P-EXPANDABLE SPACES

Heyam H. Al-Jarrah*
Khalid Y. Al Zoubi
Department of Mathematics
Faculty of science Yarmouk University
Irdid-Jordan
hiamaljarah@yahoo.com
khalidz@yu.edu.jo

Abstract. We introduce the concept of P-expandable spaces as a variation of expandable spaces. A space \((X, \tau)\) is said to be P-expandable if every locally finite collection \(F = \{F_\alpha : \alpha \in \Delta\}\) of subsets of \(X\) there exists a \(p\)-locally finite collection \(G = \{G_\alpha : \alpha \in \Delta\}\) of preopen subsets of \(X\) such that \(F \subseteq G\) for each \(\alpha \in \Delta\). We characterize P-expandable spaces and study their basic properties. We show that if a space \((X, \tau)\) is a quasi submaximal space, then \((X, \tau)\) is P-expandable if and only if it is expandable.

Keywords: preopen set, \(p\)-locally finite collection, expandable space, P-expandable space.

1. Introduction

By a space, we mean a topological space in which no separation axioms is assumed unless explicitly stated. Let \((X, \tau)\) be a space and \(A\) be a subset of \(X\). The closure of \(A\), the interior of \(A\) and the relative topology on \(A\) in \((X, \tau)\) will be denoted by \(\text{cl}(A)\), \(\text{int}(A)\) and \(\tau_A\), respectively. \(A\) is called a preopen subset of \((X, \tau)\) [3] if \(A \subseteq \text{int}(\text{cl}(A))\). The complement of a preopen set is called a preclosed set. \(A\) is called semi-open [12] (resp. \(\alpha\)-sets [13], regular closed) if \(A \subseteq \text{cl}(\text{int}(A))\) (resp. \(A \subseteq \text{int}(\text{cl}(\text{int}(A)))\), \(A = \text{cl}(\text{int}(A)))\). The family of all subsets of a space \((X, \tau)\) which are preopen (resp. preclosed, semi-open, regular closed) is denoted by \(\text{PO}(X, \tau)\) (resp. \(\text{PC}(X, \tau), \text{SO}(X, \tau), \text{RC}(X, \tau)\)). It is known that the collection of all \(\alpha\)-sets of \((X, \tau)\) forms a topology on \(X\), denoted by \(\tau^\alpha\), finer than \(\tau\) and \(\text{PO}(X, \tau) = \text{PO}(X, \tau^\alpha)\).

A space \((X, \tau)\) is called submaximal [11] if every dense subset of \((X, \tau)\) is open. It is known that \((X, \tau)\) is submaximal if and only if \(\tau = \text{PO}(X, \tau)\). In [4], Al-Nashef introduced the notion of quasi-submaximal spaces where a space \((X, \tau)\) is quasi-submaximal if \(\text{cl}(D) - D\) is nowhere dense subset for each dense subset \(D\) of \((X, \tau)\). This is equivalent to saying that \(\text{int}(D)\) is dense for each dense subset \(D\) of \((X, \tau)\) [4].

* Corresponding author
Mashhour et al. [2] used preopen sets to define $P_1$-paracompact and $P_2$-paracompact spaces. In 2007, Al-Zoubi and Al-Ghour [8] define $P_3$-paracompact space and the notion $P$-locally finite collections and study their properties. In this paper we introduce $P$-expandable spaces by using preopen sets and $p$-locally finiteness and study their topological properties. We deal with subspaces, sum, image and the inverse images of $P$-expandable.

**Lemma 1.1.** Let $A$ and $B$ be subsets of a space $(X, \tau)$.

i. If $A \in PO(X, \tau)$ and $B \in SO(X, \tau)$, then $A \cap B \in PO(B, \tau_B)$ ([6]).

ii. If $A \in PO(B, \tau_B)$ and $B \in PO(X, \tau)$, then $A \in PO(X, \tau)$ ([6]).

iii. If $A \in PO(X, \tau)$ and $B \in \tau$, then $A \cap B \in PO(X, \tau)$ ([7]).

**Definition 1.2.** A collection $\mathcal{F} = \{\alpha : \alpha \in \Delta\}$ of subsets of a space $(X, \tau)$ is called locally finite (resp. $p$-locally finite [8]) if for each $x \in X$, there exists $W_x \in \tau$ (resp. $W_x \in PO(X, \tau)$) containing $x$ and $W_x$ intersects at most finitely many members of $\mathcal{F}$.

**Corollary 1.3** ([8]). Let $(X, \tau)$ be any space:

i. Every locally finite collection subset of $X$ is $p$-locally finite collection subset of $X$.

ii. Every $p$-locally finite collection $\mathcal{F} = \{\alpha : \alpha \in \Delta\}$ of preopen subsets of a quasi- submaximal space $X$ is locally finite.

iii. Every $p$-locally finite collection of open sets ($\alpha$-sets, regular closed sets) is locally finite.

