TRANSLATION AND DENSITY OF A BIPOLAR-VALUED FUZZY SET IN UP-ALGEBRAS

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Abstract. We apply the notion of bipolar fuzzy translations of a bipolar-valued fuzzy set to UP-algebras. For any bipolar-valued fuzzy set $\varphi = (A; \varphi^-, \varphi^+)$ in a UP-algebra A, the notions of bipolar fuzzy (α, β)-translations of $\varphi = (A; \varphi^-, \varphi^+)$ of type I and of type II are introduced, their basic properties are investigated and some useful examples are discussed. The notions of extensions and of intensions of a bipolar-valued fuzzy set are also studied. Moreover, we discuss the relation between the complement of a bipolar fuzzy UP-subalgebra (resp., bipolar fuzzy UP-filter, bipolar fuzzy UP-ideal and bipolar fuzzy strongly UP-ideal) and its level cuts.
Keywords: UP-algebra, bipolar fuzzy translation, bipolar fuzzy UP-subalgebra, bipolar fuzzy UP-filter, bipolar fuzzy UP-ideal, bipolar fuzzy strongly UP-ideal.

1. Introduction

Among many algebraic structures, algebras of logic form important class of algebras. Examples of these are BCK-algebras [5], BCI-algebras [6], BCH-algebras [3], K-algebras [1], KU-algebras [16], SU-algebras [12], UP-algebras [4] and others. They are strongly connected with logic. For example, BCI-algebras introduced by Iséki [6] in 1966 have connections with BCI-logic being the BCI-system in combinatory logic which has application in the language of functional programming. BCK and BCI-algebras are two classes of logical algebras. They were introduced by Imai and Iséki [5, 6] in 1966 and have been extensively investigated by many researchers. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras.

The notion of fuzzy sets of a set was first considered by Zadeh [22] in 1965. The fuzzy set theories developed by Zadeh and others have found many appli-

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cations in the domain of mathematics and elsewhere. There are several kinds of fuzzy set extensions in the fuzzy set theory, for example, intuitionistic fuzzy sets, interval-valued fuzzy sets, vague sets, bipolar-valued fuzzy sets etc. The notion of bipolar-valued fuzzy sets was first introduced by Lee [14] in 2000, is an extension of fuzzy sets whose membership degree range is enlarged from the interval $[0, 1]$ to $[-1, 0]$. After the introduction of the notion of bipolar-valued fuzzy sets by Lee [14], several researches were conducted on the generalizations of the notion of bipolar-valued fuzzy sets and application to many logical algebras such as: In 2008, Jun and Song [10] introduced the notions of bipolar fuzzy subalgebras and bipolar fuzzy closed ideals in BCH-algebras. In 2009, Jun and Park [9] introduced the notions of bipolar fuzzy regularities, bipolar fuzzy regular subalgebras, bipolar fuzzy filters, and bipolar fuzzy closed quasi filters in BCH-algebras. In 2011, Lee and Jun [13] introduced the notion of bipolar fuzzy a-ideals of BCI-algebras. In 2012, Jun et al. [8] introduced the notions of bipolar fuzzy CI-subalgebras, bipolar fuzzy ideals and (closed) bipolar fuzzy filters in CI-algebras. In 2014, Muhiuddin [15] introduced the notions of bipolar fuzzy KU-subalgebras and bipolar fuzzy KU-ideals in KU-algebras. In 2015, Senapati [20] introduced the notion of bipolar fuzzy BG-subalgebras in BG-algebras. In 2016, Sabarinathan et al. [17] introduced the notion of bipolar valued fuzzy ideals of BF-algebras. In 2017, Sabarinathan et al. [18] introduced the notion of bipolar valued fuzzy H-ideals of BF-algebras.

Moreover, bipolar-valued fuzzy sets were extended to bipolar fuzzy translations in many algebras such as: In 2009, Jun et al. [7] introduced the notions of bipolar fuzzy translations and bipolar fuzzy S-extensions of a bipolar fuzzy subalgebra in BCK/BCI-algebras. In 2012, Sardar et al. [19] introduced the notions of bipolar valued fuzzy translations and bipolar valued fuzzy S-extensions of a bipolar valued fuzzy subsemigroup (bi-ideal) in semigroups.

In this paper, we apply the notion of bipolar fuzzy translations of a bipolar-valued fuzzy set to UP-algebras. For any bipolar-valued fuzzy set $\varphi = (A; \varphi^-, \varphi^+)$ in a UP-algebra A, the notions of bipolar fuzzy (α, β)-translations of $\varphi = (A; \varphi^-, \varphi^+)$ of type I and of type II are introduced, their basic properties are investigated and some useful examples are discussed. The notions of extensions and of intensions of a bipolar-valued fuzzy set are also studied. Moreover, we discuss the relation between the complement of a bipolar fuzzy UP-subalgebra (resp., bipolar fuzzy UP-filter, bipolar fuzzy UP-ideal and bipolar fuzzy strongly UP-ideal) and its level cuts.

2. Basic results on UP-algebras

Before we begin our study, we will introduce the definition of a UP-algebra.

An algebra $A = (A, \cdot, 0)$ of type $(2, 0)$ is called a *UP-algebra* [4] where A is a nonempty set, $\cdot$ is a binary operation on A, and 0 is a fixed element of A (i.e., a nullary operation) if it satisfies the following axioms: for any $x, y, z \in A$,

(UP-1) $(y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = 0$,

(UP-2) $0 \cdot x = x$,

(UP-3) $x \cdot 0 = 0$, and

(UP-4) $x \cdot y = 0$ and $y \cdot x = 0$ imply $x = y$.

From [4], we know that the notion of UP-algebras is a generalization of KU-algebras.

Example 2.1 ([4])**.** Let X be a universal set. Define two binary operations $\cdot$ and $*$ on the power set of X by putting $A \cdot B = B \cap A'$ and $A * B = B \cup A'$ for all $A, B \in \mathcal{P}(X)$. Then $(\mathcal{P}(X), \cdot, \emptyset)$ and $(\mathcal{P}(X), *, X)$ are UP-algebras and we shall call it the *power UP-algebra of type 1* and the *power UP-algebra of type 2*, respectively.

In what follows, let A denote a UP-algebra unless otherwise specified. The following proposition is very important for the study of UP-algebras.

Proposition 2.2 ([4])**.** *In a UP-algebra A, the following properties hold: for any $x, y, z \in A$,*

(1) $x \cdot x = 0$,

(2) $x \cdot y = 0$ *and* $y \cdot z = 0$ *imply* $x \cdot z = 0$,

(3) $x \cdot y = 0$ *implies* $(z \cdot x) \cdot (z \cdot y) = 0$,

(4) $x \cdot y = 0$ *implies* $(y \cdot z) \cdot (x \cdot z) = 0$,

(5) $x \cdot (y \cdot x) = 0$,

(6) $(y \cdot x) \cdot x = 0$ *if and only if* $x = y \cdot x$, *and*

(7) $x \cdot (y \cdot y) = 0$.

Definition 2.3 ([4])**.** A subset S of A is called a *UP-subalgebra* of A if the constant 0 of A is in S, and $(S, \cdot, 0)$ itself forms a UP-algebra.

Iampan [4] proved the useful criteria that a nonempty subset S of a UP-algebra $A = (A, \cdot, 0)$ is a UP-subalgebra of A if and only if S is closed under the $\cdot$ multiplication on A.

Definition 2.4 ([4, 21])**.** A subset S of A is called

(1) a *UP-filter* of A if

  (i) the constant 0 of A is in S, and

  (ii) for any $x, y \in A, x \cdot y \in S$ and $x \in S$ imply $y \in S$.

(2) a *UP-ideal* of A if

(i) the constant 0 of A is in S, and

(ii) for any $x, y, z \in A, x \cdot (y \cdot z) \in S$ and $y \in S$ imply $x \cdot z \in S$.

(3) a *strongly UP-ideal* of A if

(i) the constant 0 of A is in S, and

(ii) for any $x, y, z \in A, (z \cdot y) \cdot (z \cdot x) \in S$ and $y \in S$ imply $x \in S$.

Guntasow et al. [2] proved the generalization that the notion of UP-subalgebras is a generalization of UP-filters, the notion of UP-filters is a generalization of UP-ideals, and the notion of UP-ideals is a generalization of strongly UP-ideals. Moreover, they also proved that a UP-algebra A is the only one strongly UP-ideal of itself.

3. Bipolar fuzzy (α, β)-translations in UP-algebras

Let X be the universe of discourse. A *bipolar-valued fuzzy set* [13] φ in X is an object having the form

$$\varphi = \{(x, \varphi^-(x), \varphi^+(x)) \mid x \in X\}$$

where $\varphi^- : X \to [-1, 0]$ and $\varphi^+ : X \to [0, 1]$ are mappings. For the sake of simplicity, we shall use the symbol $\varphi = (X; \varphi^-, \varphi^+)$ for the bipolar-valued fuzzy set $\varphi = \{(x, \varphi^-(x), \varphi^+(x)) \mid x \in X\}$, and use the notion of bipolar fuzzy sets instead of the notion of bipolar-valued fuzzy sets.

We recall the definitions of bipolar fuzzy UP-subalgebras, bipolar fuzzy UP-filters, bipolar fuzzy UP-ideals, and bipolar fuzzy strongly UP-ideals.

Definition 3.1 ([11])**.** A bipolar fuzzy set $\varphi = (A; \varphi^-, \varphi^+)$ in A is called a *bipolar fuzzy UP-subalgebra* of A if it satisfies the following properties: for any $x, y \in A$,

(1) $\varphi^-(x \cdot y) \leq \max\{\varphi^-(x), \varphi^-(y)\}$, and

(2) $\varphi^+(x \cdot y) \geq \min\{\varphi^+(x), \varphi^+(y)\}$.

Definition 3.2 ([11])**.** A bipolar fuzzy set $\varphi = (A; \varphi^-, \varphi^+)$ in A is called a *bipolar fuzzy UP-filter* of A if it satisfies the following properties: for any $x, y \in A$,

(1) $\varphi^-(0) \leq \varphi^-(x)$,

(2) $\varphi^+(0) \geq \varphi^+(x)$,

(3) $\varphi^-(y) \leq \max\{\varphi^-(x \cdot y), \varphi^-(x)\}$, and

(4) $\varphi^+(y) \geq \min\{\varphi^+(x \cdot y), \varphi^+(x)\}$.

Definition 3.3 ([11])**.** A bipolar fuzzy set $\varphi = (A; \varphi^-, \varphi^+)$ in A is called a *bipolar fuzzy UP-ideal* of A if it satisfies the following properties: for any $x, y, z \in A$,

(1) $\varphi^-(0) \leq \varphi^-(x)$,

(2) $\varphi^+(0) \geq \varphi^+(x)$,

(3) $\varphi^-(x \cdot z) \leq \max\{\varphi^-(x \cdot (y \cdot z)), \varphi^-(y)\}$, and

(4) $\varphi^+(x \cdot z) \geq \min\{\varphi^+(x \cdot (y \cdot z)), \varphi^+(y)\}$.

Definition 3.4 ([11])**.** A bipolar fuzzy set $\varphi = (A; \varphi^-, \varphi^+)$ in A is called a *bipolar fuzzy strongly UP-ideal* of A if it satisfies the following properties: for any $x, y, z \in A$,

(1) $\varphi^-(0) \leq \varphi^-(x)$,

(2) $\varphi^+(0) \geq \varphi^+(x)$,

(3) $\varphi^-(x) \leq \max\{\varphi^-((z \cdot y) \cdot (z \cdot x)), \varphi^-(y)\}$, and

(4) $\varphi^+(x) \geq \min\{\varphi^+((z \cdot y) \cdot (z \cdot x)), \varphi^+(y)\}$.

Kawila et al. [11] proved the generalization that the notion of bipolar UP-subalgebras is a generalization of bipolar UP-filters, the notion of bipolar UP-filters is a generalization of bipolar UP-ideals, and the notion of bipolar UP-ideals is a generalization of bipolar strongly UP-ideals. Moreover, they also proved that a bipolar fuzzy set $\varphi = (A; \varphi^-, \varphi^+)$ in A is constant if and only if it is a bipolar fuzzy strongly UP-ideal of A.

3.1 Bipolar fuzzy (α, β)-translations of a bipolar fuzzy set of type I

Definition 3.5. The inclusion "$\subseteq$" is defined by setting, for any bipolar fuzzy sets $\varphi = (A; \varphi^-, \varphi^+)$ and $\psi = (A; \psi^-, \psi^+)$ in A,

$$\varphi \subseteq \psi \Leftrightarrow \varphi^-(x) \geq \psi^-(x) \text{ and } \varphi^+(x) \leq \psi^+(x) \text{ for all } x \in A.$$

We say that $\psi = (A; \psi^-, \psi^+)$ is a *bipolar fuzzy extension* of $\varphi = (A; \varphi^-, \varphi^+)$, and $\varphi = (A; \varphi^-, \varphi^+)$ is a *bipolar fuzzy intension* of $\psi = (A; \psi^-, \psi^+)$.

