NEW CONCEPTS IN INTERVAL-VALUED INTUITIONISTIC FUZZY GRAPHS

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Abstract. Intuitionistic fuzzy graphs is a highly growing research area as it is the generalization of the fuzzy graphs. In this paper, we introduce the concept of Interval-valued Intuitionistic fuzzy graphs(IVIFG), we also analyse some properties of IVIFG based on morphism such as weak isomorphism, co-weak isomorphism and some concepts on automorphism.

Keywords: IVIFG, weak isomorphism of IVIFG, co-weak isomorphism of IVIFG.

1. Introduction

Graph theory has found its importance in many real time problems. Recent applications in graph theory is quite interesting analysing any complex situations and moreover in engineering applications. It has got numerous applications

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on operations research, system analysis, network routing, transportation and many more. To analyse any complete information we make intensive use of graphs and its properties. For working on partial information or incomplete information or to handle the systems containing the elements of uncertainty we understand that fuzzy logic and its involvement in graph theory is applied. In 1975, Rosenfeld [21] discussed the concept of fuzzy graphs whose ideas are implemented by Kaufmann [18] in 1973. The fuzzy relation between fuzzy sets were also considered by Rosenfeld who developed the structure of fuzzy graphs, obtaining various analogous results of several graph theoretical concepts. Bhattacharya [4] gave some remarks of fuzzy graphs. The complement of fuzzy graphs was introduced by Mordeson [19]. Atanassov introduced the concept of intuitionistic fuzzy relation and intuitionistic fuzzy graphs [2, 3, 32, 33]. Talebi and Rashmanlou [36] studied the properties of isomorphism and complement of interval-valued fuzzy graphs. They defined isomorphism and some new operations on vague graphs [37, 38]. Borzooei and Rashmalou analysed new concepts of vague graphs [5], degree of vertices in vague graphs [6], more results on vague graphs [7], semi global domination sets in vague graphs with application [8] and degree and total degree of edges in bipolar fuzzy graphs with application [9]. Rashmanlou et al., defined the complete interval-valued fuzzy graphs [24]. Rashmanlou and Pal studied intuitionistic fuzzy graphs with categorical properties [29], some properties of highly irregular interval-valued fuzzy graphs [28], more results on highly irregular bipolar fuzzy graphs [30], balanced interval-valued fuzzy graphs [26] and antipodal interval-valued fuzzy graphs [25]. Samanta and Pal investigated fuzzy k-competition and p-competition graphs, and concept of fuzzy planar graphs in [21, 22, 31]. Also they introduced fuzzy tolerance graph [34], bipolar fuzzy hypergraphs [35] and given several properties on it. Pal and Rashmanlou [20] defined many properties of irregular interval-valued fuzzy graphs. Ganesh et al. [10, 11] analysed the properties of Regular product vague graphs and product vague line graphs. The article has been composed of four sections. Ganesh et al. [12, 13, 14, 15] has analysed some concepts on faces and dual of m-polar fuzzy graphs, regular bipolar fuzzy graphs, isomorphic properties of m-polar fuzzy graphs and novel concepts on strongly edge irregular m-polar fuzzy graphs. In section 1, we introduce the survey of Interval-valued intuitionistic fuzzy graphs. In section 2 we define the preliminaries of Intuitionistic fuzzy graphs and basic definitions, definition of IVIFG. In section 3 we define automorphic IVIFG and analyse the concepts of weak and co-weak isomorphic properties of IVIFG. For further terminologies, the readers are referred to [1-6, 12, 13].

2. Preliminaries

A fuzzy graph $G=(V, \sigma, \mu)$ where $V$ is the vertex set, $\sigma$ is a fuzzy subset of $V$ and $\mu$ is a membership value on $\sigma$ such that $\mu(u,v) \leq \sigma(u) \land \sigma(v)$ for every $u, v \in V$. The underlying crisp graph of $G$ is denoted by $G^* = (\sigma^*, \mu^*)$, where
\[ \sigma = \sup \rho(\sigma) = \{ x \in V : \sigma(x) > 0 \} \text{and} \mu = \sup \rho(\mu) = \{ (x, y) \in V \times V : \mu(x, y) > 0 \}. \]

Let \( \sigma' : X \rightarrow [0, 1] \) is a fuzzy subset and \( \mu' : X \times X \rightarrow [0, 1] \) is a fuzzy relation on \( \sigma' \) such that \( \mu(u,v) \leq \sigma(u) \land \sigma(v) \), for all \( x, y \in X \).

