beta}} = \lim_{k\to\infty} \Big( \frac{g(f(x_k)) - g(f(x_0))}{(f(x_k) - f(x_0))^\alpha} \cdot \frac{(f(x_k) - f(x_0))^\alpha}{(x_k - x_0)^{\alpha\beta}} \Big). \tag{1}$$

It can be shown that the derived Hölder number of the function $h$ with exponent $\gamma = \alpha\beta$ is realized on the sequence $\{x_k\}$.

Consider an arbitrary positive number $\delta < \gamma$ and positive numbers $\alpha_0 \le \alpha$ and $\beta_0 \le \beta$, such that $\alpha_0\beta_0 = \delta$. Substituting $\alpha_0$ and $\beta_0$ into (1) instead of $\alpha$ and $\beta$, respectively, and taking into account the definition of the derived Hölder number, it follows that for any

$$\lim_{k\to\infty} \frac{h(x_k) - h(x_0)}{(x_k - x_0)^\delta} = 0.$$

Consider now $\varepsilon > \gamma$, $\alpha_0 \ge \alpha$and $\beta_0 \ge \beta$, such that $\alpha_0\beta_0 = \varepsilon$ and repeating the arguments given above, it follows that for any $\varepsilon > \gamma$

$$\lim_{k\to\infty} \frac{h(x_k) - h(x_0)}{(x_k - x_0)^\varepsilon} = \infty.$$

The last two equalities imply that the exponent of the derived Hölder number of the function $h$ realizable on a sequence $\{x_k\}$ is equal to $\gamma$.

Now, let $\{x_k\}$ be a sequence such that $f(x_k) \ne f(x_0)$ for $x_k \ne x_0$. Without loss of generality, it can assumed that the derived Hölder numbers $\lambda_1 = \lambda_H^\alpha[g](f(x_0)) \in \Lambda_H[g](f(x_0))$ and $\lambda_2 = \lambda_H^\beta[f](x_0) \in \Lambda_H[f](x_0)$ are realized on the sequence $\{x_k\}$. It follows from the boundedness of the sets $\Lambda_H[g](f(x_0))$ and $\Lambda_H[f](x_0)$ that $\lambda_1$ and $\lambda_2$ are finite numbers. Then, considering [3] the following equality is realised:

$$\begin{aligned} \lambda_H^\gamma[h](x_0) = \lim_{k\to\infty} \frac{h(x_k) - h(x_0)}{(x_k - x_0)^\gamma} &= \lim_{k\to\infty} \frac{g(f(x_k)) - g(f(x_0))}{(f(x_k) - f(x_0))^\alpha} \\ &\cdot \left( \lim_{k\to\infty} \frac{f(x_k) - f(x_0)}{(x_k - x_0)^\beta} \right)^\alpha = \lambda_1 \cdot \lambda_2^\alpha, \end{aligned}$$

which is assirted by the theorem.

Finally, let $\{x_k\}$ be a sequence converging to $x_0$ such that $f(x_k) = f(x_0)$ for $x_k \ne x_0$. In this case for the function $f$ the derived number equal to zero is realised on the sequence $\{x_k\}$. Then, taking into account the boundedness of the set $\Lambda_H[g](f(x_0))$, it follows that for any $\lambda_1 \in \Lambda_H[g](f(x_0))$ an equality $\lambda_1 \cdot 0 = 0$ is true. But since for the function $h$ $\{x_k\}$ the derived number equal to zero is realized on a sequence, then in the considered case it can be assumed that the theorem remains valid.

**Theorem 3.** *Suppose that for some $\delta > 0$ function $f$ continuous at the point $x_0$ maps one-to-one interval $(x_0 - \delta, x_0 + \delta)$ into interval $(y_0 - \varepsilon, y_0 + \varepsilon)$, where $y_0 = f(x_0)$. Then*

$$\Lambda_H[f^{-1}](y_0) = (\Lambda_H[f](x_0))^{-1},$$

*where* $\{\Lambda_H[f](x_0)\}^{-1}$ *is obtained from the set* $\Lambda_H[f](x_0)$ *by substituting each element* $\lambda_H^\alpha \in \Lambda_H[f](x_0)$ *by an element* $\mu_H^\beta = \frac{1}{\lambda_H^\alpha[f](x_0)^\beta}$, *where* $\beta = 1/\alpha$.