Definition 3.6. For any bipolar fuzzy set $\varphi = (A; \varphi^-, \varphi^+)$ in A, we denote

$$\begin{aligned} \perp &:= -1 - \inf\{\varphi^-(x) \mid x \in A\}, \\ \top &:= 1 - \sup\{\varphi^+(x) \mid x \in A\}. \end{aligned}$$

Let $\varphi = (A; \varphi^-, \varphi^+)$ be a bipolar fuzzy set in A and $(\alpha, \beta) \in [\perp, 0] \times [0, \top]$. By a *bipolar fuzzy* (α, β)*-translation of* $\varphi = (A; \varphi^-, \varphi^+)$ *of type I*, we mean a bipolar fuzzy set ${\varphi^{\mathrm{T}_1}}_{(\alpha,\beta)} = (A; {\varphi^-}_{(\alpha,\mathrm{T}_1)}, {\varphi^+}_{(\beta,\mathrm{T}_1)})$ where

$$\begin{aligned} {\varphi^-}_{(\alpha,\mathrm{T}_1)} &: A \to [-1, 0], x \mapsto \varphi^-(x) + \alpha, \\ {\varphi^+}_{(\beta,\mathrm{T}_1)} &: A \to [0, 1], x \mapsto \varphi^+(x) + \beta. \end{aligned}$$

Theorem 3.7. *If a bipolar fuzzy set $\varphi = (A; \varphi^-, \varphi^+)$ in A is a bipolar fuzzy UP-subalgebra of A, then for all $(\alpha, \beta) \in [\perp, 0] \times [0, \top]$, a bipolar fuzzy (α, β)-translation ${\varphi^{\mathrm{T}_1}}_{(\alpha,\beta)} = (A; {\varphi^-}_{(\alpha,\mathrm{T}_1)}, {\varphi^+}_{(\beta,\mathrm{T}_1)})$ of $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-subalgebra of A.*

Proof. Assume that $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-subalgebra of A. For any $(\alpha, \beta) \in [\perp, 0] \times [0, \top]$ and for all $x, y \in A$, we have

$$\begin{aligned} \varphi^-_{(\alpha,\mathrm{T}_1)}(x \cdot y) &= \varphi^-(x \cdot y) + \alpha \\ &\leq \max\{\varphi^-(x), \varphi^-(y)\} + \alpha \\ &= \max\{\varphi^-(x) + \alpha, \varphi^-(y) + \alpha\} \\ &= \max\{\varphi^-_{(\alpha,\mathrm{T}_1)}(x), \varphi^-_{(\alpha,\mathrm{T}_1)}(y)\} \end{aligned}$$

and

$$\begin{aligned} \varphi^+_{(\beta,\mathrm{T}_1)}(x \cdot y) &= \varphi^+(x \cdot y) + \beta \\ &\geq \min\{\varphi^+(x), \varphi^+(y)\} + \beta \\ &= \min\{\varphi^+(x) + \beta, \varphi^+(y) + \beta\} \\ &= \min\{\varphi^+_{(\beta,\mathrm{T}_1)}(x), \varphi^+_{(\beta,\mathrm{T}_1)}(y)\}. \end{aligned}$$

Hence, $\varphi^{\mathrm{T}_1}_{(\alpha,\beta)} = (A; \varphi^-_{(\alpha,\mathrm{T}_1)}, \varphi^+_{(\beta,\mathrm{T}_1)})$ is a bipolar fuzzy UP-subalgebra of A. □

Theorem 3.8. *If there exists $(\alpha, \beta) \in [\perp, 0] \times [0, \top]$ such that the bipolar fuzzy (α, β)-translation ${\varphi^{\mathrm{T}_1}}_{(\alpha,\beta)} = (A; {\varphi^-}_{(\alpha,\mathrm{T}_1)}, {\varphi^+}_{(\beta,\mathrm{T}_1)})$ of $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-subalgebra of A, then $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-subalgebra of A.*

Proof. Assume that $\varphi^{\mathrm{T}_1}_{(\alpha,\beta)} = (A; \varphi^-_{(\alpha,\mathrm{T}_1)}, \varphi^+_{(\beta,\mathrm{T}_1)})$ is a bipolar fuzzy UP-subalgebra of A for $(\alpha, \beta) \in [\perp, 0] \times [0, \top]$ and for all $x, y \in A$, we have

$$\begin{aligned} \varphi^-(x \cdot y) + \alpha &= \varphi^-_{(\alpha,T_1)}(x \cdot y) \\ &\leq \max\{\varphi^-_{(\alpha,\mathrm{T}_1)}(x), \varphi^-_{(\alpha,\mathrm{T}_1)}(y)\} \\ &= \max\{\varphi^-(x) + \alpha, \varphi^-(y) + \alpha\} \\ &= \max\{\varphi^-(x), \varphi^-(y)\} + \alpha \end{aligned}$$

and

$$\begin{aligned} \varphi^+(x \cdot y) + \beta &= \varphi^+_{(\beta,\mathrm{T}_1)}(x \cdot y) \\ &\geq \{\varphi^+_{(\beta,\mathrm{T}_1)}(x), \varphi^+_{(\beta,\mathrm{T}_1)}(y)\} \\ &= \min\{\varphi^+(x) + \beta, \varphi^+(y) + \beta\} \\ &= \min\{\varphi^+(x), \varphi^+(y)\} + \beta. \end{aligned}$$

Thus $\varphi^-(x \cdot y) \leq \max\{\varphi^-(x), \varphi^-(y)\}$ and $\varphi^+(x \cdot y) \geq \min\{\varphi^+(x), \varphi^+(y)\}$. Hence, $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-subalgebra of A. □

Theorem 3.9. *If a bipolar fuzzy set $\varphi = (A; \varphi^-, \varphi^+)$ in A is a bipolar fuzzy UP-filter of A then for all $(\alpha, \beta) \in [\perp, 0] \times [0, \top]$, a bipolar fuzzy (α, β)-translation ${\varphi^{\mathrm{T}_1}}_{(\alpha,\beta)} = (A; {\varphi^-}_{(\alpha,\mathrm{T}_1)}, {\varphi^+}_{(\beta,\mathrm{T}_1)})$ of $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-filter of A.*

Proof. Assume that $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-filter of A. For any $(\alpha, \beta) \in [\perp, 0] \times [0, \top]$ and let $x \in A$. Then $\varphi^-(0) \leq \varphi^-(x)$ and $\varphi^+(0) \geq \varphi^+(x)$. Thus

$$\varphi^-_{(\alpha,\mathrm{T}_1)}(0) = \varphi^-(0) + \alpha \leq \varphi^-(x) + \alpha = \varphi^-_{(\alpha,\mathrm{T}_1)}(x)$$

and

$$\varphi^+_{(\beta,\mathrm{T}_1)}(0) = \varphi^+(0) + \beta \geq \varphi^+(x) + \beta = \varphi^+_{(\beta,\mathrm{T}_1)}(x).$$

Next, let $x, y \in A$. Then $\varphi^-(y) \leq \max\{\varphi^-(x \cdot y), \varphi^-(x)\}$ and $\varphi^+(y) \geq \min\{\varphi^+(x \cdot y), \varphi^+(x)\}$. Thus

$$\begin{aligned}
\varphi^-_{(\alpha,\mathrm{T}_1)}(y) &= \varphi^-(y) + \alpha \\
&\leq \max\{\varphi^-(x \cdot y), \varphi^-(x)\} + \alpha \\
&= \max\{\varphi^-(x \cdot y) + \alpha, \varphi^-(x)) + \alpha\} \\
&= \max\{\varphi^-_{(\alpha,\mathrm{T}_1)}(x \cdot y), \varphi^-_{(\alpha,\mathrm{T}_1)}(x)\}
\end{aligned}$$

and

$$\begin{aligned}
\varphi^+_{(\beta,\mathrm{T}_1)}(y) &= \varphi^+(y) + \beta \\
&\geq \min\{\varphi^+(x \cdot y), \varphi^+(x)\} + \beta \\
&= \min\{\varphi^+(x \cdot y) + \beta, \varphi^+(x) + \beta\} \\
&= \min\{\varphi^+_{(\beta,\mathrm{T}_1)}(x \cdot y), \varphi^+_{(\beta,\mathrm{T}_1)}(x)\}.
\end{aligned}$$

Hence, $\varphi^{\mathrm{T}_1}_{(\alpha,\beta)} = (A; \varphi^-_{(\alpha,\mathrm{T}_1)}, \varphi^+_{(\beta,\mathrm{T}_1)})$ is a bipolar fuzzy UP-filter of A. $\square$

Theorem 3.10. *If there exists $(\alpha, \beta) \in [\perp, 0] \times [0, \top]$ such that the bipolar fuzzy (α, β)-translation ${\varphi^{\mathrm{T}_1}}_{(\alpha,\beta)} = (A; {\varphi^-}_{(\alpha,\mathrm{T}_1)}, {\varphi^+}_{(\beta,\mathrm{T}_1)})$ of $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-filter of A, then $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-filter of A.*

Proof. Assume that $\varphi^{\mathrm{T}_1}_{(\alpha,\beta)} = (A; \varphi^-_{(\alpha,\mathrm{T}_1)}, \varphi^+_{(\beta,\mathrm{T}_1)})$ is a bipolar fuzzy UP-filter of A for $(\alpha, \beta) \in [\perp, 0] \times [0, \top]$ and let $x \in A$. Then

$$\varphi^-(0) + \alpha = \varphi^-_{(\alpha,\mathrm{T}_1)}(0) \leq \varphi^-_{(\alpha,\mathrm{T}_1)}(x) = \varphi^-(x) + \alpha$$

and

$$\varphi^+(0) + \beta = \varphi^+_{(\beta,\mathrm{T}_1)}(0) \geq \varphi^+_{(\beta,\mathrm{T}_1)}(x) = \varphi^+(x) + \beta.$$

Thus $\varphi^-(0) \leq \varphi^-(x)$ and $\varphi^+(0) \geq \varphi^+(x)$. Next, let $x, y \in A$. Then

$$\begin{aligned}\varphi^-(y) + \alpha &= \varphi^-_{(\alpha,\mathrm{T}_1)}(y) \\ &\leq \max\{\varphi^-_{(\alpha,\mathrm{T}_1)}(x \cdot y), \varphi^-_{(\alpha,\mathrm{T}_1)}(x)\} \\ &= \max\{\varphi^-(x \cdot y) + \alpha, \varphi^-(x) + \alpha\} \\ &= \max\{\varphi^-(x \cdot y), \varphi^-(x)\} + \alpha\end{aligned}$$

and

$$\begin{aligned}\varphi^-(y) + \beta &= \varphi^+_{(\beta,\mathrm{T}_1)}(y) \\ &\geq \min\{\varphi^+_{(\beta,\mathrm{T}_1)}(x \cdot y), \varphi^+_{(\beta,\mathrm{T}_1)}(x)\} \\ &= \min\{\varphi^-(x \cdot y) + \beta, \varphi^-(x) + \beta\} \\ &= \min\{\varphi^-(x \cdot y), \varphi^-(x)\} + \beta.\end{aligned}$$

Thus $\varphi^-(y) \leq \max\{\varphi^-(x \cdot y), \varphi^-(x)\}$ and $\varphi^-(y) \geq \min\{\varphi^-(x \cdot y), \varphi^-(x)\}$. Hence, $\varphi^- = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-filter of A. $\square$

Theorem 3.11. *If a bipolar fuzzy set $\varphi = (A; \varphi^-, \varphi^+)$ in A is a bipolar fuzzy UP-ideal of A, then for all $(\alpha, \beta) \in [\bot, 0] \times [0, \top]$, a bipolar fuzzy (α, β)-translation ${\varphi^{\mathrm{T}_1}}_{(\alpha,\beta)} = (A; {\varphi^-}_{(\alpha,\mathrm{T}_1)}, {\varphi^+}_{(\beta,\mathrm{T}_1)})$ of $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-ideal of A.*

Proof. Assume that $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-ideal of A. For any $(\alpha, \beta) \in [\bot, 0] \times [0, \top]$ and let $x \in A$. Then $\varphi^-(0) \leq \varphi^-(x)$ and $\varphi^+(0) \geq \varphi^+(x)$. Thus

$$\varphi^-_{(\alpha,\mathrm{T}_1)}(0) = \varphi^-(0) + \alpha \leq \varphi^-(x) + \alpha = \varphi^-_{(\alpha,\mathrm{T}_1)}(x)$$

and

$$\varphi^+_{(\beta,\mathrm{T}_1)}(0) = \varphi^+(0) + \beta \geq \varphi^-(x) + \beta = \varphi^-_{(\beta,\mathrm{T}_1)}(x).$$

Next, let $x, y, z \in A$. Then $\varphi^-(x \cdot z) \leq \max\{\varphi^-(x \cdot (y \cdot z)), \varphi^-(y)\}$ and $\varphi^+(x \cdot z) \geq \min\{\varphi^+(x \cdot (y \cdot z)), \varphi^+(y)\}$. Thus

$$\begin{aligned}\varphi^-_{(\alpha,\mathrm{T}_1)}(x \cdot z) &= \varphi^-(x \cdot z) + \alpha \\ &\leq \max\{\varphi^-(x \cdot (y \cdot z)), \varphi^-(y)\} + \alpha \\ &= \max\{\varphi^-(x \cdot (y \cdot z)) + \alpha, \varphi^-(y) + \alpha\} \\ &= \max\{\varphi^-_{(\alpha,\mathrm{T}_1)}(x \cdot (y \cdot z)), \varphi^-_{(\alpha,\mathrm{T}_1)}(y)\}\end{aligned}$$

and

$$\begin{aligned}\varphi^+_{(\beta,\mathrm{T}_1)}(x \cdot z) &= \varphi^+(x \cdot z) + \beta \\ &\geq \min\{\varphi^+(x \cdot (y \cdot z)), \varphi^+(y)\} + \beta \\ &= \min\{\varphi^+(x \cdot (y \cdot z)) + \beta, \varphi^+(y) + \beta\} \\ &= \min\{\varphi^+_{(\beta,\mathrm{T}_1)}(x \cdot (y \cdot z)), \varphi^+_{(\beta,\mathrm{T}_1)}(y)\}.\end{aligned}$$

Hence, $\varphi^{\mathrm{T}_1}_{(\alpha,\beta)} = (A; \varphi^-_{(\alpha,\mathrm{T}_1)}, \varphi^+_{(\beta,\mathrm{T}_1)})$ is a bipolar fuzzy UP-ideal of A. $\square$

Theorem 3.12. *If there exists $(\alpha, \beta) \in [\bot, 0] \times [0, \top]$ such that the bipolar fuzzy (α, β)-translation $\varphi^{\mathrm{T}_1}{}_{(\alpha,\beta)} = (A; \varphi^-{}_{(\alpha,\mathrm{T}_1)}, \varphi^+{}_{(\beta,\mathrm{T}_1)})$ of $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-ideal of A, then $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-ideal of A.*

Proof. Assume that $\varphi^{\mathrm{T}_1}_{(\alpha,\beta)} = (A; \varphi^-_{(\alpha,\mathrm{T}_1)}, \varphi^+_{(\beta,\mathrm{T}_1)})$ is a bipolar fuzzy UP-ideal of A for $(\alpha, \beta) \in [\bot, 0] \times [0, \top]$ and let $x \in A$. Then

$$\varphi^-(0) + \alpha = \varphi^-_{(\alpha,\mathrm{T}_1)}(0) \leq \varphi^-_{(\alpha,\mathrm{T}_1)}(x) = \varphi^-(x) + \alpha$$

and

$$\varphi^+(0) + \beta = \varphi^+_{(\beta,\mathrm{T}_1)}(0) \geq \varphi^+_{(\beta,\mathrm{T}_1)}(x) = \varphi^+(x) + \beta.$$

Thus $\varphi^-(0) \leq \varphi^-(x)$ and $\varphi^+(0) \geq \varphi^+(x)$. Next, let $x, y, z \in A$. Then

$$\begin{aligned}
\varphi^-(x \cdot z) + \alpha &= \varphi^-_{(\alpha,\mathrm{T}_1)}(x \cdot z) \\
&\leq \max\{\varphi^-_{(\alpha,\mathrm{T}_1)}(x \cdot (y \cdot z)), \varphi^-_{(\alpha,\mathrm{T}_1)}(y)\} \\
&= \max\{\varphi^-(x \cdot (y \cdot z)) + \alpha, \varphi^-(y) + \alpha\} \\
&= \max\{\varphi^-(x \cdot (y \cdot z)), \varphi^-(y)\} + \alpha
\end{aligned}$$

and

$$\begin{aligned}
\varphi^+(x \cdot z) + \beta &= \varphi^+_{(\beta,\mathrm{T}_1)}(x \cdot z) \\
&\geq \min\{\varphi^+_{(\beta,\mathrm{T}_1)}(x \cdot (y \cdot z)), \varphi^+_{(\beta,\mathrm{T}_1)}(y)\} \\
&= \min\{\varphi^+(x \cdot (y \cdot z)) + \beta, \varphi^+(y) + \beta\} \\
&= \min\{\varphi^+(x \cdot (y \cdot z)), \varphi^+(y)\} + \beta.
\end{aligned}$$