**Definition 2.1.** By an interval-valued fuzzy graph of a graph \( G \) we mean a pair \( G^* = (A, B) \) where \( A = [\mu_A^-, \mu_A^+] \) and \( B : V \times V \rightarrow [0, 1] \) are bijective such that membership value of nodes and edges are distinct and \( B(x, y) \leq \mu_v(x) \land \mu_v(y) \forall x, y \in V \).

**Definition 2.2.** An interval \([\mu - \epsilon, \mu + \epsilon]\) is said to be an \( \epsilon \)-neighborhood of any membership value (i.e., corresponding to any nodes or edges) \( \mu \) for any \( \epsilon \) satisfying the following conditions.

(i) \( \epsilon \neq \min\{\mu_v(v), \mu_v(e_{ij})\} \);

(ii) \( \epsilon \neq 1 - \max\{\mu_v(v), \mu_v(e_{ij})\} \);

(iii) \( \epsilon \neq d(\mu(x), \mu(y)) \lor \frac{1}{2}d(\mu(x), \mu(y)) \) where \( d(\mu(x), \mu(y)) = |\mu(x) - \mu(y)| \) and \( \mu(x), \mu(y) \) are the membership or nodes or edges.

**Definition 2.3.** By an interval-valued intuitionistic fuzzy graph of a graph \( G \) we mean a pair \( G^* = (A, B) \) where \( A = [\mu_A^-, \mu_A^+], (\nu_A^-, \nu_A^+) \) and \( \nu_e : V \times V \rightarrow [0, 1] \) are bijective such that true and false membership value of nodes and edges are distinct and \( \nu_e(x, y) \leq \nu_v(x) \land \nu_v(y) \forall x, y \in V \), \( \nu_e(x, y) = \nu_v(x) \lor \nu_v(y) \forall x, y \in V \).

**Definition 2.4.** An interval-valued intuitionistic fuzzy graph (IVIFG) is said to be strong for the lower and upper bounds \((\mu^-, \mu^+)\) and \((\nu^-, \nu^+)\) of the edges and vertices satisfying the following conditions \( \mu_e(x, y) = \mu_v(x) \land \mu_v(y) \forall x, y \in V \), \( \nu_e(x, y) = \nu_v(x) \lor \nu_v(y) \forall x, y \in V \).

**Definition 2.5.** Let \( G = (V, E) \) be an IVIFG. Then the degree of a vertex \( v \) is defined by \( d(v) = (d_{\mu}(v), d_{\nu}(v)) \) where \( d_{\mu}(v) = \sum_{u \neq v}(\mu_e^-(v, u), \mu_e^+(v, u)) \) and \( d_{\nu}(v) = \sum_{u \neq v}(\nu_e^-(v, u), \nu_e^+(v, u)) \).

**Definition 2.6.** Let \( G = (V, E) \) be an IVIFG. Then the total degree of a vertex \( v \) is defined by \( td(v) = (td_{\mu}(v), td_{\nu}(v)) \) where \( td_{\mu}(v) = \sum_{u \neq v}(\mu_e^-(v, u), \mu_e^+(v, u)) \) and \( td_{\nu}(v) = \sum_{u \neq v}(\nu_e^-(v, u), \nu_e^+(v, u)) \).

**Definition 2.7.** Let \( G = (V, E) \) be an IVIFG. If all the vertices of \( G \) have same degree then \( G \) is said to be regular IVIFG.

**Definition 2.8.** Let \( G = (V, E) \) be an IVIFG. Then the order of \( G \) is defined as \( O(G) = [\sum_{v \in V} \mu_e^-(v), \sum_{v \in V} \mu_e^+(v)], [\sum_{v \in V} \nu_e^-(v), \sum_{v \in V} \nu_e^+(v)] \).

**Definition 2.9.** Let \( G = (V, E) \) be an IVIFG. Then the size of \( G \) is defined as \( S(G) = [\sum_{u \neq v} \mu_e^-(v, u), \sum_{u \neq v} \mu_e^+(v, u)], [\sum_{u \neq v} \nu_e^-(v, u), \sum_{u \neq v} \nu_e^+(v, u)] \).