**Proof.** Let $\{y_k\}$ be a sequence converging to $\{y_k\}$ on which a certain derived Hölder number $\mu_H^\beta[f^{-1}](y_0)$ of a function $f^{-1}$ at a point $y_0$ is realized, and let $\{x_k\}$ be a sequence corresponding to $\{y_k\}$ given by equalities $x_k = f^{-1}(y_k)$. Note that from the continuity at the point $x_0$ and the one-to-one mapping of $f$ it follows that the sequence $\{x_k\}$ converges to $x_0$ as $k \to \infty$ and that if $y_k \neq y_0$, then $x_k \neq x_0$. Then

$$\begin{aligned}\mu_H^\beta[f^{-1}](y_0) &= \lim_{k\to\infty} \frac{f^{-1}(y_k) - f^{-1}(y_0)}{(y_k - y_0)^\beta} = \lim_{k\to\infty} \frac{x_k - x_0}{(f(x_k) - f(x_0))^\beta} \\ &= \left(\lim_{k\to\infty} \frac{f(x_k) - f(x_0)}{(x_k - x_0)^{1/\beta}}\right)^{-\beta} = \frac{1}{\lambda_H^\alpha[f](x_0)^\beta} \in (\Lambda_H[f](x_0))^{-1},\end{aligned}$$

where $\alpha = 1/\beta$.

By virtue of the arbitrariness of the sequence $\{y_k\}$ choice, it follows from this relation that $\Lambda_H[f^{-1}](y_0) \subset (\Lambda_H[f](x_0))^{-1}$. It can be shown in a similar way that the reverse inclusion can also be realised. Combining these two results, the validity of the theorem is clear.

Theorems 2 and 3 are not only a generalization of the well-known analysis theorems on differentiation of a composite function and derivative of an inverse function, but also show how the degree of a composite function smoothness depends on the smoothness of the functions included in it.

## 3. Extremum of function

It is known that the derivative of a function allows to make a conclusion about the behavior of the function itself. Similar conclusions can be made based on the values of the derived numbers of function. In this section, considering the question of extremal values of a function, some generalizations of Fermat's theorem are presented.

**Theorem 4.** *Suppose for some* $\delta > 0$ *there is a function* $f$ *defined on* $(x_0-\delta, x_0+\delta)$ *taking an extremal value at* $x_0$. *Then at this point the following inequalities are true:*

$$DH^- f(x_0) \leq 0 \leq DH_+ f(x_0),$$

*if* $x_0$ *is a local minimum point of the function* $f$, *and*

$$DH_+ f(x_0) \leq 0 \leq DH^- f(x_0),$$

*if* $x_0$ *is a local maximum point of the function* $f$.

**Proof.** Let $x_0$ be a local minimum point. Then in some neighborhood of the point $x_0$ $f(x) \geq f(x_0)$ for all $x < x_0$. Let $\{x_k\}$ be a sequence on which the left upper Dini-Holder derivative with exponent $\alpha^-$ is realised:

$$DH^- f(x_0) = \lim_{k\to\infty} \frac{f(x_k) - f(x_0)}{-|x_k - x_0|^{\alpha^-}}.$$

Since for all $k$ $\frac{f(x_k)-f(x_0)}{-|x_k-x_0|^{\alpha^-}} \leq 0$, then $DH^- f(x_0) \leq 0$.

It can be shown in a similar way that if $x_0$ is a local minimum point of the function $f$, then at this point

$$0 \leq DH_+ f(x_0).$$

The case when $x_0$ is a local maximum point of the function $f$ can be shown in the same way.

It is clear that such a statement is also valid for the Dini derivatives.

Analysis of Theorem 4 proof allows to formulate a theorem giving sufficient conditions that $x_0$ is an extremum point for the function $f$, in other words a theorem being a kind of inverse to Theorem 4.

**Theorem 5.** *Suppose that for some $\delta > 0$ on $(x_0 - \delta, x_0 + \delta)$ function $f$ is defined. Then, if*

$$D^- f(x_0) < 0 < D_+ f(x_0),$$

*then $x_0$ is a local minimum point of the function $f$, and if*

$$D_- f(x_0) > 0 > D^+ f(x_0),$$

*then $x_0$ is a local maximum point of the function $f$.*

**Proof.** Suppose $x_0$ is not an extremum of the function $f$. Then it follows from the definition of an extremum that there are two sequences $\{x_k\}$ and $\{y_k\}$ convergent to $x_0$ such that $f(x_k) > f(x_0)$ for all $k$, and $f(y_k) < f(x_0)$. Besides, without loss of generality, it can be assumed that on each of these sequences some derived number of the function $f$ at the point $x_0$ is realised. Two cases are possible here: either these sequences are monotonically increasing or decreasing, or one of them is monotonically increasing and the other one is decreasing.