Thus $\varphi^-(x \cdot z) \leq \max\{\varphi^-(x \cdot (y \cdot z)), \varphi^-(y)\}$ and $\varphi^+(x \cdot z) \geq \min\{\varphi^+(x \cdot (y \cdot z)), \varphi^+(y)\}$. Hence, $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-ideal of A. $\square$

Theorem 3.13. *If a bipolar fuzzy set $\varphi = (A; \varphi^-, \varphi^+)$ in A is a bipolar fuzzy strongly UP-ideal of A, then for all $(\alpha, \beta) \in [\bot, 0] \times [0, \top]$, a bipolar fuzzy (α, β)-translation $\varphi^{\mathrm{T}_1}{}_{(\alpha,\beta)} = (A; \varphi^-{}_{(\alpha,\mathrm{T}_1)}, \varphi^+{}_{(\beta,\mathrm{T}_1)})$ of $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy strongly UP-ideal of A.*

Proof. Assume that $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy strongly UP-ideal of A. For any $(\alpha, \beta) \in [\bot, 0] \times [0, \top]$ and let $x \in A$. Then $\varphi^-(0) \leq \varphi^-(x)$ and $\varphi^+(0) \geq \varphi^+(x)$. Thus

$$\varphi^-_{(\alpha,\mathrm{T}_1)}(0) = \varphi^-(0) + \alpha \leq \varphi^-(x) + \alpha = \varphi^-_{(\alpha,\mathrm{T}_1)}(x)$$

and

$$\varphi^+_{(\beta,T_1)}(0) = \varphi^+(0) + \beta \geq \varphi^+(x) + \beta = \varphi^+_{(\beta,\mathrm{T}_1)}(x).$$

Next, let $x, y, z \in A$. Then $\varphi^-(x) \leq \max\{\varphi^-((z \cdot y) \cdot (z \cdot x)), \varphi^-(y)\}$ and $\varphi^+(x) \geq \min\{\varphi^+((z \cdot y) \cdot (z \cdot x)), \varphi^+(y)\}$. Thus

$$\begin{aligned}
\varphi^-_{(\alpha,\mathrm{T}_1)}(x) &= \varphi^-(x) + \alpha \\
&\leq \max\{\varphi^-((z \cdot y) \cdot (z \cdot x)), \varphi^-(y)\} + \alpha \\
&= \max\{\varphi^-((z \cdot y) \cdot (z \cdot x))) + \alpha, \varphi^-(y) + \alpha\} \\
&= \max\{\varphi^-_{(\alpha,\mathrm{T}_1)}((z \cdot y) \cdot (z \cdot x)), \varphi^-_{(\alpha,\mathrm{T}_1)}(y)\}
\end{aligned}$$

and

$$\begin{aligned}
\varphi^+_{(\beta,\mathrm{T}_1)}(x) &= \varphi^+(x) + \beta \\
&\geq \min\{\varphi^+((z \cdot y) \cdot (z \cdot x)), \varphi^+(y)\} + \beta \\
&= \min\{\varphi^+((z \cdot y) \cdot (z \cdot x)) + \beta, \varphi^+(y) + \beta\} \\
&= \min\{\varphi^+_{(\beta,\mathrm{T}_1)}((z \cdot y) \cdot (z \cdot x)), \varphi^+_{(\beta,\mathrm{T}_1)}(y)\}.
\end{aligned}$$

Hence, $\varphi^{\mathrm{T}_1}_{(\alpha,\beta)} = (A; \varphi^-_{(\alpha,\mathrm{T}_1)}, \varphi^+_{(\beta,\mathrm{T}_1)})$ is a bipolar fuzzy strongly UP-ideal of A. $\square$

Theorem 3.14. *If there exists $(\alpha, \beta) \in [\perp, 0] \times [0, \top]$ such that the bipolar fuzzy (α, β)-translation ${\varphi^{\mathrm{T}_1}}_{(\alpha,\beta)} = (A; {\varphi^-}_{(\alpha,\mathrm{T}_1)}, {\varphi^+}_{(\beta,\mathrm{T}_1)})$ of $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy strongly UP-ideal of A, then $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy strongly UP-ideal of A.*

Proof. Assume that $\varphi^{\mathrm{T}_1}_{(\alpha,\beta)} = (A; \varphi^-_{(\alpha,\mathrm{T}_1)}, \varphi^+_{(\beta,\mathrm{T}_1)})$ is a bipolar fuzzy strongly UP-ideal of A for $(\alpha, \beta) \in [\perp, 0] \times [0, \top]$ and let $x \in A$. Then

$$\varphi^-(0) + \alpha = \varphi^-_{(\alpha,\mathrm{T}_1)}(0) \leq \varphi^-_{(\alpha,\mathrm{T}_1)}(x) = \varphi^-(x) + \alpha$$

and

$$\varphi^+(0) + \beta = \varphi^+_{(\beta,T_1)}(0) \geq \varphi^+_{(\beta,\mathrm{T}_1)}(x) = \varphi^+(x) + \beta.$$

Thus $\varphi^-(0) \leq \varphi^-(x)$ and $\varphi^+(0) \geq \varphi^+(x)$. Next, let $x, y, z \in A$. Then

$$\begin{aligned}
\varphi^-(x) + \alpha &= \varphi^-_{(\alpha,\mathrm{T}_1)}(x) \\
&\leq \max\{\varphi^-_{(\alpha,\mathrm{T}_1)}((z \cdot y) \cdot (z \cdot x)), \varphi^-_{(\alpha,\mathrm{T}_1)}(y)\} \\
&= \max\{\varphi^-((z \cdot y) \cdot (z \cdot x)) + \alpha, \varphi^-(y) + \alpha\} \\
&= \max\{\varphi^-((z \cdot y) \cdot (z \cdot x)), \varphi^-(y)\} + \alpha
\end{aligned}$$

and

$$\begin{aligned}
\varphi^+(x) + \beta &= \varphi^+_{(\beta,\mathrm{T}_1)}(x) \\
&\geq \min\{\varphi^+_{(\beta,\mathrm{T}_1)}((z \cdot y) \cdot (z \cdot x)), \varphi^+_{(\beta,\mathrm{T}_1)}(y)\} \\
&= \min\{\varphi^+((z \cdot y) \cdot (z \cdot x)) + \beta, \varphi^+(y) + \beta\} \\
&= \min\{\varphi^+((z \cdot y) \cdot (z \cdot x)), \varphi^+(y)\} + \beta.
\end{aligned}$$

Thus $\varphi^-(x) \leq \max\{\varphi^-((z \cdot y) \cdot (z \cdot x)), \varphi^-(y)\}$ and $\varphi^+(x) \geq \min\{\varphi^+((z \cdot y) \cdot (z \cdot x)), \varphi^+(y)\}$. Hence, $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy strongly UP-ideal of A. $\square$

Remark 3.15. If $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy set in A, then for all $(\alpha, \beta) \in [\perp, 0] \times [0, \top]$, ${\varphi^-}_{(\alpha,\mathrm{T}_1)}(x) = \varphi^-(x) + \alpha \leq \varphi^-(x)$ and ${\varphi^+}_{(\beta,\mathrm{T}_1)}(x) = \varphi^+(x) + \beta \geq \varphi^+(x)$ for all $x \in A$. Hence, the bipolar fuzzy (α, β)-translation ${\varphi^{\mathrm{T}_1}}_{(\alpha,\beta)} = (A; {\varphi^-}_{(\alpha,\mathrm{T}_1)}, {\varphi^+}_{(\beta,\mathrm{T}_1)})$ of $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy extension of $\varphi = (A; \varphi^-, \varphi^+)$ for all $(\alpha, \beta) \in [\perp, 0] \times [0, \top]$.

Lemma 3.16. *Let $\varphi = (A; \varphi^-, \varphi^+)$ and $\psi = (A; \psi^-, \psi^+)$ be bipolar fuzzy sets in A. If $\varphi^{\mathrm{T}_1}_{(\alpha_1,\beta_1)} \subseteq \psi$ for all $(\alpha_1, \beta_1) \in [\perp, 0] \times [0, \top]$, then there exists $(\alpha_2, \beta_2) \in [\perp, 0] \times [0, \top]$ with $(\alpha_1, \beta_1) \leq (\alpha_2, \beta_2)$, that is, $\alpha_1 \geq \alpha_2$ and $\beta_1 \leq \beta_2$ such that $\varphi^{\mathrm{T}_1}_{(\alpha_1,\beta_1)} \subseteq \varphi^{\mathrm{T}_1}_{(\alpha_2,\beta_2)} \subseteq \psi$.*

Proof. Assume that $\varphi^{\mathrm{T}_1}_{(\alpha_1,\beta_1)} \subseteq \psi$ for all $(\alpha_1, \beta_1) \in [\perp, 0] \times [0, \top]$. Then $\psi^-(x) \leq \varphi^-_{(\alpha_1,\mathrm{T}_1)}(x)$ and $\psi^+(x) \geq \varphi^+_{(\beta_1,\mathrm{T}_1)}(x)$ for all $x \in A$. Put $\alpha_2 = \alpha_1 + \sup\{\psi^-(x) - \varphi^-_{(\alpha_1,\mathrm{T}_1)}(x)\}$. Then

$$\begin{aligned}
\sup\{\psi^-(x) - \varphi^-_{(\alpha_1,\mathrm{T}_1)}(x)\} &= \sup\{\psi^-(x) - (\varphi^-(x) + \alpha_1)\} \\
&\geq \sup\{-1 - (\varphi^-(x) + \alpha_1)\} \\
&= -1 + \sup\{-\varphi^-(x) - \alpha_1\} \\
&= -1 + \sup\{-\varphi^-(x)\} - \alpha_1 \\
&= -1 - \inf\{\varphi^-(x)\} - \alpha_1 \\
&= \perp - \alpha_1,
\end{aligned}$$

so $\alpha_2 = \alpha_1 + \sup\{\psi^-(x) - \varphi^-_{(\alpha,\mathrm{T}_1)}(x)\} \geq \alpha_1 + \perp - \alpha_1 = \perp$. Thus $\alpha_2 \in [\perp, 0]$ and $\alpha_2 \leq \alpha_1$, so $\varphi^-_{(\alpha_2,\mathrm{T}_1)}(x) \leq \varphi^-_{(\alpha_1,\mathrm{T}_1)}(x)$ for all $x \in A$. Now for all $x \in A$, we have

$$\begin{aligned}
\varphi^-_{(\alpha_2,\mathrm{T}_1)}(x) &= \varphi^-(x) + \alpha_2 \\
&= \varphi^-(x) + \alpha_1 + \sup\{\psi^-(x) - \varphi^-_{(\alpha_1,\mathrm{T}_1)}(x)\} \\
&\geq \varphi^-_{(\alpha_1,\mathrm{T}_1)}(x) + \psi^-(x) - \varphi^-_{(\alpha_1,\mathrm{T}_1)}(x) \\
&= \psi^-(x).
\end{aligned}$$

Thus $\varphi^-_{(\alpha_1,\mathrm{T}_1)}(x) \geq \varphi^-_{(\alpha_2,\mathrm{T}_1)}(x) \geq \psi^-(x)$ for all $x \in A$. Put $\beta_2 = \beta_1 + \inf\{\psi^+(x) - \varphi^+_{(\beta_1,\mathrm{T}_1)}(x)\}$. Then

$$\begin{aligned}
\inf\{\psi^+(x) - \varphi^+_{(\beta_1,\mathrm{T}_1)}(x)\} &= \inf\{\psi^+(x) - (\varphi^+(x) + \beta_1)\} \\
&\leq \inf\{1 - (\varphi^+(x) + \beta_1)\} \\
&= 1 + \inf\{-\varphi^+(x) - \beta_1\} \\
&= 1 + \inf\{-\varphi^+(x)\} - \beta_1 \\
&= -1 - \sup\{\varphi^+(x)\} - \beta_1 \\
&= \top - \beta_1,
\end{aligned}$$

so $\beta_2 = \beta_1 + \inf\{\psi^+(x) - \varphi^+_{(\beta_1,\mathrm{T}_1)}(x)\} \leq \beta_1 + \top - \beta_1 = \top$. Thus $\beta_2 \in [0, \top]$ and $\beta_2 \geq \beta_1$, so $\varphi^+_{(\beta_2,\mathrm{T}_1)}(x) \geq \varphi^+_{(\beta_1,\mathrm{T}_1)}(x)$ for all $x \in A$. Now for all $x \in A$, we have

$$\begin{aligned}
\varphi^+_{(\beta_2,\mathrm{T}_1)}(x) &= \varphi^+(x) + \beta_2 \\
&= \varphi^+(x) + \beta_1 + \inf\{\psi^+(x) - \varphi^+_{(\beta_1,\mathrm{T}_1)}(x)\} \\
&\leq \varphi^+_{(\beta_1,\mathrm{T}_1)}(x) + \psi^+(x) - \varphi^+_{(\beta_1,\mathrm{T}_1)}(x) \\
&= \psi^+(x).
\end{aligned}$$

Thus $\varphi^+_{(\beta_1,\mathrm{T}_1)}(x) \leq \varphi^+_{(\beta_2,\mathrm{T}_1)}(x) \leq \psi^+(x)$ for all $x \in A$. Hence, $\varphi^{\mathrm{T}_1}_{(\alpha_1,\beta_1)} \subseteq \varphi^{\mathrm{T}_1}_{(\alpha_2,\beta_2)} \subseteq \psi$. □

Definition 3.17. Let $\varphi = (A; \varphi^-, \varphi^+)$ and $\psi = (A; \psi^-, \psi^+)$ be bipolar fuzzy sets in A with $\varphi \subseteq \psi$. If $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-subalgebra (resp., bipolar fuzzy UP-filter, bipolar fuzzy UP-ideal, bipolar fuzzy strongly UP-ideal) of A, then $\psi = (A; \psi^-, \psi^+)$ is a bipolar fuzzy UP-subalgebra (resp., bipolar fuzzy UP-filter, bipolar fuzzy UP-ideal, bipolar fuzzy strongly UP-ideal) of A, and we say that $\psi = (A; \psi^-, \psi^+)$ is a *bipolar fuzzy UP-subalgebra* (resp., bipolar fuzzy UP-filter, bipolar fuzzy UP-ideal, bipolar fuzzy strongly UP-ideal) *extension* of $\varphi = (A; \varphi^-, \varphi^+)$.

