CRITERION FOR NONEXISTENCE HORSESHOE-LIKE IN
$C^1$ TOPOLOGY

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Abstract. In this paper we show that if $\Lambda \subset M$ is a closed invariant set and $p \in \Lambda$ is a hyperbolic saddle periodic point satisfying condition $A$ with real and positive eigenvalues, then $\Lambda$ is not horseshoe-like.

Keywords: hyperbolic set, partially hyperbolic set, horseshoe.

1. Introduction

Bowen in his remarkable survey on Anosov diffeomorphism has proved that $C^{1+}$-diffeomorphisms do not have fat horseshoes, these are horseshoes of positive Lebesgue measure. In contrast, he gave an example of a totally disconnected horseshoe on sphere $S^2$ of positive volume. On the other hand, Bowen has proved that a basic set (locally maximal hyperbolic set with a dense orbit) of a $C^2$ diffeomorphism which attracts a set with positive volume, necessarily attracts a neighborhood of itself [3 Theorem 4.11]. In particular, the unstable manifolds through points of this set must be contained in it, and consequently $C^2$ diffeomorphisms have no horseshoes with positive volume. In this context A. Fakhari and M. Soufi proved that any partially hyperbolic horseshoe-like attractor of a $C^1$-generic diffeomorphism has zero volume [4]. As well they constructed a $C^1$-diffeomorphism with a partially hyperbolic horseshoe-like attractor of positive volume. In this paper we show that under some conditions there is no horseshoe-like in the context of $C^1$-diffeomorphisms. Indeed we show that if $\Lambda \subset M$ is a closed invariant set and $p \in \Lambda$ is a hyperbolic saddle periodic point satisfying condition $A$ with real and positive eigenvalues, then $\Lambda$ is not horseshoe-like.

Let $f : M \to M$ be a diffeomorphism of a compact connected Riemannian manifold $M$. A set $\Lambda$ is said to be invariant relative to $f$ if $f(\Lambda) = \Lambda$.

For a point $x \in M$ the stable set of $x$ is

$$W^s(x) = \{y \in M : d(f^k(x), f^k(y)) \to 0 \quad as \quad k \to +\infty\}$$

and the unstable of $x$ is

$$W^u(x) = \{y \in M : d(f^k(x), f^k(y)) \to 0 \quad as \quad k \to -\infty\}.$$
Let $O(p)$ be a hyperbolic periodic orbit of $f$, then the dimension of unstable manifold of $p$ is called index of $p$.

A compact invariant set $\Lambda$ is said to be horseshoe-like if there are local stable and local unstable manifolds through all its points which intersect $\Lambda$ in a Cantor set.

A splitting $T_\Lambda M = E \oplus F$ of the tangent bundle restricted to an invariant set $\Lambda$ is dominated splitting if there is a constant $0 < \lambda < 1$ such that for some choice of a Riemannian metric on $M$

$$\|Df|_{E_x}\| \|Df^{-1}|_{F_{f(x)}}\| \leq \lambda, \quad \text{for every } x \in \Lambda.$$ 

$\Lambda$ is partially hyperbolic, if additionally $E$ is uniformly contracting or $F$ is uniformly expanding, i.e there exists $0 < \lambda < 1$ such that

$$\|Df|_{E_x}\| \leq \lambda \quad \text{or} \quad \|Df^{-1}|_{F_{f(x)}}\| \leq \lambda.$$ 

A compact invariant set $\Lambda$ is called hyperbolic if there is a $Df$-invariant splitting $T_\Lambda M = E^s \oplus E^u$ of the tangent bundle restricted to $\Lambda$ and a constant $\lambda < 1$ such that (for some choice of a Riemannian metric on $M$) for every $x \in \Lambda$