**Remark 2.1.** In any IVIFG \( G \), we have

\[
\sum_{v \in V} d_G(v) = 2\{(\sum_{u \neq v} \mu_e^-(v, u), \sum_{u \neq v} \mu_e^+(v, u)), (\sum_{u \neq v} \nu_e^-(v, u), \sum_{u \neq v} \nu_e^+(v, u))\} = 2S(G).
\]
**Definition 2.10.** Let $G = (V, E)$ be an IVIFG. Let $e_{ij} \in E$ be an edge of $G$ where $e_{ij}$ has its lower and upper bounds $\mu^-_e, \nu^-_e$ and $\mu^+_e, \nu^+_e$. Then the degree of an edge $e_{ij}$ defined as $d_\mu(e_{ij}) = d_\mu(v_i) + d_\mu(v_j) - 2\mu_e(e_{ij})$ and $d_\nu(e_{ij}) = d_\nu(v_i) + d_\nu(v_j) - 2\nu_e(e_{ij})$, for all its vertices having the lower and upper bounds $\mu^-, \nu^-$ and $\mu^+, \nu^+$ respectively.

**Definition 2.11.** Let $G = (V, E)$ be an IVIFS. Let $e_{ij} \in E$ be an edge of $G$ where $e_{ij}$ has its lower and upper bounds $\mu^-_e, \nu^-_e$ and $\mu^+_e, \nu^+_e$. Then the total degree of an edge $e_{ij}$ defined as $td_\mu(e_{ij}) = d_\mu(e_{ij}) + \mu(e_{ij})$ and $td_\nu(e_{ij}) = d_\nu(e_{ij}) + \nu(e_{ij})$, for all the lower and upper bounds $\mu^-, \nu^-$ and $\mu^+, \nu^+$ respectively.

### 3. Automorphic IVIFG

In this section we introduce the isomorphic properties of IVIFG.

**Example 3.1.** The below figure represents the IVIFG $G$ of a crisp graph $G^*$

![Interval-valued Intuitionistic Fuzzy Graph](image)

Throughout this work $G^*$ is a crisp graph and $G$ is a IVIFG.

**Definition 3.1.** Let $G_1$ and $G_2$ be the IVIFGs. A homomorphism $f : G_1 \to G_2$ is a mapping $f : V_1 \to V_2$ which satisfies the following conditions:
1. $\mu^-_{A_1}(x_1) \leq \mu^-_{A_2}(f(x_1))$, $\mu^+_{A_1}(x_1) \leq \mu^+_{A_2}(f(x_1))$;
2. $\nu^-_{A_1}(x_1) \geq \nu^-_{A_2}(f(x_1))$, $\nu^+_{A_1}(x_1) \geq \nu^+_{A_2}(f(x_1))$;
3. $\mu^-_{B_1}x_1y_1 \leq \mu^-_{B_2}(f(x_1)f(y_1))$, $\mu^+_{B_1}x_1y_1 \leq \mu^+_{B_2}(f(x_1)f(y_1))$;
4. $\nu^-_{B_1}x_1y_1 \geq \nu^-_{B_2}(f(x_1)f(y_1))$, $\nu^+_{B_1}x_1y_1 \geq \nu^+_{B_2}(f(x_1)f(y_1))$, for all $x_1 \in V_1, x_1y_1 \in E_1$.

**Definition 3.2.** Let $G_1$ and $G_2$ be the IVIFGs. An isomorphism $f : G_1 \to G_2$ is a bijective mapping $f : V_1 \to V_2$ which satisfies the following conditions:
1. $\mu^-_{A_1}(x_1) = \mu^-_{A_2}(f(x_1))$, $\mu^+_{A_1}(x_1) = \mu^+_{A_2}(f(x_1))$;
Definition 3.3. Let $G_1$ and $G_2$ be the IVIFGs. Then a weak isomorphism $f : G_1 \rightarrow G_2$ is a bijective mapping $f : V_1 \rightarrow V_2$ which satisfies the following conditions:

(i) $f$ is a homomorphism;
(ii) $\mu^-_{A_1}(x_1) = \mu^-_{A_2}(f(x_1))$, $\mu^+_{A_1}(x_1) = \mu^+_{A_2}(f(x_1))$;
(iii) $\nu^-_{A_1}(x_1) = \nu^-_{A_2}(f(x_1))$, $\nu^+_{A_1}(x_1) = \nu^+_{A_2}(f(x_1))$.