**Definition 1.4.** A function $f : (X, \tau) \to (Y, \sigma)$ is called

i. Preirresoulte [5] if and only if $f^{-1}(A) \in PO(X, \tau)$ for each $A \in PO(Y, \sigma)$.

ii. Strongly preclosed [16] if $f(A) \in PC(Y, \sigma)$ for each $A \in PC(X, \tau)$.

iii. M-preopen [2] if $f(A) \in PO(Y, \sigma)$ for each $A \in PO(X, \tau)$.

iv. Countable perfect [10] if $f$ is continuous, closed, surjective function such that $f^{-1}(y)$ is countable compact for each $y$ in $Y$.

**Lemma 1.5.** Let $f : (X, \tau) \to (Y, \sigma)$ be a continuous function.

i. If $\mathcal{F} = \{\alpha : \alpha \in \Delta\}$ is a locally finite collection of subsets of $(Y, \sigma)$, then $f^{-1}(\mathcal{F}) = \{f^{-1}(\alpha) : \alpha \in \Delta\}$ is a locally finite collection in $(X, \tau)$ [10].

ii. Let $f$ be a countable perfect function. If $\mathcal{F} = \{\alpha : \alpha \in \Delta\}$ is a locally finite collection of subsets of $(X, \tau)$, then $f(\mathcal{F}) = \{f(\alpha) : \alpha \in \Delta\}$ is a locally finite collection in $(Y, \sigma)$ ([11]).
Recall that a space \((X, \tau)\) is called strongly compact relative to \(X\) \([2]\) if every cover of \(A\) by preopen sets of \(X\) has a finite subcover.

**Theorem 1.6** \([8]\). Let \(f : (X, \tau) \to (Y, \sigma)\) be a function:

i. If \(f\) is a preirresolute function and \(\mathcal{F} = \{F_\alpha : \alpha \in \Delta\}\) is a \(p\)-locally finite collection in \((Y, \sigma)\), then \(f^{-1}(\mathcal{F}) = \{f^{-1}(F_\alpha) : \alpha \in \Delta\}\) is a \(p\)-locally finite collection in \((X, \tau)\).

ii. If \(f\) is a strongly preclosed function such that \(f^{-1}(y)\) is strongly compact relative to \((X, \tau)\) for every \(y \in Y\) and \(\mathcal{F} = \{F_\alpha : \alpha \in \Delta\}\) is a \(p\)-locally finite collection of subsets of \((X, \tau)\), then \(f(\mathcal{F}) = \{f(F_\alpha) : \alpha \in \Delta\}\) is a \(p\)-locally finite collection in \((Y, \sigma)\).

**Corollary 1.7.** Let \((X, \tau)\) be a space, then the following are equivalent:

i. \((X, \tau)\) is expandable.

ii. For every locally finite collection \(\mathcal{F} = \{F_\alpha : \alpha \in \Delta\}\) of subsets of \(X\) there exists a \(p\)-locally finite collection \(\mathcal{G} = \{G_\alpha : \alpha \in \Delta\}\) of open subsets of \(X\) such that \(F_\alpha \subseteq G_\alpha\) for each \(\alpha \in \Delta\) \([8]\).

iii. For every locally finite collection \(\mathcal{F} = \{F_\alpha : \alpha \in \Delta\}\) of subsets of \(X\) there exists a locally finite collection \(\mathcal{G} = \{G_\alpha : \alpha \in \Delta\}\) of preopen subsets of \(X\) such that \(F_\alpha \subseteq G_\alpha\) for each \(\alpha \in \Delta\).

iv. For every locally finite collection \(\mathcal{F} = \{F_\alpha : \alpha \in \Delta\}\) of subsets of \(X\) there exists a \(p\)-locally finite \(\mathcal{G} = \{G_\alpha : \alpha \in \Delta\}\) collection of \(\alpha\)-open subsets of \(X\) such that \(F_\alpha \subseteq G_\alpha\) for each \(\alpha \in \Delta\).

**Proof.** (i → ii → iii → iv) These implication follow from definitions, Corollary 1.3 and the fact that \(\tau \subseteq \tau_\alpha\).

(iv → i) Let \(\mathcal{F} = \{F_\alpha : \alpha \in \Delta\}\) be a locally finite collection of subsets of a space \((X, \tau)\). Then, by (iv), there exists a \(p\)-locally finite collection \(\mathcal{G} = \{G_\alpha : \alpha \in \Delta\}\) of \(\alpha\)-open sets subsets of \(X\) such that \(F_\alpha \subseteq G_\alpha\) for each \(\alpha \in \Delta\). Then, by Corollary 1.3, \(\{\text{int}(\text{cl}(\text{int}(G_\alpha))) : \alpha \in \Delta\}\) is a locally finite collection of open subset of \(X\) such that \(F_\alpha \subseteq \text{int}(\text{cl}(\text{int}(G_\alpha)))\) for all \(\alpha \in \Delta\). Hence \((X, \tau)\) is expandable.