Consider the first case. Assume both sequences increase. Then immediately for all $k$

$$\frac{f(x_k) - f(x_0)}{x_k - x_0} < 0$$

and

$$\frac{f(y_k) - f(x_0)}{y_k - x_0} > 0.$$

From the estimates obtained and the assumption that the derived numbers of the function $f$ are realizable on $\{x_k\}$and $\{y_k\}$, it follows that

$$\lambda_1 = \lim_{k\to\infty} \frac{f(x_k) - f(x_0)}{x_k - x_0} \leq 0,$$
$$\lambda_2 = \lim_{k\to\infty} \frac{f(y_k) - f(x_0)}{y_k - x_0} \geq 0.$$

Since the sequences $\{x_k\}$ and $\{y_k\}$ are increasing by assumption, then $\lambda_1 \in \Lambda_-[f](x_0)$and $\lambda_2 \in \Lambda_-[f](x_0)$. Given that

$$D_- f(x_0) \leq \lambda_1 \leq 0 \leq \lambda_2 \leq D^- f(x_0),$$

it follows that in the considered case $0 \in [D_- f(x_0), D^- f(x_0)]$. Therefore none of the conditions of the theorem can be satisfied.

Consider now the second case Suppose the sequence $\{x_k\}$to increase and the sequence $\{y_k\}$ to decrease. Then for all $k$

$$\frac{f(x_k) - f(x_0)}{x_k - x_0} < 0$$

and

$$\frac{f(y_k) - f(x_0)}{y_k - x_0} < 0.$$

Repeating the arguments given in the first case analisis, it follows that $\lambda_1 \leq 0$ and $\lambda_2 \leq 0$. Since by assumption the sequence $\{x_k\}$ is increasing and the sequence $\{y_k\}$ is decreasing, then $D_- f(x_0) \leq \lambda_1 \leq 0$ and $D_+ f(x_0) \leq \lambda_2 \leq 0$. Thus, in the second case the point 0 does not divide the intervals $[D_- f(x_0), D^- f(x_0)]$ and $[D_+ f(x_0), D^+ f(x_0)]$, i.e. none of the conditions of the theorem is satisfied.

So, if $x_0$ is not an extremum point of the function $f$, then either point 0 is the derived number of the function $f$ at the point $x_0$, or both intervals $[D_- f(x_0), D^- f(x_0)]$ and $[D_+ f(x_0), D^+ f(x_0)]$ are on one side of the point 0. If the conditions of the theorem are satisfied, then neither of these two possibilities is realized at the point $x_0$, and, therefore, the point $x_0$ is the extremum point of the function $f$.

The points at which the function $f$ can take extreme values can be selected based on the behavior of any one-sided, for example, right-sided derivative. But in this case it is no longer sufficient to know the value of this derivative only at one point in order to relate this point to a set at which function $f$ can take extreme values or not. More precise representation on this phenomena is given in the following theorem.

**Theorem 6.** *Suppose for some $\delta > 0$ on $(x_0 - \delta, x_0 + \delta)$ there exists a continuous function $f$ that reaches its extremal value at the point $x_0$. If in some neighborhood of the point $x_0$ the function $f$ has a continuous right derivative $f'^{+}$, then it is necessary hat$f'^{+}(x_0) = 0$.*

**Proof.** Let the function $f$ take a maximum value at the point $x_0$ to be definite. Then for all $x > x_0$

$$\frac{f(x) - f(x_0)}{x - x_0} \leq 0.$$

Transferring to the limit for $x \to x_0 + 0$ in this inequality which exists by virtue of the assumption that there exists the right derivative at the point $x_0$, it follows that $f'^{+}(x_0) \leq 0$.

Suppose $f'^{+}(x_0) < 0$. By the continuity of the right derivative of the function $f$ in a neighborhood of the point $x_0$, there exists $\delta' > 0$, such that an inequality $f'^{+}(x) < 0$ holds for all $x \in [x_0 - \delta', x_0 + \delta']$.