It is clear that a weak isomorphism maintains only the weights of the nodes.

Example 3.2. Consider the IVIFGs $G_1$ and $G_2$ of $G^*_1$ and $G^*_2$ respectively,

A map $f : V_1 \rightarrow V_2$ defined by $f(u_1) = v_3$, $f(u_2) = v_1$ and $f(u_3) = v_2$. Then we have:

$$
\mu^-_{A_1}(u_1) = \mu^-_{A_2}(v_3), \mu^+_{A_1}(u_1) = \mu^+_{A_2}(v_3),
\nu^-_{A_1}(u_1) = \nu^-_{A_2}(v_3), \nu^+_{A_1}(u_1) = \nu^+_{A_2}(v_3),
\mu^-_{A_1}(u_2) = \mu^-_{A_2}(v_1), \mu^+_{A_1}(u_2) = \mu^+_{A_2}(v_1),
\nu^-_{A_1}(u_2) = \nu^-_{A_2}(v_1), \nu^+_{A_1}(u_2) = \nu^+_{A_2}(v_1),
\mu^-_{A_1}(u_3) = \mu^-_{A_2}(v_2), \mu^+_{A_1}(u_3) = \mu^+_{A_2}(v_2),
\nu^-_{A_1}(u_3) = \nu^-_{A_2}(v_2), \nu^+_{A_1}(u_3) = \nu^+_{A_2}(v_2).
$$
But we see that:

\[ \mu_{B_1}(u_1u_2) = \mu_{B_2}(v_1v_2), \mu_{B_1}(u_1u_2) \neq \mu_{B_2}(v_3v_1), \]
\[ \nu_{B_1}(u_1u_2) \neq \nu_{B_2}(v_1v_2), \nu_{B_1}(u_1u_2) \neq \nu_{B_2}(v_3v_1) \]
\[ \mu_{B_1}(u_1u_3) = \mu_{B_2}(v_2v_3), \mu_{B_1}(u_1u_3) \neq \mu_{B_2}(v_3v_2), \]
\[ \nu_{B_1}(u_1u_3) = \nu_{B_2}(v_2v_3), \nu_{B_1}(u_1u_3) \neq \nu_{B_2}(v_3v_2) \]
\[ \mu_{B_1}(u_2u_3) = \mu_{B_2}(v_1v_2), \mu_{B_1}(u_2u_3) \neq \mu_{B_2}(v_2v_1), \]
\[ \nu_{B_1}(u_2u_3) = \nu_{B_2}(v_2v_1) \]

Hence the map is a weak isomorphism but not an isomorphism.

**Definition 3.4.** Let \( G_1 \) and \( G_2 \) be the IVIFGs. Then a co-weak isomorphism \( f : G_1 \rightarrow G_2 \) is a bijective mapping \( f : V_1 \rightarrow V_2 \) which satisfies the following conditions:

(i) \( f \) is a homomorphism;
(ii) \( \mu_{B_1}(x_1y_1) = \mu_{B_2}(f(x_1)f(y_1)), \mu_{B_1}(x_1y_1) = \mu_{B_2}(f(x_1)f(y_1)); \)
(iii) \( \nu_{B_1}(x_1y_1) = \nu_{B_2}(f(x_1)f(y_1)), \nu_{B_1}(x_1y_1) = \nu_{B_2}(f(x_1)f(y_1)), \) for all \( x_1 \in V_1, x_1y_1 \in E_1. \)

It is clear that a co-weak isomorphism maintains only the weights of the arcs.

**Example 3.3.** Consider the IVIFGs \( G_1 \) and \( G_2 \) of \( G_1^* \) and \( G_2^* \) respectively,
But we see that
\[ \mu_{B_1}(u_1u_2) = \mu_{B_2}(v_3v_1), \mu_{B_1}^+(u_1u_2) = \mu_{B_2}^+(v_3v_1), \]
\[ \nu_{B_1}^-(u_1u_2) = \nu_{B_2}^-(v_3v_1), \nu_{B_1}^+(u_1u_2) = \nu_{B_2}^+(v_3v_1) \]
\[ \mu_{B_1}(u_1u_3) = \mu_{B_2}(v_3v_2), \mu_{B_1}^+(u_1u_3) = \mu_{B_2}^+(v_3v_2), \]
\[ \nu_{B_1}^-(u_1u_3) = \nu_{B_2}^-(v_3v_2), \nu_{B_1}^+(u_1u_3) = \nu_{B_2}^+(v_3v_2) \]
\[ \mu_{B_1}^-(u_3u_2) = \mu_{B_2}^-(v_2v_1), \mu_{B_1}^+(u_3u_2) = \mu_{B_2}^+(v_2v_1), \]
\[ \nu_{B_1}^-(u_3u_2) = \nu_{B_2}^-(v_2v_1), \nu_{B_1}^+(u_3u_2) = \nu_{B_2}^+(v_2v_1). \]

