MULTI-FUZZY GROUP INDUCED BY MULTISETS

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Abstract. Multisets can be used to represent real life problems where repetition(s) of elements is necessary. Such cases occur in database query, chemical structures and computer programming but to mention a few. In this paper, some properties of algebraic sum of multisets $\biguplus$ and some previous results on selection are mentioned. This work also introduces a new way to construct fuzzy sets and fuzzy groups structure on multiset.

Keywords: multisets, functions on multiset, selection operation, multi-fuzzy set, submultiset.

1. Introduction

Many real life problems can be represented by multisets. Such cases occur in database query, chemical structures and computer programming but to mention a few.

The term multiset can be traced back to 1888, when Dedekind in [5] stated that the element of the set may belongs to more than one. Multisets are also considered in and replaced with various notions such as bag, fireset (finitely repeated element set), heap, bunches, etc.

In the recent time, Nazmul et al [6] has put algebraic structure on multisets of a set in order to be able to consider multigroup and other related algebraic properties as in the classical group. His work was extended by Shinoj et al [8].

In this paper, we present some results on the algebraic structure of multi-fuzzy sets.

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2. Preliminaries

In what follows, we shall use $X$ to denote a non-empty set.

**Definition 2.1** ([6]). A multiset $M$ drawn from a set $X$ is denoted by the count function $C_M : X \to N$ defined by $C_M(x) = n \in N$, the multiplicity or number of occurrence of $x$ in $M$, where $N$ is the set of non-negative integers.

**Definition 2.2** ([6]). Let multisets $A$ and $B$ be drawn from $X$. $A$ is said to be a submultiset of $B$ and is denoted $A \subseteq B$ if $C_A(x) \leq C_B(x) \forall x, y \in X$.

The root set or support of a multiset $M$, which is denoted by $M^*$, is the set which contains the distinct elements in the multiset. Hence, $M^*$ is the set of $x \in M$ such that $C_M(x) > 0$.

Let us recall some notions concerning multisets. See e.g. [2] for more details.

A multiset $M$ is called a regular multiset if $C_M(x) = C_M(y) \forall x, y \in M$. The count function of the intersection of two multisets $A$ and $B$ both drawn from $X$ is denoted by $C_A(x) \cap C_B(x) = \min\{C_A(x), C_B(x)\}$ and that of their union is denoted $C_A(x) \cup C_B(x) = \max\{C_A(x), C_B(x)\}$.

Multisets $A$ and $B$ are said to be equal if and only if $C_A(x) = C_B(x)$.

Denote by $[X]^{\alpha}$, all the multisets whose elements have the multiplicity not more than $\alpha$ and $MS(X)$ the set of all multisets over $X$. An empty multiset $\phi$ is such that $C_\phi(x) = 0, \forall x \in X$. **Cardinality** of a multiset $M$ is denoted by $|M| = \sum C_M(x), \forall x \in M$. The peak element $x \in M$ is such that $C_M(x) \geq C_M(y), \forall y \in M$.

**Definition 2.3** ([10]). Let us consider $A \in MS(X)$.

i. The insertion of $x$ into $A$ results into a multiset denoted by $C = x \cup A$ which has the count function

$$C_C(y) = \begin{cases} C_A(y), & y \neq x \\ C_A(x) + 1, & y = x. \end{cases}$$

ii. The removal of $x$ from $A$ results into a multiset denoted by $D = A \ominus x$ which has the count function

$$C_D(y) = \begin{cases} \max\{C_A(y) - 1, 0\}, & y = x \\ C_A(y), & y \neq x. \end{cases}$$

Let us consider $A, B \in MS(X)$.

iii. The insertion of $A$ into $B$ or of $B$ into $A$ results into a multiset which has the count function denoted by $C(x) = C_A(x) + C_B(x)$.

iv. The removal of $B$ from $A$ results into a multiset which has the count function denoted by $C_D(x) = \max\{C_A(x) - C_B(x), 0\}$.
It is clear that the set of all multisets over \( X \) that is \( MS(X) \) is commutative and associative with respect to the sum \( \uplus \).

The removal operation is neither commutative nor associative. Apart from, it is also possible to make some kind of selection in multisets using the following operations.