2. **P-expandable spaces**

**Definition 2.1.** A space \((X, \tau)\) is said to be P-expandable (resp. \(P_1\)-expandable, pre-expandable) if every locally finite (resp. \(p\)-locally finite, \(p\)-locally finite) collection \(\mathcal{F} = \{F_\alpha : \alpha \in \Delta\}\) of subsets of \(X\) there exists a \(p\)-locally finite (resp. \(p\)-locally finite, \(p\)-locally finite) collection \(\mathcal{G} = \{G_\alpha : \alpha \in \Delta\}\) of preopen (resp. open, preopen) subsets of \(X\) such that \(F_\alpha \subseteq G_\alpha\) for each \(\alpha \in \Delta\).
It is clear (from the fact that the closure of any locally finite collection is locally finite) that a space \((X, \tau)\) is P-expandable iff every locally finite collection \(\mathcal{F} = \{F_\alpha : \alpha \in \Delta\}\) of closed subsets of \(X\), there exists a \(p\)-locally finite collection \(\mathcal{G} = \{G_\alpha : \alpha \in \Delta\}\) of preopen subsets of \(X\) such that \(F_\alpha \subseteq G_\alpha\) for each \(\alpha \in \Delta\).

The following diagram follows immediately from the definitions in which none of these implications is reversible.

\[
\begin{array}{c}
\text{Expandable} \rightarrow \text{P-expandable} \\
\uparrow \quad \quad \uparrow \\
\text{P}_1\text{-expandable} \rightarrow \text{pre-expandable}
\end{array}
\]

To show that none of these implications is reversible, In the above diagram, we consider the following examples.

**Example 2.2.** Let \(X = N \cup N^{-}\) with the topology \(\tau = \{U \subseteq X : N \subseteq U\} \cup \{\emptyset\}\) such that \(N\) is the set of all positive integers and \(N^{-}\) is the set of all negative integers. Then \(PO(X, \tau) = \{A \subseteq X : A \cap N \neq \emptyset\}\).

(i) Note that \((X, \tau)\) is not expandable since the collection \(\{\{x\} : x \in N^{-}\}\) is locally finite in \((X, \tau)\) and there exists no locally finite collection \(\{U_x : x \in N^{-}\}\) of open sets in \((X, \tau)\) such that \(x \in U_x\) for \(x \in N^{-}\).

(ii) To see that \((X, \tau)\) is pre- expandable (hence P-expandable). Let \(U = \{U_\alpha : \alpha \in \Delta\}\) be a \(p\)-locally finite collection in \((X, \tau)\). Put \(\Delta_1 = \{\alpha \in \Delta : U_\alpha \cap N \neq \emptyset\}\) and \(\Delta_2 = \{\alpha \in \Delta : U_\alpha \cap N = \emptyset\}\). Now, for \(\alpha \in \Delta_2\), choose \(x_\alpha \in U_\alpha\) and put \(U'_\alpha = U_\alpha \cup \{-x_\alpha\}\). Put \(U' = \{U_\alpha : \alpha \in \Delta_1\} \cup \{U'_\alpha : \alpha \in \Delta_2\}\).

Then, it is clear \(U'\) is a collection of preopen sets in \((X, \tau)\) such that for all \(\alpha \in \Delta\), there exists \(H_\alpha \subseteq U'\) such that \(U_\alpha \subseteq H_\alpha\) Finally, we show that \(U'\) is \(p\)-locally finite in \((X, \tau)\). Let \(x \in X\). Then, there exists a preopen set \(P_x\) in \((X, \tau)\) such that \(x \in P_x\) and a finite subset \(\Delta'_1\) of \(\Delta_1\) and a finite subset \(\Delta'_2\) of \(\Delta_2\) such that \(P_x \cap U_\alpha = \emptyset\) for all \(\alpha \in \Delta - (\Delta'_1 \cup \Delta'_2)\). Now, if \(x \in N^{-}\), put \(P_x^* = (P_x - \{x_\alpha : \alpha \in \Delta_2 - \Delta'_2\}) \cup \{-x\}\). Then \(P_x^*\) is a preopen set in \((X, \tau)\) such that \(x \in P_x^*\) and \(P_x^*\) intersect at most finitely many members of \(U'\). If \(x \in \mathbb{N}\), put \(P_x^* = (P_x - \{-x_\alpha : \alpha \in \Delta_2 - \Delta'_2\}) \cup \{x\}\). Then \(P_x^*\) is a preopen set in \((X, \tau)\) such that \(x \in P_x^*\) and \(P_x^*\) intersect at most finitely many members of \(U'\). Thus \(U'\) is a \(p\)-locally finite collection of preopen sets in \((X, \tau)\) and so \((X, \tau)\) is pre-expandable.