$$\|Df|_{E^s_x}\| < \lambda$$

and

$$\|Df^{-1}|_{E^u_x}\| < \lambda.$$ 

Alves and Pinheiro have studied nonuniformly expanding partially hyperbolic sets for $C^{1+}$ diffeomorphisms [1]. They have proved that if non-uniformly expanding condition holds for a positive Lebesgue set of points, then $\Lambda$ contains some local unstable disk. As a corollary, they deduced the non-existence of partially hyperbolic horseshoe like sets of positive volume. Also, Pacifico et al. have tried to construct Lorenz attractor of positive volume in the $C^1$-topology. The same result have obtained in the context of the volume preserving diffeomorphism. Indeed Xia proved in [2] that if an invariant set $\Lambda$ of a volume-preserving $C^{1+}$-diffeomorphism $f$ with positive volume has a dominated splitting $E \oplus F$, with $E$ is uniformly contractive, then $\Lambda$ contains stable leaves of almost every point. This argument leads to another proof of the classical result toward the ergodicity of $C^{1+}$ volume-preserving Anosov diffeomorphisms without using the Hopf argument.

2. Main theorems

In this section we present a condition that an invariant set satisfying it, is not horseshoe-like.
**Definition.** Let Λ be an invariant set. We say that a point \( p \in M \) satisfies condition \( A \) if there are a local chart \( h \) at \( p \) and sequences \( \{x_n\} \) and \( \{w_n\} \subset T_pM \), \( w_n = \sum_{i=1}^{m} \lambda_i^{w_n} v_i \) such that for \( 1 \leq i \leq m \),

\[
\begin{align*}
\lim_{n \to \infty} \frac{\lambda_i^{w_n} \lambda_i^{w_{n+1}}}{\lambda_i^{w_{n+1}}} &= 0 \\
\lambda_i^{w_{n+1}} < \lambda_i^{w_n} \\
\lim_{n \to \infty} \lambda_i^{w_n} &= 0 \\
h^{-1}(w_n) &= x_n \in \Lambda,
\end{align*}
\]

where \( \{v_1, v_2, ..., v_m\} \) is a basis of \( T_pM \).

**Remark.** In the above definition \( \lambda_i^{w_n} \) is a notation relative to \( w_n \) as a scaler. Indeed for any \( \alpha \in T_pM \), since \( \{v_1, v_2, ..., v_m\} \) is a basis of \( T_pM \), so one can write \( \alpha = \sum_{i=1}^{m} \lambda_i^{\alpha} v_i \) where \( \lambda_i^{\alpha} \) for \( 1 \leq i \leq m \), are scalers.

**Example 1.** Let \( f : M \to M \) be a \( C^1 \)-diffeomorphism on a \( C^\infty \)-manifold \( M \) with \( \dim M = 2 \) and \( p \in M \) be a hyperbolic fixed point of \( f \). Let \( f \) at \( p \) in local chart be as \( f(x, y) = (4x, \frac{1}{8}y) \) and \( \{(x_n, y_n)\} \) be a sequence such that \( x_n = \frac{1}{n} \), \( y_n \to 0 \) as the following figure.

![Diagram of Example 1](image)

If \( \Lambda \) is a closed invariant set containing \( p \) and \( \{(x_n, y_n)\} \subset \Lambda \), then \( p \) satisfies condition \( A \), since \( x_{n+1} < x_n \), \( \frac{x_n - x_{n+1}}{x_{n+1}} \to 0 \) and \( x_n \to 0 \).

**Definition.** Let \( \Lambda \) be a close invariant subset of the compact manifold \( M \). A point \( p \in \Lambda \) is said to be **topologically dense point** if

\[
\lim_{\delta \to 0} \frac{1}{\delta} \max \{ \epsilon > 0 \mid B_\epsilon(x) \cap \Lambda = \emptyset, \forall x \in B_\delta(p) \} = 0
\]

where \( B_r(z) = \{ x \in M \mid d(z, x) < r \} \).
Example 2. Let $f \in Diff^1(M)$ and $\Lambda \subset M$ be a closed invariant set which is not a periodic point containing a saddle fixed point $p \in \Lambda$ which is topologically dense and whose eigenvalues are real and positive.

$p$ is topologically dense therefore for every $m \in \mathbb{N}$ there are positive integers $\epsilon_m, \delta_m$ such that $\frac{\epsilon_m}{2\delta_m} \to 0$ as $m \to +\infty$ and $B_{\delta_m}(x) \cap \Lambda \neq \emptyset$ for every $x \in B_{2\delta_m}(p)$. Thus by induction we find sequence $\{q_m\} \subseteq \Lambda$ such that $d(q_m, q_{m+1}) = \epsilon_m$ and $d(q_m, p) = 2\delta_m$ (see the following Figure). So by taking suitable charts we can suppose that $p = 0$ and $\lim_{m \to \infty} \frac{q_m - q_{m+1}}{q_m} = 0$. Therefore $p$ satisfies condition $A$.