**Definition 2.4 ([10]).** Consider \( A \in MS(X) \) and \( B \subseteq X \).

i. The multiset \( E = A \odot B \) is such that \( E \) only contains elements of \( A \) which also occur in \( B \). The count function of \( E \) is denoted by

\[
C_E(x) = \begin{cases} 
C_A(x), & x \in B \\
0, & x \notin B.
\end{cases}
\]

ii. The multiset \( F = A \odot B \) is such that \( F \) only contains elements of \( A \) which do not occur in \( B \). The count function of \( F \) is denoted by

\[
C_F(x) = \begin{cases} 
C_A(x), & x \notin B \\
0, & x \in B.
\end{cases}
\]

Operations such as "\( \odot \)" or "\( \odot \)" are called selection operations.

**Definition 2.5 ([6]).** Let \( X \) be a group and \( e \in X \) its identity. Then, \( \forall x, y \in X \), a multiset \( M \) drawn from \( X \) is called a multigroup if

i. \( C_M(xy) \geq C_M(x) \land C_M(y) \),

ii. \( C_M(x^{-1}) \geq C_M(x) \).

The immediate consequence is that \( C_M(e) \geq C_M(x) \). The set \( MG(X) \) is called the set of all multigroups over \( X \). The next definition can be found e.g. in [2].

**Definition 2.6.** Let \( A \in MS(X) \), where \( X \) is a group.

i. \( A_n = \{ x : C_A(x) \geq n \} \);

ii. We denote a multiset containing only one element \( x \) with multiplicity. It is called \( n \) as \( [n]_x \) a simple multiset;

iii. The complement of the multiset \( M \in [X]^a \) denoted by \( M' \) is such that \( C_{M'}(x) = \alpha - C_M(x) \);

iv. \( nA = \{ x^n, \forall x \in A \} \) is the multiplicity of each element that appears in \( A \).

**Remark 2.1.** For a multigroup \( A \) over a group \( X \), \( A_n \) is a group, indeed the subgroup of \( X \) [6].
Proposition 2.1 ([6], p.645). Let $A, B \in MS(X)$ and $m, n \in \mathbb{N}$.

i. If $A \subseteq B$, then $A_n \subseteq B_n$;

ii. If $m \leq n$, then $A_m \supseteq A_n$;

iii. $(A \cap B)_n = A_n \cap B_n$;

iv. $(A \cup B)_n = A_n \cup B_n$;

v. $A = B$ if and only if $A_n = B_n$, $\forall n \in \mathbb{N}$.

Definition 2.7 ([6]). Let $X$ and $Y$ be two nonempty sets such that $f : X \rightarrow Y$ is a mapping. Consider the multisets $M \in [X]^\alpha$ and $N \in [Y]^\alpha$. Then,

i. the image of $M$ under $f$ denoted $f(M)$ has the count function

$$C_{f(M)}(y) = \begin{cases} \bigvee_{f(x) = y} C_M(x), & \text{if } f^{-1}(y) \neq \emptyset \\ 0, & \text{otherwise}; \end{cases}$$

ii. the inverse image of $N$ under $f$ denoted $f^{-1}(N)$ has the count function

$$C_{f^{-1}(N)}(x) = C_N[f(x)].$$

The following propositions were proved in [6]. But we shall later show that the items (iv), (v) and (vii) are not true and that the Proposition 2.2 needs to be restated.

Proposition 2.2 ([6]). Let $X$, $Y$ and $Z$ be three nonempty sets such that $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are mappings. If $M_i \in [X]^\alpha$, $N_i \in [Y]^\alpha$, $i \in I$ then

i. $M_1 \subseteq M_2 \Rightarrow f(M_1) \subseteq f(M_2)$;

ii. $f(\bigcup_{i \in I} M_i) = \bigcup_{i \in I} f(M_i)$;

iii. $N_1 \subseteq N_2 \Rightarrow f^{-1}(N_1) \subseteq f^{-1}(N_2)$;

iv. $f^{-1}(\bigcup_{i \in I} M_i) = \bigcup_{i \in I} f^{-1}(M_i)$;

v. $f^{-1}(\bigcap_{i \in I} M_i) = \bigcap_{i \in I} f^{-1}(M_i)$;

vi. $f(M_i) \subseteq N_j \Rightarrow M_j \subseteq f^{-1}(N_j)$;

vii. $g[f(M_i)] = [gf](M_i)$ and $f^{-1}[g^{-1}(N_j)] = [gf]^{-1}(N_j)$.