Theorem 3.18. *If $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-subalgebra (resp., bipolar fuzzy UP-filter, bipolar fuzzy UP-ideal, bipolar fuzzy strongly UP-ideal) of A, then the bipolar fuzzy (α, β)-translation $\varphi^{\mathrm{T}_1}{}_{(\alpha,\beta)} = (A; \varphi^-{}_{(\alpha,\mathrm{T}_1)}, \varphi^+{}_{(\beta,\mathrm{T}_1)})$ of $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-subalgebra (resp., bipolar fuzzy UP-filter, bipolar fuzzy UP-ideal, bipolar fuzzy strongly UP-ideal) extension of $\varphi = (A; \varphi^-, \varphi^+)$.*

Proof. It follows form Theorem 3.7 (resp., Theorem 3.9, Theorem 3.11, Theorem 3.13) and Remark 3.15. □

Theorem 3.19. *If $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-subalgebra (resp., bipolar fuzzy UP-filter, bipolar fuzzy UP-ideal, bipolar fuzzy strongly UP-ideal) of A, then the bipolar fuzzy (α_1, β_1)-translation $\varphi^{\mathrm{T}_1}{}_{(\alpha_1,\beta_1)} = (A; \varphi^-{}_{(\alpha_1,\mathrm{T}_1)}, \varphi^+{}_{(\beta_1,\mathrm{T}_1)})$ of $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-subalgebra (resp., bipolar fuzzy UP-filter, bipolar fuzzy UP-ideal, bipolar fuzzy strongly UP-ideal) extension of the bipolar fuzzy (α_2, β_2)-translation $\varphi^{\mathrm{T}_1}{}_{(\alpha_2,\beta_2)} = (A; \varphi^-{}_{(\alpha_2,\mathrm{T}_1)}, \varphi^+{}_{(\beta_2,\mathrm{T}_1)})$ of $\varphi = (A; \varphi^-, \varphi^+)$ with $(\alpha_1, \beta_1) \geq (\alpha_2, \beta_2)$.*

Proof. It follows form Theorem 3.7 (resp., Theorem 3.9, Theorem 3.11, Theorem 3.13). □

Theorem 3.20. *Let $\varphi = (A; \varphi^-, \varphi^+)$ be a bipolar fuzzy UP-subalgebra (resp., bipolar fuzzy UP-filter, bipolar fuzzy UP-ideal, bipolar fuzzy strongly UP-ideal) of A. For every bipolar fuzzy UP-subalgebra (resp., bipolar fuzzy UP-filter, bipolar* *fuzzy UP-ideal, bipolar fuzzy strongly UP-ideal) extension* $\psi = (A; \psi^-, \psi^+)$ *of the bipolar fuzzy* (α, β)*-translation* $\varphi^{\mathrm{T}_1}_{(\alpha,\beta)} = (A; \varphi^-_{(\alpha,\mathrm{T}_1)}, \varphi^+_{(\beta,\mathrm{T}_1)})$ *of* $\varphi = (A; \varphi^-, \varphi^+)$ *there exists* $(k^-, k^+) \in [\perp, 0] \times [0, \top]$ *such that* $(k^-, k^+) \geq (\alpha, \beta)$*, that is,* $k^- \leq \alpha$ *and* $k^+ \geq \beta$*, and* $\psi = (A; \psi^-, \psi^+)$ *is a bipolar fuzzy UP-subalgebra (resp., bipolar fuzzy UP-filter, bipolar fuzzy UP-ideal, bipolar fuzzy strongly UP-ideal) extension of bipolar fuzzy* (k^-, k^+)*-translation* $\varphi^{\mathrm{T}_1}_{(k^-,k^+)} = (A; \varphi^-_{(k^-,\mathrm{T}_1)}, \varphi^+_{(k^+,\mathrm{T}_1)})$ *of* $\varphi = (A; \varphi^-, \varphi^+)$*.*

Proof. It follows form Theorem 3.7 (resp., Theorem 3.9, Theorem 3.11, Theorem 3.13) and Lemma 3.16. $\square$

3.2 Bipolar fuzzy (α, β)-translations of a bipolar fuzzy set of type II

Definition 3.21. For any bipolar fuzzy set $\varphi = (A; \varphi^-, \varphi^+)$ in A, we denote

$$\begin{aligned} \pm &:= \sup\{\varphi^-(x) \mid x \in A\}, \\ \mp &:= \inf\{\varphi^+(x) \mid x \in A\}. \end{aligned}$$

Let $\varphi = (A; \varphi^-, \varphi^+)$ be a bipolar fuzzy set in A and $(\alpha, \beta) \in [\pm, 0] \times [0, \mp]$. By a *bipolar fuzzy* (α, β)*-translation of* $\varphi = (A; \varphi^-, \varphi^+)$ *of type II*, we mean a bipolar fuzzy set ${\varphi^{\mathrm{T}_2}}_{(\alpha,\beta)} = (A; {\varphi^-}_{(\alpha,\mathrm{T}_2)}, {\varphi^+}_{(\beta,\mathrm{T}_2)})$ where

$$\begin{aligned} {\varphi^-}_{(\alpha,\mathrm{T}_2)} &: A \to [-1, 0], x \mapsto \varphi^-(x) - \alpha, \\ {\varphi^+}_{(\beta,\mathrm{T}_2)} &: A \to [0, 1], x \mapsto \varphi^+(x) - \beta. \end{aligned}$$

Theorem 3.22. *If a bipolar fuzzy set* $\varphi = (A; \varphi^-, \varphi^+)$ *in* A *is a bipolar fuzzy UP-subalgebra of* A*, then for all* $(\alpha, \beta) \in [\pm, 0] \times [0, \mp]$*, a bipolar fuzzy* (α, β)*-translation* ${\varphi^{\mathrm{T}_2}}_{(\alpha,\beta)} = (A; {\varphi^-}_{(\alpha,\mathrm{T}_2)}, {\varphi^+}_{(\beta,\mathrm{T}_2)})$ *of* $\varphi = (A; \varphi^-, \varphi^+)$ *is a bipolar fuzzy UP-subalgebra of* A*.*

Proof. Assume that $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-subalgebra of A. For any $(\alpha, \beta) \in [\pm, 0] \times [0, \mp]$ and for all $x, y \in A$, we have

$$\begin{aligned} \varphi^-_{(\alpha,\mathrm{T}_2)}(x \cdot y) &= \varphi^-(x \cdot y) - \alpha \\ &\leq \max\{\varphi^-(x), \varphi^-(y)\} - \alpha \\ &= \max\{\varphi^-(x) - \alpha, \varphi^-(y) - \alpha\} \\ &= \max\{\varphi^-_{(\alpha,\mathrm{T}_2)}(x), \varphi^-_{(\alpha,\mathrm{T}_2)}(y)\} \end{aligned}$$

and

$$\begin{aligned} \varphi^+_{(\beta,\mathrm{T}_2)}(x \cdot y) &= \varphi^+(x \cdot y) - \beta \\ &\geq \min\{\varphi^+(x), \varphi^+(y)\} - \beta \\ &= \min\{\varphi^+(x) - \beta, \varphi^+(y) - \beta\} \\ &= \min\{\varphi^+_{(\beta,\mathrm{T}_2)}(x), \varphi^+_{(\beta,\mathrm{T}_2)}(y)\}. \end{aligned}$$

Hence, ${\varphi^{\mathrm{T}_2}}_{(\alpha,\beta)} = (A; {\varphi^-}_{(\alpha,\mathrm{T}_2)}, {\varphi^+}_{(\beta,\mathrm{T}_2)})$ is a bipolar fuzzy UP-subalgebra of A. $\square$

Theorem 3.23. *If there exists $(\alpha, \beta) \in [\pm, 0] \times [0, \mp]$ such that the bipolar fuzzy (α, β)-translation ${\varphi^{\mathrm{T}_2}}_{(\alpha,\beta)} = (A; {\varphi^-}_{(\alpha,\mathrm{T}_2)}, {\varphi^+}_{(\beta,\mathrm{T}_2)})$ of $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-subalgebra of A, then $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-subalgebra of A.*

Proof. Assume that $\varphi^{\mathrm{T}_2}_{(\alpha,\beta)} = (\varphi^-_{(\alpha,\mathrm{T}_2)}, \varphi^+_{(\beta,\mathrm{T}_2)})$ is a bipolar fuzzy UP-subalgebra of A for $(\alpha, \beta) \in [\pm, 0] \times [0, \mp]$. Then for all $x, y \in A$, we have

$$\begin{aligned}
\varphi^-(x \cdot y) - \alpha &= \varphi^-_{(\alpha,\mathrm{T}_2)}(x \cdot y) \\
&\leq \max\{\varphi^-_{(\alpha,\mathrm{T}_2)}(x), \varphi^-_{(\alpha,\mathrm{T}_2)}(y)\} \\
&= \max\{\varphi^-(x) - \alpha, \varphi^-(y) - \alpha\} \\
&= \max\{\varphi^-(x), \varphi^-(y)\} - \alpha
\end{aligned}$$

and

$$\begin{aligned}
\varphi^+(x \cdot y) - \beta &= \varphi^+_{(\beta,\mathrm{T}_2)}(x \cdot y) \\
&\geq \min\{\varphi^+_{(\beta,\mathrm{T}_2)}(x), \varphi^+_{(\beta,\mathrm{T}_2)}(y)\} \\
&= \min\{\varphi^+(x) - \beta, \varphi^+(y) - \beta\} \\
&= \min\{\varphi^+(x), \varphi^+(y)\} - \beta.
\end{aligned}$$

Thus $\varphi^-(x \cdot y) \leq \max\{\varphi^-(x), \varphi^-(y)\}$ and $\varphi^+(x \cdot y) \geq \min\{\varphi^+(x), \varphi^+(y)\}$. Hence, $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-subalgebra of A. □

Theorem 3.24. *If a bipolar fuzzy set $\varphi = (A; \varphi^-, \varphi^+)$ in A is a bipolar fuzzy UP-filter of A, then for all $(\alpha, \beta) \in [\pm, 0] \times [0, \mp]$, a bipolar fuzzy (α, β)-translation ${\varphi^{\mathrm{T}_2}}_{(\alpha,\beta)} = (A; {\varphi^-}_{(\alpha,\mathrm{T}_2)}, {\varphi^+}_{(\beta,\mathrm{T}_2)})$ of $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-filter of A.*

Proof. Assume that $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-filter of A. For any $(\alpha, \beta) \in [\pm, 0] \times [0, \mp]$ and let $x \in A$. Then $\varphi^-(0) \leq \varphi^-(x)$ and $\varphi^+(0) \geq \varphi^+(x)$. Thus

$$\varphi^-_{(\alpha,\mathrm{T}_2)}(0) = \varphi^-(0) - \alpha \leq \varphi^-(x) - \alpha = \varphi^-_{(\alpha,\mathrm{T}_2)}(x)$$

and

$$\varphi^+_{(\beta,\mathrm{T}_2)}(0) = \varphi^+(0) - \beta \geq \varphi^+(x) - \beta = \varphi^+_{(\beta,\mathrm{T}_2)}(x).$$

Next, let $x, y \in A$. Then $\varphi^-(y) \leq \max\{\varphi^-(x \cdot y), \varphi^-(x)\}$ and $\varphi^+(y) \geq \min\{\varphi^+(x \cdot y), \varphi^+(x)\}$. Thus

$$\begin{aligned}
\varphi^-_{(\alpha,\mathrm{T}_2)}(y) &= \varphi^-(y) - \alpha \\
&\leq \max\{\varphi^-(x \cdot y), \varphi^-(x)\} - \alpha \\
&= \max\{\varphi^-(x \cdot y) - \alpha, \varphi^-(x) - \alpha\} \\
&= \max\{\varphi^-_{(\alpha,\mathrm{T}_2)}(x \cdot y), \varphi^-_{(\alpha,\mathrm{T}_2)}(x)\}
\end{aligned}$$

and

$$\begin{aligned}\varphi^+_{(\beta,\mathrm{T}_2)}(y) &= \varphi^+(y) - \beta \\ &\geq \min\{\varphi^+(x \cdot y), \varphi^+(x)\} - \beta \\ &= \min\{\varphi^+(x \cdot y) - \beta, \varphi^+(x) - \beta\} \\ &= \min\{\varphi^+_{(\beta,\mathrm{T}_2)}(x \cdot y), \varphi^+_{(\beta,\mathrm{T}_2)}(x)\}.\end{aligned}$$

Hence, $\varphi^{\mathrm{T}_2}_{(\alpha,\beta)} = (A; \varphi^-_{(\alpha,\mathrm{T}_2)}, \varphi^+_{(\beta,\mathrm{T}_2)})$ is a bipolar fuzzy UP-filter of A. □

Theorem 3.25. *If there exists $(\alpha, \beta) \in [\pm, 0] \times [0, \mp]$ such that the bipolar fuzzy (α, β)-translation ${\varphi^{\mathrm{T}_2}}_{(\alpha,\beta)} = (A; {\varphi^-}_{(\alpha,\mathrm{T}_2)}, {\varphi^+}_{(\beta,\mathrm{T}_2)})$ of $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-filter of A, then $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-filter of A.*

Proof. Assume that $\varphi^{\mathrm{T}_2}_{(\alpha,\beta)} = (A; \varphi^-_{(\alpha,\mathrm{T}_2)}, \varphi^+_{(\beta,\mathrm{T}_2)})$ is a bipolar fuzzy UP-filter of A for $(\alpha, \beta) \in [\pm, 0] \times [0, \mp]$ and let $x \in A$. Then

$$\varphi^-(0) - \alpha = \varphi^-_{(\alpha,\mathrm{T}_2)}(0) \leq \varphi^-_{(\alpha,\mathrm{T}_2)}(x) = \varphi^-(x) - \alpha$$

and

$$\varphi^+(0) - \beta = \varphi^+_{(\beta,\mathrm{T}_2)}(0) \geq \varphi^+_{(\beta,\mathrm{T}_2)}(x) = \varphi^+(x) - \beta.$$

Thus $\varphi^-(0) \leq \varphi^-(x)$ and $\varphi^+(0) \geq \varphi^+(x)$. Next, let $x, y \in A$. Then

$$\begin{aligned}\varphi^-(y) - \alpha &= \varphi^-_{(\alpha,\mathrm{T}_2)}(y) \\ &\leq \max\{\varphi^-_{(\alpha,\mathrm{T}_2)}(x \cdot y), \varphi^-_{(\alpha,\mathrm{T}_2)}(x)\} \\ &= \max\{\varphi^-(x \cdot y) - \alpha, \varphi^-(x) - \alpha\} \\ &= \max\{\varphi^-(x \cdot y), \varphi^-(x)\} - \alpha\end{aligned}$$

and

$$\begin{aligned}\varphi^+(y) - \beta &= \varphi^+_{(\beta,\mathrm{T}_2)}(y) \\ &\geq \min\{\varphi^+_{(\beta,\mathrm{T}_2)}(x \cdot y), \varphi^+_{(\beta,\mathrm{T}_2)}(x)\} \\ &= \min\{\varphi^+(x \cdot y) - \beta, \varphi^+(x) - \beta\} \\ &= \min\{\varphi^+(x \cdot y), \varphi^+(x)\} - \beta.\end{aligned}$$

Thus $\varphi^-(y) \leq \max\{\varphi^-(x), \varphi^-(x \cdot y)\}$ and $\varphi^+(y) \geq \min\{\varphi^+(x), \varphi^+(x \cdot y)\}$. Hence, $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-filter of A. □

Theorem 3.26. *If a bipolar fuzzy set $\varphi = (A; \varphi^-, \varphi^+)$ in A is a bipolar fuzzy UP-ideal of A, then for all $(\alpha, \beta) \in [\pm, 0] \times [0, \mp]$, a bipolar fuzzy (α, β)-translation ${\varphi^{\mathrm{T}_2}}_{(\alpha,\beta)} = (A; {\varphi^-}_{(\alpha,\mathrm{T}_2)}, {\varphi^+}_{(\beta,\mathrm{T}_2)})$ of $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-ideal of A.*