The following theorem shows that Example 2 is a prototype structures for an invariant set to be not horseshoe-like.

Theorem 1. Let $f \in Diff^1(M)$ and $\Lambda \subset M$ be a closed invariant set which is not a periodic point. Suppose $\Lambda$ contains a saddle fixed point $p$ satisfying condition $A$ with real and positive eigenvalues. Then $\Lambda$ is not horseshoe-like.

Proof. We show that there is a connected component in $\Lambda$ which is not consist of a single point. So $\Lambda$ is not a Cantor set and hence it is not horseshoe-like. Since $p$ is a hyperbolic point, there is an $\epsilon_0 > 0$ and a homeomorphism $h : B_{\epsilon_0}(p) \to T_pM$ such that

1. $h(p) = 0$
2. $D_pf h = h f$.

There exists $\epsilon' > 0$ such that

$$\{v \in T_pM \mid \|v\| < \epsilon'\} = h(B_{\epsilon_0}(p)).$$

Let $\{\lambda_i \mid 1 \leq i \leq s\}$ be the set of all eigenvalues of $D_p f$ which norm greater than 1. Denote by $\{\lambda_i \mid s + 1 \leq i \leq m\}$ the set of all eigenvalues of $D_p f$ which norm less than 1 and let $\{v_1, \ldots, v_s\}$ and $\{v_{s+1}, \ldots, v_m\}$ be the set of eigenvectors of $\{\lambda_i \mid 1 \leq i \leq s\}$ and $\{\lambda_i \mid s + 1 \leq i \leq m\}$ respectively. Put

$$L = \left\{ \sum_{i=1}^{m} \lambda_i v_i \mid 0 < \lambda_i < \frac{\epsilon'}{2m} \right\}$$
and
\[ L = \left\{ \sum_{i=1}^{s} \lambda_i v_i \mid 0 < \lambda_i < \frac{\epsilon'}{2s} \right\}. \]

We can see that \( h^{-1}(\bar{L}) \) and \( h^{-1}(L) \subset B_\epsilon(p) \). Since \( p \) is a saddle point with condition \( A \), there are sequences \( \{x_n\} \) and \( \{w_n\} \subset T_p M \), \( w_n = \sum_{i=1}^{m} \lambda_i^{w_n} v_i \) such that for \( 1 \leq i \leq s \), we have
\[
\begin{align*}
\text{(1.3)} & \quad \begin{cases}
\lim_{n \to \infty} \frac{\lambda_i^{w_n} - \lambda_i^{w_{n+1}}}{\lambda_i^{w_n}} = 0 \\
\lambda_i^{w_{n+1}} < \lambda_i^{w_n} \\
\lim_{n \to \infty} \lambda_i^{w_n} = 0 \\
h^{-1}(w_n) = x_n \in \Lambda
\end{cases}
\end{align*}
\]

Let \( z \in L \). For every \( s + 1 \leq i \leq m \) there exists \( N_i \in \mathbb{N} \) such that for any \( n \geq N_i \), \( 0 < \lambda_i^{w_n} < \frac{\delta}{2M(m-s)} \) where
\[ M = \max\{\tilde{\lambda}_i \mid 1 \leq i \leq m\}. \]

Put \( N_0 = \max\{N_i \mid s+1 \leq i \leq m\} \). Since \( z \in L \), we can consider \( z = \sum_{i=1}^{s} \lambda_i^z v_i \) such that \( 0 < \lambda_i^z < \frac{\epsilon'}{2s} \). For every \( k > N_0 \) put
\[ m_k = \min\left\{ m \mid \lambda_i^{w_{m+1}} \leq \frac{\lambda_i^z + \frac{\delta}{2s}}{\lambda_i^k} \right\} \]
for \( 1 \leq i \leq s \). So
\[ \lambda_i^{w_{m_k+1}} \leq \frac{\lambda_i^z + \frac{\delta}{2s}}{\lambda_i^k} \leq \lambda_i^{w_{m_k}} \]
for \( 1 \leq i \leq s \).