**Example 2.3.** Let \(X = \mathbb{R}\) with the topology \(\tau = \{U : U \subseteq \mathbb{Q}\} \cup \{\mathbb{R}\}\). Note that \(PO(X, \tau) = \{U : U \subseteq \mathbb{Q}\} \cup \{U : \mathbb{Q} \subseteq U\}\) and every locally finite collection is finite. Hence \((X, \tau)\) is expandable (and so P-expandable). On the other hand, \((X, \tau)\) is not pre-expandable since the collection \(\{\{x\} : x \in \mathbb{R} - \mathbb{Q}\}\) is \(p\)-locally finite in \(X\) but there does not exists a \(p\)-locally finite collection of preopen set \(\{G_x : x \in \mathbb{R} - \mathbb{Q}\}\) in \((X, \tau)\) such that \(x \in G_x\) for each \(x \in \mathbb{R} - \mathbb{Q}\).

If \(G = \{G_x : x \in \mathbb{R} - \mathbb{Q}\}\) is \(p\)-locally finite collection of preopen sets, then \(\{x\} \cup \mathbb{Q} \subseteq G_x\) for all \(x \in \mathbb{R} - \mathbb{Q}\). Choose \(x_0 \in \mathbb{Q}\) and \(p_0 \in PO(X, \tau)\) such that \(x_0 \in p_0\). Then \(p_0 \cap G_x \neq \phi\) for all \(x\).
Example 2.4. Let $X = \mathbb{R}$ with the topology $\tau = \{\emptyset, \mathbb{R}, \mathbb{Q}, \mathbb{R} - \mathbb{Q}\}$. Note that $PO(X, \tau) = \tau_{dis}$ and so $(X, \tau)$ is pre-expandable. On the other hand, every locally finite is finite, therefore $(X, \tau)$ is expandable. To show that $(X, \tau)$ is not $P_1$-expandable, we consider the collection $U = \{\{x\} : x \in \mathbb{Q}\}$. $U$ is $p$-locally finite in $X$ but there does not exists a locally finite collection of open set $\{G_x : x \in \mathbb{Q}\}$ in $(X, \tau)$ such that $x \in G_x$ for each $x \in \mathbb{Q}$. Note that, if $G_x \in \tau$ such that $\{x\} \subseteq G_x$ then either $G_x = \mathbb{Q}$ or $G_x = \mathbb{R}$ and so $\{G_x\}$ is not locally finite.

Note that Example 2.2 and Example 2.3 shows that expandable and pre-expandable spaces are independent notions.

Proposition 2.5. Let $(X, \tau)$ be a space, then the following are equivalent:

i. $(X, \tau)$ is $P_1$-expandable.

ii. For every $p$-locally finite collection $\mathcal{F} = \{F_\alpha : \alpha \in \Delta\}$ of subsets of $X$ there exists a $p$-locally finite collection $\mathcal{G} = \{G_\alpha : \alpha \in \Delta\}$ of open subset of $X$ such that $F_\alpha \subseteq G_\alpha$ for all $\alpha \in \Delta$.

iii. For every $p$-locally finite collection $\mathcal{F} = \{F_\alpha : \alpha \in \Delta\}$ of subsets of $X$ there exists locally finite collection $\mathcal{G} = \{G_\alpha : \alpha \in \Delta\}$ of preopen subset of $X$ such that $F_\alpha \subseteq G_\alpha$ for all $\alpha \in \Delta$.

Proof. (i$\Rightarrow$ii$\Rightarrow$iii) These implication follow from the definition and Corollary 1.3.

(iii$\Rightarrow$i) Let $\mathcal{U} = \{U_\alpha : \alpha \in \Delta\}$ be a $p$-locally finite collection of subsets of $X$. Then, by (iii) there exists a locally finite collection $\mathcal{G} = \{G_\alpha : \alpha \in \Delta\}$ of preopen subset of $X$ such that $F_\alpha \subseteq G_\alpha$ for all $\alpha \in \Delta$. Then, $\{\text{int}(\text{cl}(G_\alpha)) : \alpha \in \Delta\}$ is a locally finite collection of open subset of $X$ such that $U_\alpha \subseteq \text{int}(\text{cl}(G_\alpha))$ for all $\alpha \in \Delta$. Hence $(X, \tau)$ is $P_1$-expandable.

Proposition 2.6. Let $(X, \tau)$ be any space:

i. If $(X, \tau)$ is a quasi-submaximal, then $(X, \tau)$ is expandable iff it is $P$-expandable.

ii. If $(X, \tau)$ is a submaximal, then $(X, \tau)$ is expandable iff it is pre-expandable.

Proof. The easy proof is left to the reader.

Proposition 2.7. Let $(X, \tau)$ be a countably $P$-compact space. Then $(X, \tau)$ is expandable if and only if it is $P$-expandable.
Proof. The necessity is clear and we need only prove the sufficiency. Let \( \mathcal{U} = \{ U_\alpha : \alpha \in \Delta \} \) be a locally finite collection of \( X \). Then there exists a P-locally
finite collection \( \mathcal{G} = \{ G_\alpha : \alpha \in \Delta \} \) of preopen subset of \( X \) such that \( U_\alpha \subseteq G_\alpha \) for each \( \alpha \in \Delta \). Since \((X, \tau)\) is countably P-compact then \( \{ \text{int}(\text{cl}(G_\alpha)) : \alpha \in \Delta \} \)
is a locally finite collection of open subset of \( X \) such that \( U_\alpha \subseteq \text{int}(\text{cl}(G_\alpha)) \) for all \( \alpha \in \Delta \). Hence \((X, \tau)\) is expandable. \( \square \)

Recall that a space \((X, \tau)\) is called P\(_1\)-paracompact [2], (resp. P\(_2\)-paracompact [2], P\(_3\)-paracompact [8]) if every preopen (resp. preopen, open) cover of \( X \) has a locally finite open (resp. locally finite preopen, \( p \)-locally finite preopen) refinement.