Proof. Assume that $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-ideal of A. For any $(\alpha, \beta) \in [\pm, 0] \times [0, \mp]$ and let $x \in A$. Then $\varphi^-(0) \leq \varphi^-(x)$ and $\varphi^+(0) \geq \varphi^+(x)$. Thus

$$\varphi^-_{(\alpha,\mathrm{T}_2)}(0) = \varphi^-(0) - \alpha \leq \varphi^-(x) - \alpha = \varphi^-_{(\alpha,\mathrm{T}_2)}(x)$$

and

$$\varphi^+_{(\beta,\mathrm{T}_2)}(0) = \varphi^+(0) - \beta \geq \varphi^+(x) - \beta = \varphi^+_{(\beta,\mathrm{T}_2)}(x).$$

Next, let $x, y, z \in A$. Then $\varphi^-_{(\alpha,\mathrm{T}_2)}(x \cdot z) \leq \max\{\varphi^-_{(\alpha,\mathrm{T}_2)}(x \cdot (y \cdot z)), \varphi^-_{(\alpha,\mathrm{T}_2)}(y)\}$ and $\varphi^+_{(\beta,\mathrm{T}_2)}(x \cdot z) \geq \min\{\varphi^+_{(\beta,\mathrm{T}_2)}(x \cdot (y \cdot z)), \varphi^+_{(\beta,\mathrm{T}_2)}(y)\}$. Thus

$$\begin{aligned}
\varphi^-_{(\alpha,\mathrm{T}_2)}(x \cdot z) &= \varphi^-(x \cdot z) - \alpha \\
&\leq \max\{\varphi^-(x \cdot (y \cdot z)), \varphi^-(y)\} - \alpha \\
&= \max\{\varphi^-(x \cdot (y \cdot z)) - \alpha, \varphi^-(y) - \alpha\} \\
&= \max\{\varphi^-_{(\alpha,\mathrm{T}_2)}(x \cdot (y \cdot z)), \varphi^-_{(\alpha,\mathrm{T}_2)}(y)\}
\end{aligned}$$

and

$$\begin{aligned}
\varphi^+_{(\beta,\mathrm{T}_2)}(x \cdot z) &= \varphi^+(x \cdot z) - \beta \\
&\geq \min\{\varphi^+(x \cdot (y \cdot z)), \varphi^+(y)\} - \beta \\
&= \min\{\varphi^+(x \cdot (y \cdot z)) - \beta, \varphi^+(y) - \beta\} \\
&= \min\{\varphi^+_{(\beta,\mathrm{T}_2)}(x \cdot (y \cdot z)), \varphi^+_{(\beta,\mathrm{T}_2)}(y)\}.
\end{aligned}$$

Hence, $\varphi^{\mathrm{T}_2}_{(\alpha,\beta)} = (A; \varphi^-_{(\alpha,\mathrm{T}_2)}, \varphi^+_{(\beta,\mathrm{T}_2)})$ is a bipolar fuzzy UP-ideal of A. $\square$

Theorem 3.27. *If there exists $(\alpha, \beta) \in [\pm, 0] \times [0, \mp]$ such that the bipolar fuzzy (α, β)-translation ${\varphi^{\mathrm{T}_2}}_{(\alpha,\beta)} = (A; {\varphi^-}_{(\alpha,\mathrm{T}_2)}, {\varphi^+}_{(\beta,\mathrm{T}_2)})$ of $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-ideal of A, then $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-ideal of A.*

Proof. Assume that $\varphi^{\mathrm{T}_2}_{(\alpha,\beta)} = (A; \varphi^-_{(\alpha,\mathrm{T}_2)}, \varphi^+_{(\beta,\mathrm{T}_2)})$ is a bipolar fuzzy UP-ideal of A for $(\alpha, \beta) \in [\pm, 0] \times [0, \mp]$ and let $x \in A$. Then

$$\varphi^-(0) - \alpha = \varphi^-_{(\alpha,\mathrm{T}_2)}(0) \leq \varphi^-_{(\alpha,\mathrm{T}_2)}(x) = \varphi^-(x) - \alpha$$

and

$$\varphi^+(0) - \beta = \varphi^+_{(\beta,\mathrm{T}_2)}(0) \geq \varphi^+_{(\beta,\mathrm{T}_2)}(x) = \varphi^+(x) - \beta.$$

Thus $\varphi^-(0) \leq \varphi^-(x)$ and $\varphi^+(0) \geq \varphi^+(x)$. Next, $x, y, z \in A$. Then

$$\begin{aligned}
\varphi^-(x \cdot z) - \alpha &= \varphi^-_{(\alpha,\mathrm{T}_2)}(x \cdot z) \\
&\leq \max\{\varphi^-_{(\alpha,\mathrm{T}_2)}(x \cdot (y \cdot z)), \varphi^-_{(\alpha,\mathrm{T}_2)}(y)\} \\
&= \max\{\varphi^-(x \cdot (y \cdot z)) - \alpha, \varphi^-(y) - \alpha\} \\
&= \max\{\varphi^-(x \cdot (y \cdot z)), \varphi^-(y)\} - \alpha
\end{aligned}$$

and

$$\begin{aligned}\varphi^+(x\cdot z)-\beta &= \varphi^+_{(\beta,\mathrm{T}_2)}(x\cdot z)\\ &\geq \min\{\varphi^+_{(\beta,\mathrm{T}_2)}(x\cdot(y\cdot z)),\varphi^+_{(\beta,\mathrm{T}_2)}(y)\}\\ &= \min\{\varphi^+(x\cdot(y\cdot z))-\beta,\varphi^+(y)-\beta\}\\ &= \min\{\varphi^+(x\cdot(y\cdot z)),\varphi^+(y)\}-\beta.\end{aligned}$$

Thus $\varphi^-(x\cdot z)\leq \max\{\varphi^-(x\cdot(y\cdot z)),\varphi^-(y)\}$ and $\varphi^+(x\cdot z)\geq \min\{\varphi^+(x\cdot(y\cdot z)),\varphi^+(y)\}$. Hence, $\varphi=(A;\varphi^-,\varphi^+)$ is a bipolar fuzzy UP-ideal of A. $\square$

Theorem 3.28. *If a bipolar fuzzy set $\varphi=(A;\varphi^-,\varphi^+)$ in A is a bipolar fuzzy strongly UP-ideal of A, then for all $(\alpha,\beta)\in[\pm,0]\times[0,\mp]$, a bipolar fuzzy (α,β)-translation ${\varphi^{\mathrm{T}_2}}_{(\alpha,\beta)}=(A;{\varphi^-}_{(\alpha,\mathrm{T}_2)},{\varphi^+}_{(\beta,\mathrm{T}_2)})$ of $\varphi=(A;\varphi^-,\varphi^+)$ is a bipolar fuzzy strongly UP-ideal of A.*

Proof. Assume that $\varphi=(A;\varphi^-,\varphi^+)$ is a bipolar fuzzy strongly UP-ideal of A. For any $(\alpha,\beta)\in[\pm,0]\times[0,\mp]$ and let $x\in A$. Then $\varphi^-(0)\leq\varphi^-(x)$ and $\varphi^+(0)\geq\varphi^+(x)$. Thus

$$\varphi^-_{(\alpha,\mathrm{T}_2)}(0)=\varphi^-(0)-\alpha\leq\varphi^-(x)-\alpha=\varphi^-_{(\alpha,\mathrm{T}_2)}(x)$$

and

$$\varphi^+_{(\beta,\mathrm{T}_2)}(0)=\varphi^+(0)-\beta\geq\varphi^+(x)-\beta=\varphi^+_{(\beta,\mathrm{T}_2)}(x).$$

Next, let $x,y,z\in A$. Then $\varphi^-(x)\leq\max\{\varphi^-((z\cdot y)\cdot(z\cdot x)),\varphi^-(y)\}$ and $\varphi^+(x)\geq\min\{\varphi^+((z\cdot y)\cdot(z\cdot x)),\varphi^+(y)\}$. Thus

$$\begin{aligned}\varphi^-_{(\alpha,\mathrm{T}_2)}(x) &= \varphi^-(x)-\alpha\\ &\leq \max\{\varphi^-((z\cdot y)\cdot(z\cdot x)),\varphi^-(y)\}-\alpha\\ &= \max\{\varphi^-((z\cdot y)\cdot(z\cdot x))-\alpha,\varphi^-(y)-\alpha\}\\ &= \max\{\varphi^-_{(\alpha,\mathrm{T}_2)}((z\cdot y)\cdot(z\cdot x)),\varphi^-_{(\alpha,\mathrm{T}_2)}(y)\}\end{aligned}$$

and

$$\begin{aligned}\varphi^+_{(\beta,\mathrm{T}_2)}(x) &= \varphi^+(x)-\beta\\ &\geq \min\{\varphi^+((z\cdot y)\cdot(z\cdot x)),\varphi^+(y)\}-\beta\\ &= \min\{\varphi^+((z\cdot y)\cdot(z\cdot x))-\beta,\varphi^+(y)-\beta\}\\ &= \min\{\varphi^+_{(\beta,\mathrm{T}_2)}((z\cdot y)\cdot(z\cdot x)),\varphi^+_{(\beta,\mathrm{T}_2)}(y)\}.\end{aligned}$$

Hence, $\varphi^{\mathrm{T}_2}_{(\alpha,\beta)}=(A;\varphi^-_{(\alpha,\mathrm{T}_2)},\varphi^+_{(\beta,\mathrm{T}_2)})$ is a bipolar fuzzy strongly UP-ideal of A. $\square$

Theorem 3.29. *If there exists $(\alpha,\beta) \in [\pm,0] \times [0,\mp]$ such that the bipolar fuzzy (α,β)-translation ${\varphi^{\mathrm{T}_2}}_{(\alpha,\beta)} = (A; {\varphi^-}_{(\alpha,\mathrm{T}_2)}, {\varphi^+}_{(\beta,\mathrm{T}_2)})$ of $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy strongly UP-ideal of A, then $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy strongly UP-ideal of A.*

Proof. Assume that ${\varphi^{\mathrm{T}_2}}_{(\alpha,\beta)} = (A; {\varphi^-}_{(\alpha,\mathrm{T}_2)}, {\varphi^+}_{(\beta,\mathrm{T}_2)})$ is a bipolar fuzzy strongly UP-ideal of A for $(\alpha,\beta) \in [\pm,0] \times [0,\mp]$ and let $x \in A$. Then

$$\varphi^-(0) - \alpha = \varphi^-_{(\alpha,\mathrm{T}_2)}(0) \leq \varphi^-_{(\alpha,\mathrm{T}_2)}(x) = \varphi^-(x) - \alpha$$

and

$$\varphi^+(0) - \beta = \varphi^+_{(\beta,\mathrm{T}_2)}(0) \geq \varphi^+_{(\beta,\mathrm{T}_2)}(x) = \varphi^+(x) - \beta.$$

Thus $\varphi^-(0) \leq \varphi^-(x)$ and $\varphi^+(0) \geq \varphi^+(x)$. Next, let $x, y, z \in A$. Then

$$\begin{aligned}
\varphi^-(x) - \alpha &= \varphi^-_{(\alpha,\mathrm{T}_2)}(x) \\
&\leq \max\{\varphi^-_{(\alpha,\mathrm{T}_2)}((z \cdot y) \cdot (z \cdot x)), \varphi^-_{(\alpha,\mathrm{T}_2)}(y)\} \\
&= \max\{\varphi^-((z \cdot y) \cdot (z \cdot x)) - \alpha, \varphi^-(y) - \alpha\} \\
&= \max\{\varphi^-((z \cdot y) \cdot (z \cdot x)), \varphi^-(y)\} - \alpha
\end{aligned}$$

and

$$\begin{aligned}
\varphi^+(x) - \beta &= \varphi^+_{(\beta,\mathrm{T}_2)}(x) \\
&\geq \min\{\varphi^+_{(\beta,\mathrm{T}_2)}((z \cdot y) \cdot (z \cdot x)), \varphi^+_{(\beta,\mathrm{T}_2)}(y)\} \\
&= \min\{\varphi^+((z \cdot y) \cdot (z \cdot x)) - \beta, \varphi^+(y) - \beta\} \\
&= \min\{\varphi^+((z \cdot y) \cdot (z \cdot x)), \varphi^+(y)\} - \beta.
\end{aligned}$$

Thus $\varphi^-(x) \leq \max\{\varphi^-((z \cdot y) \cdot (z \cdot x)), \varphi^-(y)\}$ and $\varphi^+(x) \geq \min\{\varphi^+((z \cdot y) \cdot (z \cdot x)), \varphi^+(y)\}$. Hence, $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy strongly UP-ideal of A. $\square$

Remark 3.30. If $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy set in A and $(\alpha,\beta) \in [\pm,0] \times [0,\mp]$, then ${\varphi^-}_{(\alpha,\mathrm{T}_2)}(x) = \varphi^-(x) - \alpha \geq \varphi^-(x)$ and ${\varphi^+}_{(\beta,\mathrm{T}_2)}(x) = \varphi^+(x) - \beta \leq \beta^+(x)$ for all $x \in A$. Hence, the bipolar fuzzy (α,β)-translation ${\varphi^{\mathrm{T}_2}}_{(\alpha,\beta)} = (A; {\varphi^-}_{(\alpha,\mathrm{T}_2)}, {\varphi^+}_{(\beta,\mathrm{T}_2)})$ of $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy intension of $\varphi = (A; \varphi^-, \varphi^+)$ for all $(\alpha,\beta) \in [\pm,0] \times [0,\mp]$.