Hence the map is a co-weak isomorphism but not an isomorphism.

**Remark 3.1.**
1. If \( G_1 = G_2 = G \), then the homomorphism \( f \) over itself is called an endomorphism. An Isomorphism \( f \) over \( G \) is called an automorphism.

2. Let \( A = (\mu_A^-, \mu_A^+, \nu_A^-, \nu_A^+) \) be an IVIFG with an underlying set \( V \). Let \( Aut(G) \) be the set of all bipolar intuitionistic automorphism of \( G \). Let \( e : G \to G \) be a map defined by \( e(x) = x \), for all \( x \in V \) clearly \( e \in Aut(G) \).

3. If \( G_1 = G_2 \), then the weak and co-weak isomorphisms actually become isomorphic.

4. If \( f : V_1 \to V_2 \) is a bijective map then \( f^{-1} : V_1 \to V_2 \) is also a bijective map.

**Definition 3.5.** An Interval-valued intuitionistic fuzzy set \( A = (\mu_A^-, \mu_A^+, \nu_A^-, \nu_A^+) \) in a semigroup \( S \) is called a interval-valued intuitionistic subsemigroup of \( S \) if it satisfies the following conditions:

\[ \mu_B^-(xy) \leq (\mu_A^-(x) \land \mu_A^-(y)), \mu_B^+(xy) \leq (\mu_A^+(x) \land \mu_A^+(y)) \]
\[ \nu_B^-(xy) \geq (\nu_A^- (x) \lor \nu_A^-(y)), \nu_B^+(xy) \geq (\nu_A^+ (x) \lor \nu_A^+(y)), \text{ for all } x, y \in S. \]

**Definition 3.6.** An Interval-valued intuitionistic fuzzy set \( A = (\mu_A^-, \mu_A^+, \nu_A^-, \nu_A^+) \) in a group \( G \) is called a interval-valued intuitionistic fuzzy subgroup of \( G \) if it is a interval-valued intuitionistic fuzzy sub-semigroup of \( G \) and satisfies \( \mu_A(x^{-1}) = \mu_A^-(x), \mu_A^+(x^{-1}) = \mu_A^+(x), \nu_A(x^{-1}) = \nu_A^-(x), \nu_A^+(x^{-1}) = \nu_A^+(x) \).

We now show how to associate an interval-valued intuitionistic fuzzy group with a interval-valued intuitionistic fuzzy graph in a natural way.

**Proposition 3.1.** Let \( G = (A, B) \) be an IVIFG and let \( Aut(G) \) be the set of all automorphisms of \( G \). Then \( (Aut(G), \circ) \) forms a group.

**Proof.** We have the following conditions:

\[ \mu_A^-(\phi \circ \psi(x)) = \mu_A^-(\phi(\psi(x))) \leq \mu_A^-(\phi(x)) \geq \mu_A^-(x), \]
\[ \mu_A^+(\phi \circ \psi(x)) = \mu_A^+(\phi(\psi(x))) \leq \mu_A^+(\phi(x)) \geq \mu_A^+(x), \]
\[ \nu_A^-(\phi \circ \psi(x)) = \nu_A^-(\phi(\psi(x))) \geq \nu_A^-(\phi(x)) \geq \nu_A^-(x), \]
\[ \nu_A^+(\phi \circ \psi(x)) = \nu_A^+(\phi(\psi(x))) \geq \nu_A^+(\phi(x)) \geq \nu_A^+(x), \]