Definition 2.8 ([11]). A fuzzy set $A$ of a non-empty set $X$ is a class of objects in $X$ with the associated (or characteristic) membership function $\mu : X \rightarrow [0, 1]$ which assigns to every $x \in X$ a real value between 0 and 1.

The value of $\mu(x)$ restricted to $A$ is actually the degree of membership of $x$ in $A$. If $\mu(x) = 0$, it represents complete non-membership while $\mu(x) = 1$ represents complete membership. But since $\mu$ characterizes the fuzzy set $A$, we can simply refer to $\mu$ as fuzzy subset.
Definition 2.9 ([7]). Let $\mu$ be a fuzzy subset of $G$. Then, $\mu$ is called a fuzzy subgroup of $G$ if $\forall x, y \in G$

(i) $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$

(ii) $\mu(x) = \mu(x^{-1})$

Proposition 2.3 ([7]). A fuzzy subset $\mu$ of $G$ is a fuzzy subgroup of $G$ if and only if $\mu(xy^{-1}) \geq \min\{\mu(x), \mu(y)\}$ for every $x, y \in G$.

In paper [5] defined a kind of multiset (Dedekind’s multiset) denoted by $M_f$ as follows: if there are $n$ elements in a domain $X$ (of a function $f$ mapping $X$ to $Y$) which are mapped to an element $y \in Y$, then $y$ has frequency $n$ so that it is an $n$-fold element of $Y$.

But, on multisets (mentioned as Definition 2.7(i)) fails for Dedekind’s multisets. So let us redefine Nazmul et al’s definition of function as follows:

Definition 2.10 ([2]). Let $X$ and $Y$ be two non-empty sets and $f : X \rightarrow Y$ a mapping such that $M \in [X]^\alpha$. Then, $C_f(M)(y) = \sum_{f^{-1}(y) \neq \emptyset} f(x) = y C_M(x)$.

For more details concerning this problem see [2].

3. Induced fuzzy group

In the Introduction we have recalled that $[X]^\alpha$ is the collection of all multisets drawn from $X$ so that if $A \in [X]^\alpha$, then $C_A(x) \leq \alpha \forall x \in A$. Now, let $[[X]]^\alpha$ be a subset of $[X]^\alpha$ such that if $B \in [[X]]^\alpha$, $C_B(x) = \alpha \forall x \in B$, meaning that $B$ is a regular multiset in which each element has the multiplicity $\alpha$.

Proposition 3.1. Let $A$ be a regular multiset over a group $X$. Then $A$ is a multigroup if and only if $A^*$ is a group (i.e. a subgroup of $X$).

Proof. Assume that $A^*$ is a group. Then for any $x, y \in A^*, xy^{-1} \in A^*$. Then, $C_A(xy^{-1}) > 0$. Since $A$ is regular, $C_A(xy^{-1}) = C_A(x) = C_A(y^{-1}) = C_A(y)$. Without loss of generality, $C_A(xy^{-1}) \geq C_A(x) \land C_A(y)$.

Conversely, assume that $A$ is a multigroup. Since $A_1 = A^*$, by Remark 2.1, the proof is complete. $\square$

Proposition 3.2. Let $A$ be a multigroup over a group $X$ such that $A \in [X]^w$. Then $A'$ is a regular multigroup over $X$ if and only if $A$ is a regular.

Proof. Assume that $A$ is a multigroup over $X$ and regular. Then $\forall x, y \in A, C_A(x) = C_A(y)$. Since $A$ is a multigroup, $C_A(xy^{-1}) \geq C_A(x) \land C_A(y) = C_A(x) = C_A(y)$. Also, since $C_A(x) \neq 0 \neq C_A(y)$, then $C_A(xy^{-1}) \neq 0$. Hence, we have $xy^{-1} \in A$. But, $A$ is regular, $C_A(xy^{-1}) = C_A(x) = C_A(y)$. Without loss of generality, $C_A(xy^{-1}) \geq C_A(x) \land C_A(y)$ and $w - C_A(xy^{-1}) \geq w - C_A(x) \land w - C_A(y)$. Thus, $C_{A'}(xy^{-1}) \geq C_{A'}(x) \land C_{A'}(y)$.