Let us take an arbitrary point $x_1$ in $[x_0 - \delta', x_0)$. Since the function $f$ is continuous, it reaches its minimum at some point $x_2$ on $[x_1, x_0]$.

Let us show that $x_2 \neq x_0$. Indeed, otherwise for all $x \in [x_1, x_2]$ equality $f(x) = f(x_0)$ must be realised by the fact that at the point $x_0$ the function $f$ reaches its maximum. But such a conclusion is incompatible with the assumption that $f'^{+}(x) < 0$ for all $x \in [x_1, x_2]$.

So, now it is prooved that $x_2 \in (x_1, x_0)$. Then there exists a monotonically decreasing sequence $\{y_k\}$ converging to $x_2$, such that for all $k$

$$\frac{f(y_k) - f(x_2)}{y_k - x_2} \geq 0.$$

But by assumption that $f'^{+}(x_2) < 0$ and for all $x > x_2$ and sufficiently close to $x_2$ the following is true

$$\frac{f(x) - f(x_2)}{x - x_2} < 0.$$

Th arguments given above imply that a sequence $\{y_k\}$ with the properties listed above does not exist.

Thus, the assumption that the inequality $f'^{+}(x_0) < 0$ is satisfied leads to a contradiction and$f'^{+}(x_0) = 0$.

## 4. A theorem on a convex function

The function $f$ is called convex if from condition $x = \alpha x_1 + (1-\alpha)x_2$, $\alpha \in [0,1]$, the validity of inequality follows $f(x) \leq \alpha f(x_1) + (1-\alpha) f(x_2)$. If for all $\alpha \in (0,1)$ there is a strict inequality, i.e. $f(x) < \alpha f(x_1) + (1-\alpha) f(x_2)$, then the function $f$ is called strictly convex.

The main purpose of this section is to prove a theorem giving necessary and sufficient conditions for the function $f$ to be convex. But before proceeding with the proof of this theorem, let us first prove an auxiliary statement on the reconstruction of a function from the values of its right derivative.

**Theorem 7.** *Let a continuous function $f$ have a right derivative $f'^{+}(x)$ at each point $x \in [a,b]$. If $f'^{+}$ is bounded on $[a,b]$, then it is integrable on $[a,b]$ and for any $x \in [a,b]$*

$$f(x) = f(a) + \int_a^x f'^{+}(y)\,dy.$$

**Proof.** Let us construct a function $g$ by setting that

$$g(x) = f(x), x \in [a,b] f(b) + (x-b) f'^{+}(b), x > b.$$

It is obvious that the function $g$ is continuous and has a finite right derivative on $[a, b+1]$. Let us introduce the following function for $x \in [a, b]$ and $n = 1, 2$

$$\phi_n(x) = n[g(x + \frac{1}{n}) - g(x)].$$

At each point $x \in [a, b]$

$$\lim_{n\to\infty} \phi_n(x) = g^{'+}(x) = f^{'+}(x),$$

and since each of the continuous functions $\phi_n$ is measurable, then $f^{'+}$ is also measurable, which implies the integrability of this function due to the condition of boundedness. Further, by Theorem 6 [27]

$$g^{'+}(x + \frac{\theta'}{n}) \leq \phi_n(x) = [g(x + \frac{1}{n}) - g(x)] \leq g^{'+}(x + \frac{\theta^{''}}{n}), \quad \theta^{'}, \theta'' \in (0, 1),$$

so that all of the functions $\phi_n$ are bounded by one number and, by the Lebesgue theorem on the passage to the limit under the integral sign

$$\int_a^b f^{'+}(x)\,dx = \int_a^b g^{'+}(x)\,dx = \lim_{n\to\infty} \int_a^b \phi_n(x)\,dx.$$

But

$$\begin{aligned}\int_a^b \phi_n(x)\,dx &= n\int_a^b g(x + \frac{1}{n})\,dx - n\int_a^b g(x)\,dx \\ &= n\int_{a+\frac{1}{n}}^{b+\frac{1}{n}} g(x)\,dx - n\int_a^b g(x)\,dx \\ &= n\int_b^{b+\frac{1}{n}} g(x)\,dx - n\int_a^{a+\frac{1}{n}} g(x)\,dx.\end{aligned}$$

Applying the mean-value theorem to each of the last two integrals, the following is obtained:

$$\int_a^b \phi_n(x)dx = g(b + \frac{\theta_n^{'}}{n}) - g(a + \frac{\theta_n^{''}}{n}), \quad \theta_n^{'}, \theta_n^{''} \in (0, 1),$$

which on the basis of the continuity of the function $g$ implies the following

$$\lim_{n\to\infty} \int_a^b \phi_n(x)dx = \int_a^b g^{'+}(x)dx = \int_a^b f^{'+}(x)dx = g(b) - g(a) = f(b) - f(a).$$

Replacing $b$ by an arbitrary $x \in [a, b]$, the equality required in the condition of the theorem is obtained.