Lemma 3.31. *Let $\varphi = (A; \varphi^-, \varphi^+)$ and $\psi = (A; \psi^-, \psi^+)$ be bipolar fuzzy sets in A. If $\psi \subseteq \varphi^{\mathrm{T}_2}_{(\alpha_1,\beta_1)}$ for $(\alpha_1,\beta_1) \in [\pm,0] \times [0,\mp]$, then there exists $(\alpha_2,\beta_2) \in [\pm,0] \times [0,\mp]$ with $(\alpha_1,\beta_1) \leq (\alpha_2,\beta_2)$ such that $\psi \subseteq \varphi^{\mathrm{T}_2}_{(\alpha_2,\beta_2)} \subseteq \varphi^{\mathrm{T}_2}_{(\alpha_1,\beta_1)}$.*

Proof. Assume that $\psi \subseteq \varphi^{\mathrm{T}_2}_{(\alpha_1,\beta_1)}$ for $(\alpha_1,\beta_1) \in [\pm,0] \times [0,\mp]$. Then $\varphi^-_{(\alpha_1,\mathrm{T}_1)}(x) \leq \psi^-(x)$ and $\varphi^+_{(\beta_1,\mathrm{T}_1)}(x) \geq \psi^+(x)$ for all $x \in A$. Put $\alpha_2 = \alpha_1 + \sup\{\varphi^-_{(\alpha_1,\mathrm{T}_2)}(x) -$

$\psi^-(x)\}$. Then

$$\begin{aligned}\sup\{\varphi^-_{(\alpha_1,\mathrm{T}_2)}(x)-\psi^-(x)\} &\geq \sup\{\varphi^-_{(\alpha_1,\mathrm{T}_2)}(x)\}\\ &= \sup\{\varphi^-(x)-\alpha_1\}\\ &= \sup\{\varphi^-(x)\}-\alpha_1\\ &= \pm-\alpha_1,\end{aligned}$$

so $\alpha_2=\alpha_1+\sup\{\varphi^-_{(\alpha_1,\mathrm{T}_2)}(x)-\psi^-(x)\}\geq\alpha_1+\pm-\alpha_1=\pm$. Thus $\alpha_2\in[\pm,0]$ and $\alpha_2\leq\alpha_1$, so $\varphi^-_{(\alpha_2,\mathrm{T}_2)}(x)\geq\varphi^-_{(\alpha_1,\mathrm{T}_2)}(x)$ for all $x\in A$. Now for all $x\in A$, we have

$$\begin{aligned}\varphi^-_{(\alpha_2,\mathrm{T}_2)}(x) &= \varphi^-(x)-\alpha_2\\ &= \varphi^-(x)-(\alpha_1+\sup\{\varphi^-_{(\alpha_1,\mathrm{T}_2)}(x)-\psi^-(x)\})\\ &= \varphi^-(x)-\alpha_1-\sup\{\varphi^-_{(\alpha_1,\mathrm{T}_2)}(x)-\psi^-(x)\}\\ &= \varphi^-(x)-\alpha_1+\inf\{\psi^-(x)-\varphi^-_{(\alpha_1,\mathrm{T}_2)}(x)\}\\ &\leq \varphi^-_{(\alpha_1,\mathrm{T}_2)}(x)+\psi^-(x)-\varphi^-_{(\alpha_1,\mathrm{T}_2)}(x)\\ &= \psi^-(x).\end{aligned}$$

Thus $\varphi^-_{(\alpha_1,\mathrm{T}_2)}(x)\leq\varphi^-_{(\alpha_2,\mathrm{T}_2)}(x)\leq\psi^-(x)$ for all $x\in A$. Put $\beta_2=\beta_1+\inf\{\varphi^+_{(\beta_1,\mathrm{T}_2)}(x)-\psi^+(x)\}$. Then

$$\begin{aligned}\inf\{\varphi^+_{(\beta_1,\mathrm{T}_2)}(x)-\psi^+(x)\} &\leq \inf\{\varphi^+_{(\beta_1,\mathrm{T}_2)}(x)\}\\ &= \inf\{\varphi^+(x)-\beta_1\}\\ &= \inf\{\varphi^+(x)\}-\beta_1\\ &= \mp-\beta_1,\end{aligned}$$

so $\beta_2=\beta_1+\inf\{\varphi^+_{(\beta_1,\mathrm{T}_2)}(x)-\psi^+(x)\}\leq\beta_1+\mp-\beta_1=\mp$. Thus $\beta_2\in[0,\mp]$ and $\beta_2\geq\beta_1$, so $\varphi^+_{(\beta_2,\mathrm{T}_1)}(x)\leq\varphi^+_{(\beta_1,\mathrm{T}_2)}(x)$ for all $x\in A$. Now for all $x\in A$, we have

$$\begin{aligned}\varphi^+_{(\beta_2,\mathrm{T}_2)}(x) &= \varphi^+(x)-\beta_2\\ &= \varphi^+(x)-(\beta_1+\inf\{\varphi^+_{(\beta_1,\mathrm{T}_2)}(x)-\psi^+(x)\})\\ &= \varphi^+(x)-\beta_1-\inf\{\varphi^+_{(\beta_1,\mathrm{T}_2)}(x)-\psi^+(x)\}\\ &= \varphi^+(x)-\beta_1+\sup\{\psi^+(x)-\varphi^+_{(\beta_1,\mathrm{T}_2)}(x)\}\\ &\geq \varphi^+_{(\beta_1,\mathrm{T}_2)}(x)+\psi^+(x)-\varphi^+_{(\beta_1,\mathrm{T}_2)}(x)\\ &= \psi^+(x).\end{aligned}$$

Thus $\psi^+(x)\leq\varphi^+_{(\beta_2,\mathrm{T}_2)}(x)\leq\varphi^+_{(\beta_1,\mathrm{T}_2)}(x)$ for all $x\in A$. Hence, $\psi\subseteq\varphi^{\mathrm{T}_2}_{(\alpha_2,\beta_2)}\subseteq\varphi^{\mathrm{T}_2}_{(\alpha_1,\beta_1)}$. □

Definition 3.32. Let $\varphi = (A; \varphi^-, \varphi^+)$ and $\psi = (A; \psi^-, \psi^+)$ be bipolar fuzzy sets in A with $\psi \subseteq \varphi$. If $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-subalgebra (resp., bipolar fuzzy UP-filter, bipolar fuzzy UP-ideal, bipolar fuzzy strongly UP-ideal) of A, then $\psi = (A; \psi^-, \psi^+)$ is a bipolar fuzzy UP-subalgebra (resp., bipolar fuzzy UP-filter, bipolar fuzzy UP-ideal, bipolar fuzzy strongly UP-ideal) of A, and we say that $\psi = (A; \psi^-, \psi^+)$ is a *bipolar fuzzy UP-subalgebra* (resp., bipolar fuzzy UP-filter, bipolar fuzzy UP-ideal, bipolar fuzzy strongly UP-ideal) *intension* of $\varphi = (A; \varphi^-, \varphi^+)$.

Theorem 3.33. *If $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-subalgebra (resp., bipolar fuzzy UP-filter, bipolar fuzzy UP-ideal, bipolar fuzzy strongly UP-ideal) of A, then the bipolar fuzzy (α, β)-translation ${\varphi^{\mathrm{T}_2}}_{(\alpha,\beta)} = (A; {\varphi^-}_{(\alpha,\mathrm{T}_2)}, {\varphi^+}_{(\beta,\mathrm{T}_2)})$ of $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-subalgebra (resp., bipolar fuzzy UP-filter, bipolar fuzzy UP-ideal, bipolar fuzzy strongly UP-ideal) intension of $\varphi = (A; \varphi^-, \varphi^+)$.*

Proof. It follows form Theorem 3.22 (resp., Theorem 3.24, Theorem 3.26, Theorem 3.28) and Remark 3.30. □

Theorem 3.34. *If $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-subalgebra (resp., bipolar fuzzy UP-filter, bipolar fuzzy UP-ideal, bipolar fuzzy strongly UP-ideal) of A, then the bipolar fuzzy (α_1, β_1)-translation ${\varphi^{\mathrm{T}_2}}_{(\alpha_1,\beta_1)} = (A; {\varphi^-}_{(\alpha_1,\mathrm{T}_2)}, {\varphi^+}_{(\beta_1,\mathrm{T}_2)})$ of $\varphi = (A; \varphi^-, \varphi^+)$ is a bipolar fuzzy UP-subalgebra (resp., bipolar fuzzy UP-filter, bipolar fuzzy UP-ideal, bipolar fuzzy strongly UP-ideal) intension of the bipolar fuzzy (α_2, β_2)-translation ${\varphi^{\mathrm{T}_2}}_{(\alpha_2,\beta_2)} = (A; {\varphi^-}_{(\alpha_2,\mathrm{T}_2)}, {\varphi^+}_{(\beta_2,\mathrm{T}_2)})$ of $\varphi = (A; \varphi^-, \varphi^+)$ with $(\alpha_1, \beta_1) \leq (\alpha_2, \beta_2)$.*

Proof. It follows form Theorem 3.22 (resp., Theorem 3.24, Theorem 3.26, Theorem 3.28). □

Theorem 3.35. *Let $\varphi = (A; \varphi^-, \varphi^+)$ be a bipolar fuzzy UP-subalgebra (resp., bipolar fuzzy UP-filter, bipolar fuzzy UP-ideal, bipolar fuzzy strongly UP-ideal) of A. For every bipolar fuzzy UP-subalgebra (resp., bipolar fuzzy UP-filter, bipolar fuzzy UP-ideal, bipolar fuzzy strongly UP-ideal) intension $\psi = (A; \psi^-, \psi^+)$ of the bipolar fuzzy (α, β)-translation $\varphi^{\mathrm{T}_2}_{(\alpha,\beta)} = (A; \varphi^-_{(\alpha,\mathrm{T}_2)}, \varphi^+_{(\beta,\mathrm{T}_2)})$ of $\varphi = (A; \varphi^-, \varphi^+)$ there exists $(k^-, k^+) \in [\pm, 0] \times [0, \mp]$ such that $(k^-, k^+) \leq (\alpha, \beta)$, that is, $k^- \geq \alpha$ and $k^+ \leq \beta$, and $\psi = (A; \psi^-, \psi^+)$ is a bipolar fuzzy UP-subalgebra (resp., bipolar fuzzy UP-filter, bipolar fuzzy UP-ideal, bipolar fuzzy strongly UP-ideal) intension of bipolar fuzzy (k^-, k^+)-translation $\varphi^{\mathrm{T}_2}_{(k^-,k^+)} = (A; \varphi^-_{(k^-,\mathrm{T}_2)}, \varphi^+_{(k^+,\mathrm{T}_2)})$ of $\varphi = (A; \varphi^-, \varphi^+)$.*

Proof. It follows form Theorem 3.22 (resp., Theorem 3.24, Theorem 3.26, Theorem 3.28) and Lemma 3.31. □

3.3 Complement of a bipolar fuzzy set

In this part, we discuss the relation between the complement of a bipolar fuzzy UP-subalgebra (resp., bipolar fuzzy UP-filter, bipolar fuzzy UP-ideal and bipolar fuzzy strongly UP-ideal) and its level cuts.

Definition 3.36. Let $\varphi = (A; \varphi^-, \varphi^+)$ be a bipolar fuzzy set in A. The bipolar fuzzy set $\overline{\varphi} = (A; \overline{\varphi^-}, \overline{\varphi^+})$ defined by: for all $x \in A$,

$$\begin{array}{rcl}\overline{\varphi^-}(x) & = & -1 - \varphi^-(x), \\ \overline{\varphi^+}(x) & = & 1 - \varphi^+(x),\end{array}$$

is called the *complement* of $\varphi = (A; \varphi^-, \varphi^+)$ in A.

Definition 3.37. Let $\varphi = (A; \varphi^-, \varphi^+)$ be a bipolar fuzzy set in A and for any $(t^-, t^+) \in [-1, 0] \times [0, 1]$. The sets

$$N_L(\varphi; t^-) = \{x \in A \mid \varphi^-(x) \leq t^-\}$$

and

$$N_U(\varphi; t^-) = \{x \in A \mid \varphi^-(x) \geq t^-\}$$

are called the *negative lower t^--cut* and the *negative upper t^--cut* of $\varphi = (A; \varphi^-, \varphi^+)$, respectively. The sets

$$P_L(\varphi; t^+) = \{x \in A \mid \varphi^+(x) \leq t^+\}$$

and

$$P_U(\varphi; t^+) = \{x \in A \mid \varphi^+(x) \geq t^+\}$$

are called the *positive lower t^+-cut* and the *positive upper t^+-cut* of $\varphi = (A; \varphi^-, \varphi^+)$, respectively.

Lemma 3.38. *Let $a, b, c \in \mathbb{R}$. Then the following statements hold:*

(1) $a - \min\{b, c\} = \max\{a - b, a - c\}$, *and*

(2) $a - \max\{b, c\} = \min\{a - b, a - c\}$.

Proof. (1) If $\min\{b, c\} = b$, then $c \geq b$. Thus $a - c \leq a - b$, so $\max\{a - b, a - c\} = a - b = a - \min\{b, c\}$. Similarly, if $\min\{b, c\} = c$, then

$$\max\{a - b, a - c\} = a - c = a - \min\{b, c\}.$$

(2) If $\max\{b, c\} = b$, then $b \geq c$. Thus $a - b \leq a - c$, so $\min\{a - b, a - c\} = a - b = a - \max\{b, c\}$. Similarly, if $\max\{b, c\} = c$, then

$$\min\{a - b, a - c\} = a - c = a - \max\{b, c\}.$$

□

Theorem 3.39. *Let $\varphi = (A; \varphi^-, \varphi^+)$ be a bipolar fuzzy set in A. Then $\overline{\varphi} = (A; \overline{\varphi^-}, \overline{\varphi^+})$ is a bipolar fuzzy UP-subalgebra of A if and only if for all $(t^-, t^+) \in [-1, 0] \times [0, 1], N_U(\varphi; t^-)$ and $P_L(\varphi; t^+)$ are UP-subalgebras of A if $N_U(\varphi; t^-)$ and $P_L(\varphi; t^+)$ are nonempty.*

Proof. Assume that $\overline{\varphi} = (A; \overline{\varphi^-}, \overline{\varphi^+})$ is a bipolar fuzzy UP-subalgebra of A. Let $(t^-, t^+) \in [-1, 0] \times [0, 1]$ be such that $N_U(\varphi; t^-)$ and $P_L(\varphi; t^+)$ are nonempty.

(i) Let $x, y \in N_U(\varphi; t^-)$. Then $\varphi^-(x) \geq t^-$ and $\varphi^-(y) \geq t^-$, so t^- is a lower bound of $\{\varphi^-(x), \varphi^-(y)\}$. Since $\overline{\varphi} = (A; \overline{\varphi^-}, \overline{\varphi^+})$ is a bipolar fuzzy UP-subalgebra of A, we have $\overline{\varphi^-}(x \cdot y) \leq \max\{\overline{\varphi^-}(x), \overline{\varphi^-}(y)\}$. By Lemma 3.38 (1), we have $-1 - \varphi^-(x \cdot y) \leq \max\{-1 - \varphi^-(x), -1 - \varphi^-(y)\} = -1 - \min\{\varphi^-(x), \varphi^-(y)\}$. Thus $\varphi^-(x \cdot y) \geq \min\{\varphi^-(x), \varphi^-(y)\} \geq t^-$ and so $x \cdot y \in N_U(\varphi; t^-)$. Therefore, $N_U(\varphi; t^-)$ is a UP-subalgebra of A.

(ii) Let $x, y \in P_L(\varphi; t^+)$. Then $\varphi^+(x) \leq t^+$ and $\varphi^+(y) \leq t^+$, so t^+ is an upper bound of $\{\varphi^+(x), \varphi^+(y)\}$. Since $\overline{\varphi} = (A; \overline{\varphi^-}, \overline{\varphi^+})$ is a bipolar fuzzy UP-subalgebra of A, we have $\overline{\varphi^+}(x \cdot y) \geq \min\{\overline{\varphi^+}(x), \overline{\varphi^+}(y)\}$. By Lemma 3.38 (2), we have $1 - \varphi^+(x \cdot y) \geq \min\{1 - \varphi^+(x), 1 - \varphi^+(y)\} = 1 - \max\{\varphi^+(x), \varphi^+(y)\}$. Thus $\varphi^+(x \cdot y) \leq \max\{\varphi^+(x), \varphi^+(y)\} \leq t^+$ and so $x \cdot y \in P_L(\varphi; t^+)$. Therefore, $P_L(\varphi; t^+)$ is a UP-subalgebra of A.