Theorem 2.8. Every P\(_3\)-paracompact space is P-expandable.

Proof. Let \( \mathcal{F} = \{ F_\alpha : \alpha \in \Delta \} \) be a locally finite collection of closed subsets of \( X \). Let \( \Delta' \) be the collection of all finite subsets of \( \Delta \). For \( \beta \in \Delta' \), let \( V_\beta = X - \cup \{ F_\alpha : \alpha \notin \beta \} \). Because \( \mathcal{F} \) is the locally finite collection, \( V_\beta \) is open. Also, \( V_\beta \) meets only finitely many elements of \( \mathcal{F} \). Let \( \mathcal{V} = \{ V_\beta : \beta \in \Delta' \} \). Then \( \mathcal{V} \) is an open cover of \( X \). Since \( X \) is P\(_3\)-paracompact, \( \mathcal{V} \) has a \( p \)-locally finite preopen refinements, say \( \mathcal{W} = \{ W_\gamma : \gamma \in \Delta \} \). Set \( U_\alpha = \cup \{ W_\gamma \in \mathcal{W} : W_\gamma \cap F_\alpha \neq \emptyset \} \) for each \( \alpha \in \Delta \). Because arbitrary unions of preopen sets are preopen set, \( U_\alpha \) is preopen and \( F_\alpha \subseteq U_\alpha \) for each \( \alpha \in \Delta \). Now, we shall try to show that \( \{ U_\alpha : \alpha \in \Delta \} \) is \( p \)-locally finite. Since \( \mathcal{W} \) is \( p \)-locally finite, for each \( x \in X \), there exists a preopen set \( U_x \) in \((X, \tau)\) containing \( x \) and \( U_x \) intersects at most finitely many members of \( \mathcal{W} \). Also, by the definition of \( U_\alpha \), we say that \( U_x \cap U_\alpha \neq \emptyset \) if and only if \( U_x \cap W_\gamma \neq \emptyset \) and \( W_\gamma \cap F_\alpha \neq \emptyset \) for some \( \gamma \in \Delta \). Since \( \mathcal{W} \) is refinement of \( \mathcal{V} \), there is number \( V_\beta \) of \( \mathcal{V} \) containing \( W_\gamma \) for each number \( W_\gamma \) of \( \mathcal{W} \). Then \( W_\gamma \) meets only finitely many \( F_\alpha \) for each \( \gamma \in \Delta \). Thus, \( \{ U_\alpha : \alpha \in \Delta \} \) is \( p \)-locally finite. \( \square \)

Corollary 2.9. Every P\(_1\)-paracompact (reps. P\(_2\)-paracompact) space is P-expandable.

The following example shows that the converse of the above corollary need not be true.

Example 2.10. Let \( \omega_1 \) denote the first uncountable ordinal and let \( X = [0, \omega_1) \) with the usual order topology. Then, from [9], \( X \) is countable compact but not paracompact since the collection \( \{ [0, \alpha) : \alpha < \omega_1 \} \) is an open cover of \( X \) which has no open locally finite refinement. Hence \( X \) is P-expandable but neither P\(_1\)-paracompact nor P\(_2\)-paracompact.

Theorem 2.11. Let \((X, \tau)\) be a space:

i. If \((X, \tau^\alpha)\) is P-expandable, then \((X, \tau)\) is P-expandable.

ii. If \((X, \tau)\) is P-expandable submaximal space, then \((X, \tau^\alpha)\) is P-expandable.
This follows immediately from the definitions and the facts that $\tau \subseteq \tau^\alpha$ and $PO(X, \tau^\alpha) = PO(X, \tau)$.

The converse of part (i) of Theorem 2.11 is not true in general as the following example shows.

**Example 2.12.** Let $X$ be an infinite set and $q \in X$. Let $\tau = \{\phi, X, \{q\}\}$. Then $(X, \tau)$ is P-expandable. But $(X, \tau^\alpha)$ is not P-expandable and not submaximal. Since $\tau^\alpha = PO(X, \tau) = PO(X, \tau^\alpha) = \{\phi\} \cup \{U \subseteq X : q \in U\}$. Now, the collection $\{|\{x\} : x \in X - \{q\}\}$ is locally finite in $(X, \tau^\alpha)$ and there is no $p$-locally finite collection $\{G_\alpha : x \in X - \{q\}\}$ of preopen subset in $(X, \tau^\alpha)$ such that $x \in G_\alpha$ and $x \in X - \{q\}$.