In conclusion, note that substitution of variable in the integral and the application of the mean-value theorem in the proof of the theorem are admissible, since in both cases the continuous function are under integral.

**Theorem 8.** *In order for a function $f$ bounded on $(a, b)$ to be strictly convex, it is necessary and sufficient that it be continuous and have a strictly increasing right derivative $f'^{+}$ bounded at each point.*

**Proof.** *Necessity.* If the function $f$ is convex and bounded on $(a, b)$, then it is continuous on any interval $[p, q] \subset (a, b)$, and hence it is continuous on $(a, b)$, too. Further, at each point of the open interval $(a, b)$ the continuous convex function has a bounded right derivative. Let us show that this derivative is strictly increasing function, if $f$ is strictly convex.

Consider two arbitrary points $x$ and $y > x$. Assume $\alpha = \frac{1}{2}(y - x)$ and let $z$ denote a point $x + \alpha = y - \alpha$. For the right derivative of a convex function at each point $x_0 \in (a, b)$ the following the estimates are realised:

$$\frac{f(x_0) - f(x_0 - \beta)}{\beta} \leq f'^{+}(x_0) \leq \frac{f(x_0 + \beta) - f(x_0)}{\beta},$$

where $\beta > 0$ so little that $[x_0 - \beta, x_0 + \beta] \subset (a, b)$.
Applying these estimates to the function $f$ at the points $x$ and $y$ for $\beta = \alpha$, the following two inequalities are obtained:

$$f'^{+}(x) \leq \alpha^{-1}(f(z) - f(x)) = u,$$
$$f'^{+}(y) \geq \alpha^{-1}(f(y) - f(z)) = v.$$

By assumption, $f$ it is strictly convex, and by virtue of this fact the folowing inequality is valid:

$$u - v = \alpha^{-1}(2f(z) - f(x) - f(y)) < 0,$$

i.e. this proves the inequality $u < v$. But, as noted above, $f'^{+}(x) \leq u$ and $v \leq f'^{+}(y)$, which implies that $f'^{+}(x) \leq u < v \leq f'^{+}(y)$, and therefore $f'^{+}(x) < f'^{+}(y)$, which proves a strict increase of the function $f$due to the arbitrariness of the points $x$ and $y$.

*Sufficiency.* Suppose $f$ is continuous on $(a, b)$ and at each of its points has a bounded right derivative, which is strictly a increasing function $(a, b)$. First of all, note that $f'^{+}$ is bounded on each interval $[p, q] \subset (a, b)$. Indeed, consider an arbitrary point $x_1$ on $(a, p)$, and an arbitrary point $x_2$ on $(q, b)$. Then, by the monotonicity of the function $f'^{+}$, for any $x \in [p, q]$ the following estimation is true:

$$f'^{+}(x_1) < f'^{+}(x) < f'^{+}(x_2).$$

The note that the function $f'^{+}$ takes finite values at the points $x_1$ and $x_2$ proves the validity of the assertion.

Thus, it is shown that all the conditions of Theorem 13 are satisfied on an arbitrary interval $[p, q] \subset (a, b)$, and therefore for any $x \in [p, q]$ the followng representation holds:

$$f(x) = f(p) + \int_p^x f'^{+}(y)\, dy.$$

By condition, the function $f'^{+}$ is strictly increasing, and hence the function $f$ is strictly convex on $[p, q]$. Since $[p, q]$ is an arbitrary interval belonging to $(a, b)$, it is strictly convex on $(a, b)$.

**Remark.** The boundedness of the function $f$ on $(a, b)$ is used only to prove its continuity. Thus, if it is known in advance that the function $f$ is continuous on $(a, b)$, then the requirement of its boundedness on this interval can be omitted.

**Conclusion.** The method of derived numbers to study periodic and almost periodic solutions of ordinary differential equations is developed. Necessary and sufficient conditions for the convexity of one variable functions are presented.

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