Conversely, assume that $A'$ is a regular multigroup over $X$. Since $C_A'(xy^{-1}) = C_A'(x) = C_A'(y)$, then we can have both $C_A(xy^{-1}) \geq C_A(x) \land C_A(y)$ and $C_A(x) = C_A(y) \forall x, y \in A$. $\square$
There are some exceptions to what happens in classical algebra of set in multiset. If \( A \in [X]^w \) is regular, it is possible to have the following:

(i) \( A = A' \); 
(ii) \( A \cap A' \neq \emptyset \); 
(iii) \( A \subseteq A' \) or \( A' \subseteq A \); 
(iv) Also, \( A \cup A' = B \in [[X]]^w \).

**Example 3.1.** Let \( X = \mathbb{Z}_6 \), \( A = \{0, 0, 2, 2, 4, 4\} \) and \( B = \{0, 0, 0, 2, 2, 4, 4, 4\} \). Then the complement \( A' = \{0, 0, 0, 2, 2, 2, 4, 4, 4\} \) and \( B' = \{0, 0, 0, 2, 2, 4, 4, 4\} \) is \( B \).

**Definition 3.1.** Let \( \tilde{B} \in [X]^\alpha \) and \( B \in [[X]]^\alpha \) such that \( \tilde{B} \subseteq B \). Then the degree to which the multiplicity of \( b \in B \) is near to the multiplicity of \( b \in B \) or how near \( \tilde{B} \) is to \( B \) is defined by \( \gamma = \frac{C(B)}{\alpha} \).

**Remark 3.1.** Then, any such \( \tilde{B} \) and \( C_B \) induce a multi-fuzzy group structure with membership function \( \mu_B(x) = \gamma \).

For simplicity, we shall use \( \mu_B \) in place of \( \mu_B \) and \( \mu_B \) in place of \( \mu_B \). This multiset has a structure similar to that defined by Syropoulos [9] in that every \( x \in X \) which has a multiplicity in \( B \) is such that it has only one membership degree and one multiplicity. The support of \( \mu_B \) yields a fuzzy subset of \( X \).

It should be noted that if \( X \) is a group with identity \( e \) and \([X]^{\alpha}\) is the collection of all multigroups drawn from \( X \) such that for any \( \tilde{A} \in [X]^{\alpha} \) and \( \forall x \in A, C_{\tilde{A}}(x) \leq \alpha \). Let \( A \in [[X]]^{\alpha} \) be the collection of all regular multigroups drawn from \( X \). Then, any such multigroup \( \tilde{A} \) from \([X]^{\alpha}\) induces a multi-fuzzy group with membership degree \( \mu_A \). It can be said that \( \mu_A \) is fuzzy relative to \( A \). The support of \( \mu_A \) is a fuzzy subgroup of \( X \). All the operations on fuzzy set such as intersection, union, inclusion and complement can be seen to hold for \( \mu_A \).

**Example 3.2.** Let

\[ X = S_3 = \{e, (12) = a, (13) = b, (23) = c, (123) = d, (132) = f\} \]

be the group with identity. Let \( A \in [X]^d \) be \( A = \{e, e, e, d, d, d, f, f, f\} \), \( \tilde{A} = \{e, e, e, d, d, f, f\} \) and \( \tilde{A} = \{e, d, d, f\} \). Observe the following:

(i) \( \mu_A = \{(e, 0.75), (e, 0.75), (e, 0.75), (d, 0.5), (d, 0.5), (f, 0.5), (f, 0.5)\} \) is a multi-fuzzy group.

(ii) \( \mu_A = \{(e, 0.25), (d, 0.5), (d, 0.5), (f, 0.25)\} \) is a multi-fuzzy set.