**Definition 2.13.** A space $(X, \tau)$ is said to be $\omega$-P-expandable if every locally finite collection $\mathcal{F} = \{F_\alpha : \alpha \in \Delta, |\Delta| \leq \omega\}$ of subsets of $X$ there exists a $p$-locally finite collection $\mathcal{G} = \{G_\alpha : \alpha \in \Delta\}$ of preopen subsets of $X$ such that $F_\alpha \subseteq G_\alpha$ for each $\alpha \in \Delta$.

**Theorem 2.14.** Let $(X, \tau)$ be a space. Then $(X, \tau)$ is $\omega$-P-expandable if and only if every countable open cover of $X$ has a p-locally finite preopen refinement.

**Proof.** Sufficiency is similar to the proof of Theorem 2.8.

To prove necessity, let $\mathcal{U} = \{U_i : i \in \mathbb{N}\}$ be a countable open cover of $X$. Put $A_i = \bigcup\{U_j : j \leq i\}$ for each $i \in \mathbb{N}$. Let $B_1 = A_1$ and $B_i = A_i - A_{i-1}$ such that $i = 2, 3, 4, \ldots$. Therefore $B_i \subseteq U_i$ for each $i \in \mathbb{N}$. For $x \in X$, let $i(x) = \min\{i \in \mathbb{N} : x \in U_i\}$. Then $x \in B_{i(x)}$. Put $\mathcal{A} = \{B_i : i \in \mathbb{N}\}$. Then, $\mathcal{A}$ is a refinement of $\mathcal{U}$ and $\mathcal{A}$ is locally finite since $U_i \cap B_i = \phi$ for $j > i$. Because $X$ is $\omega$-P-expandable, there exists a $p$-locally finite collection $\{G_i : i \in \mathbb{N}\}$ of preopen subsets of $X$ such that $B_i \subseteq G_i$ for each $i \in \mathbb{N}$. Let $V_i = U_i \cap G_i$ for each $i \in \mathbb{N}$. By Lemma 1.1, $V_i$ is preopen set in $(X, \tau)$ for each $i \in \mathbb{N}$. Let $\mathcal{V} = \{V_i : i \in \mathbb{N}\}$. Since $\{G_i : i \in \mathbb{N}\}$ is $p$-locally finite, $\mathcal{V}$ is $p$-locally finite. Because $\mathcal{A}$ is a cover of $X$, there exists some $i \in \mathbb{N}$ such that $x \in B_i$ for each $x \in X$. Since $B_i \subseteq V_i$, $x \in V_i$. Thus, $\mathcal{V}$ is a $p$-locally finite preopen refinement of $\mathcal{U}$.

**3. Operations**

In this section we study some basic operation on P-expandable spaces.

**Definition 3.1.** A subset $A$ of a space $(X, \tau)$ is called an:

i. $\alpha$ P-expandable set in $(X, \tau)$ if every locally finite (in $X$) collection $\mathcal{F} = \{F_\alpha : \alpha \in \Delta\}$ of subsets of $A$ there exists a $p$-locally finite collection $\mathcal{G} = \{G_\alpha : \alpha \in \Delta\}$ of preopen subset of $(X, \tau)$ such that $F_\alpha \subseteq G_\alpha$ for all $\alpha \in \Delta$.

ii. $\beta$ P-expandable set in $(X, \tau)$ if and only if $(A, \tau_A)$ is P-expandable.
Note that $\alpha P$-expandable and $\beta P$-expandable sets are linearly independent. To see that we give the following examples.

**Example 3.2.** Let $X = \mathbb{R}$ with the topology $\tau = \{U : 1 \in U\} \cup \{\phi\}$. Note that the set of all $PO(X, \tau) = \tau$. Put $A = \mathbb{R} - \{1\}$. Then $A \notin PO(X, \tau)$ and $\tau_A = \tau_{\text{dis}}$. Therefore, $A$ is $\beta P$-expandable but not $\alpha P$-expandable.

**Example 3.3.** Let $(X, \tau)$ be as in Example 2.2 and let $A = \mathbb{N}^- \cup \{1\}$. Then $\tau_A = \{\{1\} \cup H : H \subseteq \mathbb{N}^-\} \cup \{\phi\}$ and $PO(A, \tau_A) = \tau_A$. To show that $(A, \tau_A)$ is not $\beta P$-expandable we consider the collection $U = \{\{x\} : x \in \mathbb{N}^-\}$. Then $U$ is a locally finite collection of subsets of $A$ and note that if $U$ is a locally finite (this equivalent $p$-locally finite) collection of open (preopen) subsets of $(A, \tau_A)$, then $U$ is finite. Therefore, $(A, \tau_A)$ is not $\beta P$-expandable. On the other hand, $A$ is $\alpha P$-expandable. Indeed, let $U = \{U_\alpha : \alpha \in \Delta\}$ be a locally finite (in $X$) collection of subsets of $A$. Then $1 \notin U_\alpha$ for every $\alpha \in \Delta$. As in Example part (2) we show that there exists a $p$-locally finite collection $\rho = \{P_\alpha : \alpha \in \Delta\}$ of preopen subsets of $X$ such that $U_\alpha \subseteq P_\alpha$ for all $\alpha \in \Delta$. Thus $A$ is $\alpha P$-expandable.