We claim that there exists \( k_0 > N_0 \) such that for \( 1 \leq i \leq s \),
\[ \lambda_i^{w_{m_{k_0}+1}} > \frac{\lambda_i^z - \frac{\delta}{2s}}{\lambda_i^{k_0}}. \]

Suppose our claim is not true. Hence for every \( k > N_0 \) and some \( 1 \leq i \leq s \)
\[ \lambda_i^{w_{m_k+1}} \leq \frac{\lambda_i^z - \frac{\delta}{2s}}{\lambda_i^k} < \frac{\lambda_i^z + \frac{\delta}{2s}}{\lambda_i^k} \leq \lambda_i^{w_{m_k}}. \]

So
\[ \frac{\lambda_i^{w_{m_k}} - \lambda_i^{w_{m_{k+1}}}}{\lambda_i^{w_{m_{k+1}}}} \geq \frac{\frac{2\delta}{2s\lambda_i^k}}{\lambda_i^{w_{m_{k+1}}}} = \frac{2\delta}{2s\lambda_i^k} + \frac{\delta}{2s} > 0 \]
that contradicts (1.3). Hence there exists \( n_0 > N_0 \) such that
\[ \frac{\lambda_i^z - \frac{\delta}{2s}}{\lambda_i^{n_0}} < \lambda_i^{w_{m_{n_0}+1}} < \frac{\lambda_i^z + \frac{\delta}{2s}}{\lambda_i^{n_0}}. \]
for $1 \leq i \leq s$. This shows that
\[
\sum_{i=1}^{s} \left| \tilde{\lambda}_i^{n_0} \lambda_i^{w_{m_0}+1} - \lambda_i^2 \right| < \delta \frac{2}{\delta}.
\]
Hence we have
\[
\left\| Df^{n_0} \left( \sum_{i=1}^{m} \lambda_i^{w_{m_0}+1} v_i \right) - z \right\|
= \left\| \sum_{i=1}^{s} \tilde{\lambda}_i^{n_0} \lambda_i^{w_{m_0}+1} v_i + \sum_{i=s+1}^{m} \tilde{\lambda}_i^{n_0} \lambda_i^{w_{m_0}+1} v_i - \sum_{i=1}^{s} \lambda_i^2 v_i \right\|
\leq \left\| \sum_{i=1}^{s} \tilde{\lambda}_i^{n_0} \lambda_i^{w_{m_0}+1} v_i - \sum_{i=1}^{s} \lambda_i^2 v_i \right\| + \left\| \sum_{i=s+1}^{m} \tilde{\lambda}_i^{n_0} \lambda_i^{w_{m_0}+1} v_i \right\| =: B
\]
Since for $s + 1 \leq i \leq m$, $\tilde{\lambda}_i^{n_0} < 1$ and $0 < \lambda_i^{w_{m_0}+1} < \frac{\delta}{2M(m-s)}$. Hence
\[
B \leq \sum_{i=1}^{s} \left| \tilde{\lambda}_i^{n_0} \lambda_i^{w_{m_0}+1} - \lambda_i^2 \right| + \sum_{i=s+1}^{m} \left| \tilde{\lambda}_i^{n_0} \lambda_i^{w_{m_0}+1} \right|
\leq \frac{\delta}{2} + \frac{(m-s)\delta}{2(m-s)M} < \delta.
\]
This shows that $Df^{n_0} \left( \sum_{i=1}^{m} \lambda_i^{w_{m_0}+1} v_i \right) \in B_\delta(z)$. This shows that for every $x \in h^{-1}(L)$ there is sequence such that
\[
D_p f^m(v_{nm}) \to h(x)
\{h(v_{nm}) = x_{nm}\} \subset \Lambda
\]
since $h^{-1} \circ D_p \circ h = f$ so
\[
f^m(x_{nm}) \to x.
\]
$\Lambda$ is closed and invariant so we have $x \in \Lambda$. Hence $h^{-1}(L) \subset \Lambda$. Note that $h^{-1}(L)$ is connected component. Hence $\Lambda$ is not like horseshoe. □

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