(iii) \( \tilde{A} \) is a multigroup and \( \mu_A \) is a multi-fuzzy group. The support of \( \mu_A \) is the fuzzy subgroup \( \{(e, 0.75), (d, 0.5), (f, 0.5)\} \) of \( X \).
Considering \( \mu_A \) and \( \mu_{\bar{A}} \) as multi-fuzzy sets,

\[
\mu_A \cap \mu_{\bar{A}} = \{(e, 0.25), (d, 0.5), (d, 0.5), (f, 0.25)\}
\]

and

\[
\mu_A \cup \mu_{\bar{A}} = \{(e, 0.75), (e, 0.75), (e, 0.75), (d, 0.5), (d, 0.5), (f, 0.5), (f, 0.5)\}.
\]

**Proposition 3.3.** Let \( X \) be a group, \( \bar{A} \in [X]^n \) be a multigroup over \( X \) and \( \mu_A \) the multi-fuzzy group induced by \( \bar{A} \) and \( C_{\bar{A}} \). Then, \( \forall x \in X \) the following hold:

(i) \( \mu_A(e) \geq \mu_A(x) \),

(ii) \( \mu_A(x^{-1}) = \mu_A(x) \).

**Proof.** (i) Since \( \bar{A} \) is a multigroup, \( C_{\bar{A}}(e) = C_{\bar{A}}(x) \). This implies that \( C_{\bar{A}}(x) \leq C_{\bar{A}}(e) \). Thus, \( \mu_A(e) \geq \mu_A(x) \).

(ii) The proof is similar. \( \square \)

**Proposition 3.4.** Let \( X \) be a group, \( A \in [X]^n \) and \( \bar{A} \in [A]^n \). Then, \( \mu_A \) has a fuzzy group structure.

**Proof.** Let \( x, y^{-1} \in \bar{A} \). Since \( \bar{A} \) is a multigroup, \( C_A(x) = C_A(y^{-1}) = \alpha \). But, \( \mu_A(x) = C_A(x) \) and \( \mu_A(y) = C_A(y^{-1}) = \alpha \). Since \( \bar{A} \) is a multigroup, \( C_A(xy) = \min\{C_A(x), C_A(y^{-1})\} \). Since \( \alpha \) is a natural number, we can have that \( C_A(xy) \geq \min\{C_A(x), C_A(y^{-1})\} \). Hence,

\[
\mu_A(xy^{-1}) \geq \min\{\mu_A(x), \mu_A(y^{-1})\}. \quad \square
\]

The intersection of this multi-fuzzy group is also a multi-fuzzy group but the union is not necessarily a multi-fuzzy group. These properties are established by the following examples.

**Example 3.3.** Let \( X = \{e, a, b, ab\} \) the Klein’s 4-group. Consider the multigroups \( A \in [X]^3 \) such that \( A = \{e, e, e, a, a, a, b, b, ab, ab, ab, ab\} \). Let \( \bar{A} = \{e, e, e, a\} \) and \( \bar{A} = \{e, e, b, b\} \). \( \mu_A = \{(e, 1), (e, 1), (e, 1), (a, 2/3), (a, 2/3)\} \) and \( \mu_{\bar{A}} = \{(e, 2/3), (e, 2/3), (b, 2/3), (b, 2/3)\} \). The intersection of \( \mu_A \) and \( \mu_{\bar{A}} \) is \( \{(e, 2/3), (e, 2/3)\} \), which is a trivial multi-fuzzy group. Their union is

\[
\{(e, 1), (e, 1), (e, 1), (a, 2/3), (a, 2/3), (b, 2/3), (b, 2/3)\},
\]

which is obviously not a fuzzy group, since

\[
0 = \mu_{A \cup \bar{A}}(ab) \neq \min\{\mu_{A \cup \bar{A}}(a), \mu_{A \cup \bar{A}}(b)\} = \frac{2}{3}.
\]
Conclusion

It is known that the theory of multisets is an important generalization of classical set theory which has emerged by violating a basic property of classical sets that an element can belong to a set just once. Given a regular multiset $A \in [X]$, where $X$ is a group, we can fuzzify, relative to $A$, any multiset $B$ which is a submultiset of $A$. Another interesting approach to multisets can be found e.g. in [1, 3, 4].

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