A subset $A$ of a space $(X, \tau)$ is called pre-clopen if $A$ is preopen and preclosed.

**Theorem 3.4.** Let $A$ and $B$ be subsets of a space $(X, \tau)$ such that $A \subseteq B$.

i. If $B$ is pre-clopen in $(X, \tau)$ and $A$ is $\alpha P$-expandable in $(B, \tau_B)$ then $A$ is $\alpha P$-expandable in $(X, \tau)$.

ii. If $B$ is semi-open in $(X, \tau)$ and $A$ is $\alpha P$-expandable in $(X, \tau)$, then $A$ is $\alpha P$-expandable in $(B, \tau_B)$.

**Proof.** i) Let $U = \{U_\alpha : \alpha \in \Delta\}$ be a locally finite collection of subsets of $A$. Then there exists a $p$-locally finite collection $G = \{G_\alpha : \alpha \in \Delta\}$ of preopen subsets of $(B, \tau_B)$ such that $F_\alpha \subseteq G_\alpha$ for all $\alpha \in \Delta$. Since $B$ is pre-clopen subset in $(X, \tau)$, then, by Lemma 1.1, $G$ is $p$-locally finite collection of preopen subsets of $(X, \tau)$. For, let $x \in X$. Then either $x \in B$ or $x \notin B$. If $x \in B$, then there exists a preopen set $W$ in $(B, \tau_B)$ containing $x$ such that $W$ intersects at most finitely many members of $G$. Since $B$ is preopen in $(X, \tau)$ then $W$ is preopen in $(X, \tau)$, by Lemma 1.1 and hence $G$ is $p$-locally finite collection in $(X, \tau)$. However, if $x \notin B$, then $X - B$ is preopen set in $(X, \tau)$ containing $x$ which intersects no member of $G$. Hence $G$ is a $p$-locally finite collection in $(X, \tau)$.

ii) Let $F = \{F_\alpha : \alpha \in \Delta\}$ be a locally finite collection of subsets of $A$. Then there exists a $p$-locally finite collection $G = \{G_\alpha : \alpha \in \Delta\}$ of preopen subset of $(X, \tau)$ such that $F_\alpha \subseteq G_\alpha$ for all $\alpha \in \Delta$. Now consider $G^* = \{G_\alpha \cap B : \alpha \in \Delta\}$, by Lemma 1.1, $G^*$ is a $p$-locally finite collection of preopen subset of $(B, \tau_B)$ such that $F_\alpha \subseteq G_\alpha \cap B$ for all $\alpha \in \Delta$. Thus $A$ is $\alpha P$-expandable in $(B, \tau_B)$.

**Corollary 3.5.** Let $A$ be a subset of a space $(X, \tau)$.

i. If $A$ is pre-clopen in $(X, \tau)$ and $\beta P$-expandable, then $A$ is $\alpha P$-expandable.
ii. If A is semi-open in $(X, \tau)$ and $\alpha$ P-expandable, then A is $\beta$ P-expandable.

Note that Example 3.2 shows that the assumption A is pre-clopen in Corollary 3.5 cannot be replaced by the statement A is preclosed.

**Lemma 3.6.** If A is a closed subset of a space $(X, \tau)$, then any locally finite collection of subsets of A is a locally finite collection in X.

**Proposition 3.7.** Let $(X, \tau)$ be a P-expandable space, then:

i. Every regular closed subset of $(X, \tau)$ is $\beta$ P-expandable.

ii. Every closed subsets of $(X, \tau)$ is $\alpha$ P-expandable.

**Proof.** i) Let A be a regular closed subset of a P-expandable space $(X, \tau)$. Let $\mathcal{F} = \{F_\alpha : \alpha \in \Delta\}$ be a locally finite collection of subset of A. Since A is closed by Lemma 3.6, $\mathcal{F}$ is locally finite in $(X, \tau)$, so there exists a $p$-locally finite collection of preopen subset of $(X, \tau)$, say $\mathcal{G} = \{G_\alpha : \alpha \in \Delta\}$ such that $F_\alpha \subseteq G_\alpha$, for each $\alpha \in \Delta$. Consider $\mathcal{G}^* = \{G_\alpha \cap A : \alpha \in \Delta\}$. Then, by Lemma 1.1 and the fact that $RC(X, \tau) \subseteq SO(X, \tau)$, $\mathcal{G}^*$ is a $p$-locally finite collection of preopen subsets of A such that $F_\alpha \subseteq G_\alpha \cap A$ for each $\alpha \in \Delta$. Thus A is $\beta$ P-expandable.

ii) It is follow from Lemma 3.6.