Conversely, assume that for all $(t^-, t^+) \in [-1, 0] \times [0, 1], N_U(\varphi; t^-)$ and $P_L(\varphi; t^+)$ are UP-subalgebras of A if $N_U(\varphi; t^-)$ and $P_L(\varphi; t^+)$ are nonempty.

(i) Let $x, y \in A$. Then $\varphi^-(x), \varphi^-(y) \in [-1, 0]$. Choose $t^- = \min\{\varphi^-(x), \varphi^-(y)\}$. Thus $\varphi^-(x) \geq t^-$ and $\varphi^-(y) \geq t^-$, so $x, y \in N_U(\varphi; t^-) \neq \emptyset$. By assumption, we have $N_U(\varphi; t^-)$ is a UP-subalgebra of A and so $x \cdot y \in N_U(\varphi; t^-)$. Thus $\varphi^-(x \cdot y) \geq t^- = \min\{\varphi^-(x), \varphi^-(y)\}$. By Lemma 3.38 (1), we have

$$\begin{aligned}
\overline{\varphi^-}(x \cdot y) &= -1 - \varphi^-(x \cdot y) \\
&\leq -1 - \min\{\varphi^-(x), \varphi^-(y)\} \\
&= \max\{-1 - \varphi^-(x), -1 - \varphi^-(y)\} \\
&= \max\{\overline{\varphi^-}(x), \overline{\varphi^-}(y)\}.
\end{aligned}$$

(ii) Let $x, y \in A$. Then $\varphi^+(x), \varphi^+(y) \in [0, 1]$. Choose $t^+ = \max\{\varphi^+(x), \varphi^+(y)\}$. Thus $\varphi^+(x) \leq t^+$ and $\varphi^+(y) \leq t^+$, so $x, y \in P_L(\varphi; t^+) \neq \emptyset$. By assumption, we have $P_L(\varphi; t^+)$ is a UP-subalgebra of A and so $x \cdot y \in P_L(\varphi; t^+)$. Thus $\varphi^+(x \cdot y) \leq t^+ = \max\{\varphi^+(x), \varphi^+(y)\}$. By Lemma 3.38 (2), we have

$$\begin{aligned}
\overline{\varphi^+}(x \cdot y) &= 1 - \varphi^+(x \cdot y) \\
&\geq 1 - \max\{\varphi^+(x), \varphi^+(y)\} \\
&= \min\{1 - \varphi^+(x), 1 - \varphi^+(y)\} \\
&= \min\{\overline{\varphi^+}(x), \overline{\varphi^+}(y)\}.
\end{aligned}$$

Hence, $\overline{\varphi} = (A; \overline{\varphi^-}, \overline{\varphi^+})$ is a bipolar fuzzy UP-subalgebra of A. □

Theorem 3.40. *Let $\varphi = (A; \varphi^-, \varphi^+)$ be a bipolar fuzzy set in A. Then $\overline{\varphi} = (A; \overline{\varphi^-}, \overline{\varphi^+})$ is a bipolar fuzzy UP-filter of A if and only if for all $(t^-, t^+) \in [-1, 0] \times [0, 1], N_U(\varphi; t^-)$ and $P_L(\varphi; t^+)$ are UP-filters of A if $N_U(\varphi; t^-)$ and $P_L(\varphi; t^+)$ are nonempty.*

Proof. Assume that $\overline{\varphi} = (A; \overline{\varphi^-}, \overline{\varphi^+})$ is a bipolar fuzzy UP-filter of A. Let $(t^-, t^+) \in [-1, 0] \times [0, 1]$ be such that $N_U(\varphi; t^-)$ and $P_L(\varphi; t^+)$ are nonempty.

(i) Let $a \in N_U(\varphi; t^-)$. Then $\varphi^-(a) \geq t^-$. Since $\overline{\varphi} = (A; \overline{\varphi^-}, \overline{\varphi^+})$ is a bipolar fuzzy UP-filter of A, we have $\overline{\varphi^-}(0) \leq \overline{\varphi^-}(a)$. Thus $-1 - \varphi^-(0) \leq -1 - \varphi^-(a)$, so $\varphi^-(0) \geq \varphi^-(a) \geq t^-$. Hence, $0 \in N_U(\varphi; t^-)$. Next, let $x, y \in A$ be such that $x \cdot y \in N_U(\varphi; t^-)$ and $x \in N_U(\varphi; t^-)$. Then $\varphi^-(x \cdot y) \geq t^-$ and $\varphi^-(x) \geq t^-$, so t^- is a lower bound of $\{\varphi^-(x \cdot y), \varphi^-(x)\}$. Since $\overline{\varphi} = (A; \overline{\varphi^-}, \overline{\varphi^+})$ is a bipolar fuzzy UP-filter of A, we have $\overline{\varphi^-}(y) \leq \max\{\overline{\varphi^-}(x \cdot y), \overline{\varphi^-}(x)\}$. By Lemma 3.38 (1), we have $-1 - \varphi^-(y) \leq \max\{-1 - \varphi^-(x \cdot y), -1 - \varphi^-(x)\} = -1 - \min\{\varphi^-(x \cdot y), \varphi^-(x)\}$. Thus $\varphi^-(y) \geq \min\{\varphi^-(x \cdot y), \varphi^-(x)\} \geq t^-$ and so $y \in N_U(\varphi; t^-)$. Therefore, $N_U(\varphi; t^-)$ is a UP-filter of A.

(ii) Let $b \in P_L(\varphi; t^+)$. Then $\varphi^+(b) \leq t^+$. Since $\overline{\varphi} = (A; \overline{\varphi^-}, \overline{\varphi^+})$ is bipolar fuzzy UP-filter of A, we have $\overline{\varphi^+}(0) \geq \overline{\varphi^+}(b)$. Thus $1 - \varphi^+(0) \geq 1 - \varphi^+(b)$, so $\varphi^+(0) \leq \varphi^+(b) \leq t^+$. Hence, $0 \in P_L(\varphi; t^+)$. Next, let $x, y \in A$ be such that $x \cdot y \in P_L(\varphi; t^+)$ and $x \in P_L(\varphi; t^+)$. Then $\varphi^+(x \cdot y) \leq t^+$ and $\varphi^+(x) \leq t^+$, so t^+ is an upper bound of $\{\varphi^+(x \cdot y), \varphi^+(x)\}$. Since $\overline{\varphi} = (A; \overline{\varphi^-}, \overline{\varphi^+})$ is a bipolar fuzzy UP-filter of A, we have $\overline{\varphi^+}(y) \geq \min\{\overline{\varphi^+}(x \cdot y), \overline{\varphi^+}(x)\}$. By Lemma 3.38 (2), we have $1 - \varphi^+(y) \geq \min\{1 - \varphi^+(x \cdot y), 1 - \varphi^+(x)\} = 1 - \max\{\varphi^-(x \cdot y), \varphi^+(x)\}$. Thus $\varphi^+(y) \leq \max\{\varphi^+(x \cdot y), \varphi^+(x)\} \leq t^+$ and so $y \in P_L(\varphi; t^+)$. Therefore, $P_L(\varphi; t^+)$ is a UP-filter of A.

Conversely, assume that for all $(t^-, t^+) \in [-1, 0] \times [0, 1], N_U(\varphi; t^-)$ and $P_L(\varphi; t^+)$ are UP-filters of A if $N_U(\varphi; t^-)$ and $P_L(\varphi; t^+)$ are nonempty.

(i) Let $x \in A$. Then $\varphi^-(x) \in [-1, 0]$. Choose $t^- = \varphi^-(x)$. Thus $\varphi^-(x) \geq t^-$, so $x \in N_U(\varphi; t^-) \neq \emptyset$. By assumption, we have $N_U(\varphi; t^-)$ is a UP-filter of A and so $0 \in N_U(\varphi; t^-)$. Thus $\varphi^-(0) \geq t^- = \varphi^-(x)$ and so $\overline{\varphi^-}(0) = -1 - \varphi^-(0) \leq -1 - \varphi^-(x) = \overline{\varphi^-}(x)$.

(ii) Let $x, y \in A$. Then $\varphi^-(x \cdot y), \varphi^-(x) \in [-1, 0]$. Choose $t^- = \min\{\varphi^-(x \cdot y), \varphi^-(x)\}$. Thus $\varphi^-(x \cdot y) \geq t^-$ and $\varphi^-(x) \geq t^-$, so $x \cdot y, x \in N_U(\varphi; t^-) \neq \emptyset$. By assumption, we have $N_U(\varphi; t^-)$ is a UP-filter of A and so $y \in N_U(\varphi; t^-)$. Thus $\varphi^-(y) \geq t^- = \min\{\varphi^-(x \cdot y), \varphi^-(x)\}$. By Lemma 3.38 (1), we have

$$\begin{aligned}
\overline{\varphi^-}(y) &= -1 - \varphi^-(y) \\
&\leq -1 - \min\{\varphi^-(x \cdot y), \varphi^-(x)\} \\
&= \max\{-1 - \varphi^-(x \cdot y), -1 - \varphi^-(x)\} \\
&= \max\{\overline{\varphi^-}(x \cdot y), \overline{\varphi^-}(x)\}.
\end{aligned}$$

(iii) Let $x \in A$. Then $\varphi^+(x) \in [0, 1]$. Choose $t^+ = \varphi^+(x)$. Thus $\varphi^+(x) \leq t^+$, so $x \in P_L(\varphi; t^+) \neq \emptyset$. By assumption, we have $P_L(\varphi; t^+)$ is a UP-filter of A and so $0 \in P_L(\varphi; t^+)$. Thus $\varphi^+(0) \leq t^+ = \varphi^+(x)$ and so $\overline{\varphi^+}(0) = 1 - \varphi^+(0) \geq 1 - \varphi^+(x) = \overline{\varphi^+}(x)$.

(iv) Let $x, y \in A$. Then $\varphi^+(x \cdot y), \varphi^+(x) \in [0, 1]$. Choose $t^+ = \max\{\varphi^+(x \cdot y), \varphi^+(x)\}$. Thus $\varphi^+(x \cdot y) \leq t^+$ and $\varphi^+(x) \leq t^+$, so $x \cdot y, x \in P_L(\varphi; t^+) \neq \emptyset$. By assumption, we have $P_L(\varphi; t^+)$ is a UP-filter of A and so $y \in P_L(\varphi; t^+)$. Thus $\varphi^+(y) \leq t^+ = \max\{\varphi^+(x \cdot y), \varphi^+(x)\}$. By Lemma 3.38 (2), we have

$$\begin{aligned}
\overline{\varphi^+}(y) &= 1-\varphi^+(y) \\
&\geq 1-\max\{\varphi^+(x\cdot y),\varphi^+(x)\} \\
&= \min\{1-\varphi^+(x\cdot y),1-\varphi^+(x)\} \\
&= \min\{\overline{\varphi^+}(x\cdot y),\overline{\varphi^+}(x)\}.
\end{aligned}$$

Hence, $\overline{\varphi}=(A;\overline{\varphi^-},\overline{\varphi^+})$ is a bipolar fuzzy UP-filter of A. □

Theorem 3.41. *Let $\varphi=(A;\varphi^-,\varphi^+)$ be a bipolar fuzzy set in A. Then $\overline{\varphi}=(A;\overline{\varphi^-},\overline{\varphi^+})$ is a bipolar fuzzy UP-ideal of A if and only if for all $(t^-,t^+)\in[-1,0]\times[0,1], N_U(\varphi;t^-)$ and $P_L(\varphi;t^+)$ are UP-ideals of A if $N_U(\varphi;t^-)$ and $P_L(\varphi;t^+)$ are nonempty.*

Proof. Assume that $\overline{\varphi}=(A;\overline{\varphi^-},\overline{\varphi^+})$ is a bipolar fuzzy UP-ideal of A. Let $(t^-,t^+)\in[-1,0]\times[0,1]$ be such that $N_U(\varphi;t^-)$ and $P_L(\varphi;t^+)$ are nonempty.

(i) Let $a\in N_U(\varphi;t^-)$. Then $\varphi^-(a)\geq t^-$. Since $\overline{\varphi}=(A;\overline{\varphi^-},\overline{\varphi^+})$ is a bipolar fuzzy UP-ideal of A, we have $\overline{\varphi^-}(0)\leq\overline{\varphi^-}(a)$. Thus $-1-\varphi^-(0)\leq-1-\varphi^-(a)$, so $\varphi^-(0)\geq\varphi^-(a)\geq t^-$. Hence, $0\in N_U(\varphi;t^-)$.

(ii) Let $x,y,z\in A$ be such that $x\cdot(y\cdot z)\in N_U(\varphi;t^-)$ and $y\in N_U(\varphi;t^-)$. Then $\varphi^-(x\cdot(y\cdot z))\geq t^-$ and $\varphi^-(y)\geq t^-$, so t^- is a lower bound of $\{\varphi^-(x\cdot(y\cdot z)),\varphi^-(y)\}$. Since $\overline{\varphi}=(A;\overline{\varphi^-},\overline{\varphi^+})$ is a bipolar fuzzy UP-ideal of A, we have $\overline{\varphi^-}(x\cdot z)\leq\max\{\overline{\varphi^-}(x\cdot(y\cdot z)),\overline{\varphi^-}(y)\}$. By Lemma 3.38 (1), we have $-1-\varphi^-(x\cdot z)\leq\max\{-1-\varphi^-(x\cdot(y\cdot z)),-1-\varphi^-(y)\}=-1-\min\{\varphi^-(x\cdot(y\cdot z)),\varphi^-(y)\}$. Thus $\varphi^-(x\cdot z)\geq\min\{\varphi^-(x\cdot(y\cdot z)),\varphi^-(y)\}\geq t^-$ and so $x\cdot z\in N_U(\varphi;t^-)$. Therefore, $N_U(\varphi;t^-)$ is a UP-ideal of A.

(iii) Let $b\in P_L(\varphi;t^+)$. Then $\varphi^+(b)\leq t^+$. Since $\overline{\varphi}=(A;\overline{\varphi^-},\overline{\varphi^+})$ is a bipolar fuzzy UP-ideal of A, we have $\overline{\varphi^+}(0)\geq\overline{\varphi^+}(b)$. Thus $1-\varphi^+(0)\geq1-\varphi^+(b)$, so $\varphi^+(0)\leq\varphi^+(b)\leq t^+$. Hence, $0\in P_L(\varphi;t^+)$.

(iv) Let $x,y,z\in A$ be such that $x\cdot(y\cdot z)\in P_L(\varphi;t^+)$ and $y\in P_L(\varphi;t^+)$. Then $\varphi^+(x\cdot(y\cdot z))\leq t^+$ and $\varphi^+(y)\leq t^+$, so t^+ is an upper bound of $\{\varphi^+(x\cdot(y\cdot z)),\varphi^+(y)\}$. Since $\overline{\varphi}=(A;\overline{\varphi^-},\overline{\varphi^+})$ is a bipolar fuzzy UP-ideal of A, we have $\overline{\varphi^+}(x\cdot z)\geq\min\{\overline{\varphi^+}(x\cdot(y\cdot z)),\overline{\varphi^+}(y)\}$. By Lemma 3.38 (2), we have $1-\varphi^+(x\cdot z)\geq\min\{1-\varphi^+(x\cdot(y\cdot z)),1-\varphi^+(y)\}=1-\max\{\varphi^+(x\cdot(y\cdot z)),\varphi^+(y)\}$. Thus $\varphi^+(x\cdot z)\leq\max\{\varphi^+(x\cdot(y\cdot z)),\varphi^+(y)\}\leq t^+$ and so $x\cdot z\in P_L(\varphi;t^+)$. Therefore, $P_L(\varphi;t^+)$ is a UP-ideal of A.