**Theorem 3.8.** The topological sum $\bigoplus_{\alpha \in \Delta} X_\alpha$ is P-expandable if and only if $(X_\alpha, \tau_\alpha)$ is P-expandable, for each $\alpha \in \Delta$.

**Proof.** Necessity follows from Proposition 3.7. To prove sufficiency, let $\mathcal{U}$ be a locally finite collection of $\bigoplus_{\alpha \in \Delta} X_\alpha$. For each $\alpha \in \Delta$ the family $U_\alpha = \{U \cap X_\alpha : U \in \mathcal{U}\}$ is a locally finite collection of the P-expandable space $(X_\alpha, \tau_\alpha)$. Therefore there exists a $p$-locally finite collection $\mathcal{G}_\alpha = \{G_{U_\alpha} : U \in \mathcal{U}\}$ of a preopen subsets of $(X_\alpha, \tau_\alpha)$ such that for all $\alpha \in \Delta$, $U \cap X_\alpha \subseteq G_{U_\alpha}$ for all $U \in \mathcal{U}$. Put $G_U = \bigcup_{\alpha \in \Delta} G_{U_\alpha}$ and $\mathcal{G}^* = \{G_U : U \in \mathcal{U}\}$. We note that (i) $G_U$ is preopen in X for each $U \in \mathcal{U}$ (by Lemma 1.1) (ii) $\mathcal{G}^*$ is $p$-locally finite in X. Let $x \in X$. Then there exists $\alpha_0 \in \Delta$ such that $x \in X_{\alpha_0}$. So there exists a preopen subset $W_{\alpha_0}$ of $X_{\alpha_0}$ such that $W_{\alpha_0}$ intersects at most finitely many member of $\mathcal{G}_{\alpha_0}$, say $G_{U_{\alpha_0}}(a), G_{U_2(a_0)}, \ldots G_{U_n(a_0)}$. Note that $G_{U_\beta} \cap W_{\alpha_0} = \phi$ for each $U \in \mathcal{U}$ and so for every $U \in \mathcal{U} - \{U_1, \ldots, U_n\}$, $W_{\alpha_0} \cap G_U = \phi$. Thus $\mathcal{G}^*$ is $p$-locally finite in X such that for each $U \in \mathcal{U}$, $U = U \cap X = U \cap (\bigcup_{\beta \in \Delta} X_\beta) \subseteq G_U$. 

$\square$
**Theorem 3.9.** Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be a \( M \)-preopen and strongly preclosed surjective continuous function such that \( f^{-1}(y) \) is strongly compact relative to \((X, \tau)\) for every \( y \in Y \). If \((X, \tau)\) is \( P \)-expandable then \((Y, \sigma)\) is \( P \)-expandable.

**Proof.** Assume that \((X, \tau)\) is \( P \)-expandable and \( \mathcal{F} = \{F_\alpha : \alpha \in \Delta\} \) is a locally finite collection of subsets of \( Y \). Then \( f^{-1}(\mathcal{F}) = \{f^{-1}(F_\alpha) : \alpha \in \Delta\} \) is a locally finite collection of subsets of the \( P \)-expandable \((X, \tau)\) and so there is a \( p \)-locally finite \( \{G_\alpha : \alpha \in \Delta\} \) of preopen subsets of \( X \) such that \( f^{-1}(F_\alpha) \subseteq G_\alpha \) for each \( \alpha \in \Delta \). Since \( f \) is \( M \)-preopen and by Theorem 1.6, the collection \( f(G_\alpha) \) is \( p \)-locally finite collection of preopen subsets of \( Y \) such that \( F_\alpha \subseteq f(G_\alpha) \).

**Theorem 3.10.** Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be a countably perfect preirresolute continuous function. If \((Y, \sigma)\) is \( P \)-expandable, then so is \((X, \tau)\).

**Proof.** Let \( \mathcal{F} = \{F_\alpha : \alpha \in \Delta\} \) be a locally finite collection of subsets of \( X \), by Lemma 1.5, \( \{f(F_\alpha) : \alpha \in \Delta\} \) is a locally finite collection in \( Y \). Hence there is a \( p \)-locally finite collection \( \mathcal{G} = \{G_\alpha : \alpha \in \Delta\} \) of preopen subsets of \( Y \) such that \( f(F_\alpha) \subseteq G_\alpha \) for each \( \alpha \in \Delta \). Then, by Theorem 1.6, \( F_\alpha \subseteq f^{-1}(f(F_\alpha) \subseteq f^{-1}(G_\alpha) \) and \( \{f^{-1}(G_\alpha) : \alpha \in \Delta\} \) is a \( p \)-locally finite collection of preopen subsets of \( X \).

It clear that every continuous open function is preirresolute and \( M \)-preopen [8].

**Corollary 3.11.** Let \((X, \tau)\) be compact and \((Y, \sigma)\) be \( P \)-expandable. Then the product space \( (X, \tau) \times (Y, \sigma) \) is \( P \)-expandable.

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References


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