Conversely, assume that for all $(t^-,t^+)\in[-1,0]\times[0,1], N_U(\varphi;t^-)$ and $P_L(\varphi;t^+)$ are UP-ideals of A if $N_U(\varphi;t^-)$ and $P_L(\varphi;t^+)$ are nonempty.

(i) Let $x\in A$. Then $\varphi^-(x)\in[-1,0]$. Choose $t^-=\varphi^-(x)$. Thus $\varphi^-(x)\geq t^-$, so $x\in N_U(\varphi;t^-)\neq\emptyset$. By assumption, we have $N_U(\varphi;t^-)$ is a UP-ideal of A and so $0\in N_U(\varphi;t^-)$. Thus $\varphi^-(0)\geq t^-=\varphi^-(x)$ and so $\overline{\varphi^-}(0)=-1-\varphi^-(0)\leq-1-\varphi^-(x)=\overline{\varphi^-}(x)$.

(ii) Let $x,y,z\in A$. Then $\varphi^-(x\cdot(y\cdot z)),\varphi^-(y)\in[-1,0]$. Choose $t^-=\min\{\varphi^-(x\cdot(y\cdot z)),\varphi^-(y)\}$. Thus $\varphi^-(x\cdot(y\cdot z))\geq t^-$ and $\varphi^-(y)\geq t^-$, so $x\cdot(y\cdot z),y\in N_U(\varphi;t^-)\neq\emptyset$. By assumption, we have $N_U(\varphi;t^-)$ is a UP-ideal of A and so $x\cdot z\in N_U(\varphi;t^-)$. Thus $\varphi^-(x\cdot z)\geq t^-=\min\{\varphi^-(x\cdot(y\cdot z)),\varphi^-(y)\}$. By Lemma 3.38 (1), we have

$$\begin{aligned}\overline{\varphi^-}(x\cdot z) &= -1-\varphi^-(x\cdot z)\\ &\leq -1-\min\{\varphi^-(x\cdot(y\cdot z)),\varphi^-(y)\}\\ &= \max\{-1-\varphi^-(x\cdot(y\cdot z)),-1-\varphi^-(y)\}\\ &= \max\{\overline{\varphi^-}(x\cdot(y\cdot z)),\overline{\varphi^-}(y)\}.\end{aligned}$$

(iii) Let $x \in A$. Then $\varphi^+(x) \in [0,1]$. Choose $t^+ = \varphi^+(x)$. Thus $\varphi^+(x) \leq t^+$, so $x \in P_L(\varphi;t^+) \neq \emptyset$. By assumption, we have $P_L(\varphi;t^+)$ is a UP-ideal of A and so $0 \in P_L(\varphi;t^+)$. Thus $\varphi^+(0) \leq t^+ = \varphi^+(x)$ and so $\overline{\varphi^+}(0) = 1-\varphi^+(0) \geq 1-\varphi^+(x) = \overline{\varphi^+}(x)$.

(iv) Let $x,y,z \in A$. Then $\varphi^+(x\cdot(y\cdot z)),\varphi^+(y) \in [0,1]$. Choose $t^+ = \max\{\varphi^+(x\cdot(y\cdot z)),\varphi^+(y)\}$. Thus $\varphi^+(x\cdot(y\cdot z)) \leq t^+$ and $\varphi^+(y) \leq t^+$, so $x\cdot(y\cdot z), y \in P_L(\varphi;t^+) \neq \emptyset$. By assumption, we have $P_L(\varphi;t^+)$ is a UP-ideal of A and so $x\cdot z \in P_L(\varphi;t^+)$. Thus $\varphi^+(x\cdot z) \leq t^+ = \max\{\varphi^+(x\cdot(y\cdot z)),\varphi^+(y)\}$. By Lemma 3.38 (2), we have

$$\begin{aligned}\overline{\varphi^+}(x\cdot z) &= 1-\varphi^+(x\cdot z)\\ &\geq 1-\max\{\varphi^+(x\cdot(y\cdot z)),\varphi^+(y)\}\\ &= \min\{1-\varphi^+(x\cdot(y\cdot z)),1-\varphi^+(y)\}\\ &= \min\{\overline{\varphi^+}(x\cdot(y\cdot z)),\overline{\varphi^+}(y)\}.\end{aligned}$$

Hence, $\overline{\varphi} = (A;\overline{\varphi^-},\overline{\varphi^+})$ is a bipolar fuzzy UP-ideal of A. □

Theorem 3.42. *Let $\varphi = (A;\varphi^-,\varphi^+)$ be a bipolar fuzzy set in A. Then $\overline{\varphi} = (A;\overline{\varphi^-},\overline{\varphi^+})$ is a bipolar fuzzy strongly UP-ideal of A if and only if for all $(t^-,t^+) \in [-1,0]\times[0,1]$, $N_U(\varphi;t^-)$ and $P_L(\varphi;t^+)$ are strongly UP-ideals of A if $N_U(\varphi;t^-)$ and $P_L(\varphi;t^+)$ are nonempty.*

Proof. Assume that $\overline{\varphi} = (A;\overline{\varphi^-},\overline{\varphi^+})$ is a bipolar fuzzy strongly UP-ideal of A. Let $(t^-,t^+) \in [-1,0]\times[0,1]$ be such that $N_U(\varphi;t^-)$ and $P_L(\varphi;t^+)$ are nonempty.

(i) Let $a \in N_U(\varphi;t^-)$. Then $\varphi^-(a) \geq t^-$. Since $\overline{\varphi} = (A;\overline{\varphi^-},\overline{\varphi^+})$ is a bipolar fuzzy strongly UP-ideal of A, we have $\overline{\varphi^-}(0) \leq \overline{\varphi^-}(a)$. Thus $-1-\varphi^-(0) \leq -1-\varphi^-(a)$, so $\varphi^-(0) \geq \varphi^-(a) \geq t^-$. Hence, $0 \in N_U(\varphi;t^-)$.

(ii) Let $x,y,z \in A$ be such that $(z\cdot y)\cdot(z\cdot x) \in N_U(\varphi;t^-)$ and $y \in N_U(\varphi;t^-)$. Then $\varphi^-((z\cdot y)\cdot(z\cdot x)) \geq t^-$ and $\varphi^-(y) \geq t^-$, so t^- is a lower bound of $\{\varphi^-((z\cdot y)\cdot(z\cdot x)),\varphi^-(y)\}$. Since $\overline{\varphi} = (A;\overline{\varphi^-},\overline{\varphi^+})$ is a bipolar fuzzy strongly UP-ideal of A, we have $\overline{\varphi^-}(x) \leq \max\{\overline{\varphi^-}((z\cdot y)\cdot(z\cdot x)),\overline{\varphi^-}(y)\}$. By Lemma 3.38 (1), we have $-1-\varphi^-(x) \leq \max\{-1-\varphi^-((z\cdot y)\cdot(z\cdot x)),-1-\varphi^-(y)\} = -1-\min\{\varphi^-((z\cdot y)\cdot(z\cdot x)),\varphi^-(y)\}$. Thus $\varphi^-(x) \geq \min\{\varphi^-((z\cdot y)\cdot(z\cdot x)),\varphi^-(y)\} \geq t^-$ and so $x \in N_U(\varphi;t^-)$. Therefore, $N_U(\varphi;t^-)$ is a strongly UP-ideal of A.

(iii) Let $b \in P_L(\varphi;t^+)$. Then $\varphi^+(b) \leq t^+$. Since $\overline{\varphi} = (A;\overline{\varphi^-},\overline{\varphi^+})$ is a bipolar fuzzy strongly UP-ideal of A, we have $\overline{\varphi^+}(0) \geq \overline{\varphi^+}(b)$. Thus $1-\varphi^+(0) \geq 1-\varphi^+(b)$, so $\varphi^+(0) \leq \varphi^+(b) \leq t^+$. Hence, $0 \in P_L(\varphi;t^+)$.

(iv) Let $x,y,z \in A$ be such that $(z\cdot y)\cdot(z\cdot x) \in P_L(\varphi;t^+)$ and $y \in P_L(\varphi;t^+)$. Then $\varphi^+((z\cdot y)\cdot(z\cdot x)) \leq t^+$ and $\varphi^+(y) \leq t^+$, so t^+ is an upper bound of $\{\varphi^+((z\cdot y)\cdot(z\cdot x)),\varphi^+(y)\}$. Since $\overline{\varphi} = (A;\overline{\varphi^-},\overline{\varphi^+})$ is a bipolar fuzzy strongly UP-ideal of A, we have $\overline{\varphi^+}(x) \geq \min\{\overline{\varphi^+}((z\cdot y)\cdot(z\cdot x)),\overline{\varphi^+}(y)\}$. By Lemma 3.38

(2), we have $1-\varphi^+(x) \geq \min\{1-\varphi^+((z\cdot y)\cdot(z\cdot x)), 1-\varphi^+(y)\} = 1-\max\{\varphi^+((z\cdot y)\cdot(z\cdot x)), \varphi^+(y)\}$. Thus $\varphi^+(x) \leq \max\{\varphi^+((z\cdot y)\cdot(z\cdot x)), \varphi^+(y)\} \leq t^+$ and so $x \in P_L(\varphi; t^+)$. Therefore, $P_L(\varphi; t^+)$ is a strongly UP-ideal of A.

Conversely, assume that for all $(t^-, t^+) \in [-1,0]\times[0,1], N_U(\varphi; t^-)$ and $P_L(\varphi; t^+)$ are strongly UP-ideals of A if $N_U(\varphi; t^-)$ and $P_L(\varphi; t^+)$ are nonempty.

(i) Let $x \in A$. Then $\varphi^-(x) \in [-1,0]$. Choose $t^- = \varphi^-(x)$. Thus $\varphi^-(x) \geq t^-$, so $x \in N_U(\varphi; t^-) \neq \emptyset$. By assumption, we have $N_U(\varphi; t^-)$ is a strongly UP-ideal of A and so $0 \in N_U(\varphi; t^-)$. Thus $\varphi^-(0) \geq t^- = \varphi^-(x)$ and so $\overline{\varphi^-}(0) = -1-\varphi^-(0) \leq -1-\varphi^-(x) = \overline{\varphi^-}(x)$.

(ii) Let $x, y, z \in A$. Then $\varphi^-((z\cdot y)\cdot(z\cdot x)), \varphi^-(y) \in [-1,0]$. Choose $t^- = \min\{\varphi^-((z\cdot y)\cdot(z\cdot x)), \varphi^-(y)\}$. Thus $\varphi^-((z\cdot y)\cdot(z\cdot x)) \geq t^-$ and $\varphi^-(y) \geq t^-$, so $(z\cdot y)\cdot(z\cdot x), y \in N_U(\varphi; t^-) \neq \emptyset$. By assumption, we have $N_U(\varphi; t^-)$ is a strongly UP-ideal of A and so $x \in N_U(\varphi; t^-)$. Thus $\varphi^-(x) \geq t^- = \min\{\varphi^-((z\cdot y)\cdot(z\cdot x)), \varphi^-(y)\}$. By Lemma 3.38 (1), we have

$$\begin{aligned}\overline{\varphi^-}(x) &= -1-\varphi^-(x)\\ &\leq -1-\min\{\varphi^-((z\cdot y)\cdot(z\cdot x)), \varphi^-(y)\}\\ &= \max\{-1-\varphi^-((z\cdot y)\cdot(z\cdot x)), -1-\varphi^-(y)\}\\ &= \max\{\overline{\varphi^-}((z\cdot y)\cdot(z\cdot x)), \overline{\varphi^-}(y)\}.\end{aligned}$$

(iii) Let $x \in A$. Then $\varphi^+(x) \in [0,1]$. Choose $t^+ = \varphi^+(x)$. Thus $\varphi^+(x) \leq t^+$, so $x \in P_L(\varphi; t^+) \neq \emptyset$. By assumption, we have $P_L(\varphi; t^+)$ is a strongly UP-ideal of A and so $0 \in P_L(\varphi; t^+)$. Thus $\varphi^+(0) \leq t^+ = \varphi^+(x)$ and so $\overline{\varphi^+}(0) = 1-\varphi^+(0) \geq 1-\varphi^+(x) = \overline{\varphi^+}(x)$.

(iv) Let $x, y, z \in A$. Then $\varphi^+((z\cdot y)\cdot(z\cdot x)), \varphi^+(y) \in [0,1]$. Choose $t^+ = \max\{\varphi^+((z\cdot y)\cdot(z\cdot x)), \varphi^+(y)\}$. Thus $\varphi^+((z\cdot y)\cdot(z\cdot x))) \leq t^+$ and $\varphi^+(y) \leq t^+$, so $(z\cdot y)\cdot(z\cdot x), y \in P_L(\varphi; t^+) \neq \emptyset$. By assumption, we have $P_L(\varphi; t^+)$ is a strongly UP-ideal of A and so $x \in P_L(\varphi; t^+)$. Thus $\varphi^+(x) \leq t^+ = \max\{\varphi^+((z\cdot y)\cdot(z\cdot x)), \varphi^+(y)\}$. By Lemma 3.38 (2), we have

$$\begin{aligned}\overline{\varphi^+}(x) &= 1-\varphi^+(x)\\ &\geq 1-\max\{\varphi^+((z\cdot y)\cdot(z\cdot x)), \varphi^+(y)\}\\ &= \min\{1-\varphi^+((z\cdot y)\cdot(z\cdot x)), 1-\varphi^+(y)\}\\ &= \min\{\overline{\varphi^+}((z\cdot y)\cdot(z\cdot x)), \overline{\varphi^+}(y)\}.\end{aligned}$$

Hence, $\overline{\varphi} = (A; \overline{\varphi^-}, \overline{\varphi^+})$ is a bipolar fuzzy strongly UP-ideal of A. □

4. Conclusions and future work

In the present paper, we have introduced the notions of bipolar fuzzy (α, β)-translations of $\varphi = (A; \varphi^-, \varphi^+)$ of type I and of type II for a bipolar-valued fuzzy set $\varphi = (A; \varphi^-, \varphi^+)$ in a UP-algebra A. The notions of extensions and of intensions of a bipolar-valued fuzzy set are also studied. We think this work would enhance the scope for further study in UP-algebras and related algebraic systems. It is our hope that this work would serve as a foundation for the further study in a new concept of UP-algebras.

In our future study of UP-algebras, may be the following topics should be considered:

- To get more results in bipolar fuzzy translations of a bipolar-valued fuzzy set in UP-algebras.
- To define bipolar-valued fuzzy sets with thresholds in UP-algebras.
- To define bipolar-valued fuzzy soft sets in UP-algebras.

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