\[ (\phi \circ \psi)(x) = \phi(\psi(x)), \phi \circ (\psi \circ \psi)(x) = (\phi \circ \psi)(\psi(x)) = (\phi \circ \psi)(\phi(\psi(x))) = (\phi \circ \psi)(\phi(x)) = \phi(\psi(x)) = \phi(\psi(\psi(x))) = (\phi \circ \psi)(x). \]
\[ \mu_B((\phi \circ \psi)(x))(\phi \circ \psi)(y)) = \mu_B(\phi(\psi(x)))\phi(\psi(y)) \leq \mu_B(\phi(x)\phi(y)) \leq \mu_B(xy), \]
\[ \mu_B^+(\phi \circ \psi)(x))(\phi \circ \psi)(y))) = \mu_B^+(\phi(\psi(x)))\phi(\psi(y)) \leq \mu_B^+(\phi(x)\phi(y)) \leq \mu_B^+(xy), \]
\[ \nu_B((\phi \circ \psi)(x))(\phi \circ \psi)(y))) = \nu_B(\phi(\psi(x)))\phi(\psi(y)) \geq \nu_B(\phi(x)\phi(y)) \geq \nu_B(xy), \]
\[ \nu_B^+(\phi \circ \psi)(x))(\phi \circ \psi)(y))) = \nu_B^+(\phi(\psi(x)))\phi(\psi(y)) \geq \nu_B^+(\phi(x)\phi(y)) \geq \nu_B^+(xy). \]

Thus \( \phi \circ \psi \in Aut(G) \). Clearly, \( Aut(G) \) satisfies associativity under the operation \( \circ \), \( \phi \circ e = e \circ \phi \).

\[ \mu_A^-(\phi^{-1}) = \mu_A^{-1}(\phi), \mu_A^+(\phi^{-1}) = \mu_A^+(\phi), \nu_A^-(\phi^{-1}) = \nu_A^-(\phi), \nu_A^+(\phi^{-1}) = \nu_A^+(\phi), \]
for all \( \phi \in Aut(G) \).

Hence \( (Aut(G), \circ) \) forms a group. \( \Box \)

Now we state some propositions without their proofs as follows.

**Proposition 3.2.** Let \( G = (A, B) \) be an IVIFG and let \( Aut(G) \) be the set of all automorphisms of \( G \). Let \( g = (\mu_g^-, \mu_g^+, \nu_g^-, \nu_g^+) \) be an interval-valued intuitionistic fuzzy set in \( Aut(G) \) defined by

\[ \mu_g^-(\phi) = \inf\{\mu_B^{-}(\phi(x), \phi(y)) : (x, y) \in V \times V\}, \]
\[ \mu_g^+(\phi) = \inf\{\mu_B^{+}(\phi(x), \phi(y)) : (x, y) \in V \times V\}, \]
\[ \nu_g^-(\phi) = \sup\{\nu_B^{-}(\phi(x), \phi(y)) : (x, y) \in V \times V\}, \]
\[ \nu_g^+(\phi) = \sup\{\nu_B^{+}(\phi(x), \phi(y)) : (x, y) \in V \times V\}, \]

for all \( \phi \in Aut(G) \). Then \( g = (\mu_g^-, \mu_g^+, \nu_g^-, \nu_g^+) \) is an interval-valued intuitionistic fuzzy group on \( Aut(G) \).

**Proposition 3.3.** Every interval-valued intuitionistic fuzzy group has an embedding into the interval-valued intuitionistic fuzzy group of the group of automorphisms of some IVIFG.

We now prove that the isomorphism (weak isomorphism) between IVIFG is an equivalence solution (partial order relation).

**Proposition 3.4.** Let \( G_1, G_2, G_3 \) be IVIFGs. Then the isomorphism between these IVIFGs is an equivalence relation.

**Proof.** Reflexivity property is obvious. To prove the symmetry, let \( f : V_1 \rightarrow V_2 \) be an isomorphism of \( G_1 \) onto \( G_2 \). Then \( f \) is bijective map defined by

\[ f(x_1) = x_2, \quad \forall x_1 \in V_1 \]

satisfying the following conditions:

(i) \( \mu_{A_1}(x_1) = \mu_{A_2}(f(x_1)), \mu_{A_1}(x_1) = \mu_{A_2}(f(x_1)) \);
(ii) \( \nu_{A_1}(x_1) = \nu_{A_2}(f(x_1)), \nu_{A_1}(x_1) = \nu_{A_2}(f(x_1)) \);
(iii) \( \mu_{B_1}(x_1y_1) = \mu_{B_2}(f(x_1)f(y_1)), \mu_{B_1}(x_1y_1) = \mu_{B_2}(f(x_1)f(x_2)) \);
(iv) \( \nu_{B_1}(x_1y_1) = \nu_{B_2}(f(x_1)f(y_1)), \nu_{B_1}(x_1y_1) = \nu_{B_2}(f(x_1)f(x_2)) \), for all \( x_1 \in V_1, x_1y_1 \in E_1 \).
Since $f$ is bijective from $3.1$ it follows that: $f^{-1}(x_2) = x_1$, for all $x_2 \in V_2$. Thus:

(i) $\mu_{A_1}^{-1}(f^{-1}(x_2)) = \mu_{A_2}^{-1}(x_2), \mu_{A_1}^+(f^{-1}(x_2)) = \mu_{A_2}^+(x_2)$;
(ii) $\nu_{A_1}^{-1}(f^{-1}(x_2)) = \nu_{A_2}^{-1}(x_2), \nu_{A_1}^+(f^{-1}(x_2)) = \nu_{A_2}^+(x_2)$, for all $x_2 \in V_2$;
(iii) $\mu_{B_1}^{-1}(f^{-1}(x_2 y_2)) = \mu_{A_2}^{-1}(x_2 y_2), \mu_{A_1}^+(f^{-1}(x_2 y_2)) = \mu_{A_2}^+(x_2 y_2)$;
(iv) $\nu_{B_1}^{-1}(f^{-1}(x_2 y_2)) = \nu_{A_2}^{-1}(x_2 y_2), \nu_{A_1}^+(f^{-1}(x_2 y_2)) = \nu_{A_2}^+(x_2 y_2)$, for all $x_2 y_2 \in E_2$.

Hence a bijective map $f^{-1} : V_2 \to V_1$ is an isomorphism from $G_2$ onto $G_2$.

To prove the transitivity, let $f : V_1 \to V_2$ and $g : V_2 \to V_3$ be the isomorphisms of $G_1$ onto $G_2$ and $G_2$ onto $G_3$, respectively. Then $g \circ f : V_1 \to V_3$ is a bijective map from $V_1$ and $V_3$, where $(g \circ f)(x_1) = g(f(x_1))$, for all $x_1 \in V_1$. Since a map $f : V_1 \to V_2$ defined by $f(x_1) = x_2$, for all $x_1 \in V_1$ is an isomorphism, so we have

$$
\begin{align*}
\mu_{A_1}^{-1}(x_1) &= \mu_{A_2}^{-1}(f(x_1)) = \mu_{A_3}^{-1}(x_2), \\
\mu_{A_1}^+(x_1) &= \mu_{A_2}^+(f(x_1)) = \mu_{A_3}^+(x_2), \\
\nu_{A_1}^{-1}(x_1) &= \nu_{A_2}^{-1}(f(x_1)) = \nu_{A_3}^{-1}(x_2), \\
\nu_{A_1}^+(x_1) &= \nu_{A_2}^+(f(x_1)) = \nu_{A_3}^+(x_2), \forall x_1 \in V_1.
\end{align*}
$$

$$
\begin{align*}
\mu_{B_1}^{-1}(x_1 y_1) &= \mu_{B_2}^{-1}(f(x_1) f(y_1)) = \mu_{B_3}^{-1}(x_2 y_2), \\
\mu_{B_1}^+(x_1 y_1) &= \mu_{B_2}^+(f(x_1) f(y_1)) = \mu_{B_3}^+(x_2 y_2), \\
\nu_{B_1}^{-1}(x_1 y_1) &= \nu_{B_2}^{-1}(f(x_1) f(y_1)) = \nu_{B_3}^{-1}(x_2 y_2), \\
\nu_{B_1}^+(x_1 y_1) &= \nu_{B_2}^+(f(x_1) f(y_1)) = \nu_{B_3}^+(x_2 y_2), \forall x_1 y_1 \in E_2.
\end{align*}
$$

Since a map $g : V_2 \to V_3$ defined by $g(x_2) = x_3$ for $x_2 \in V_2$ is an isomorphism, We have

$$
\begin{align*}
\mu_{A_2}^{-1}(x_2) &= \mu_{A_3}^{-1}(g(x_2)) = \mu_{A_3}^{-1}(x_3), \\
\mu_{A_2}^+(x_2) &= \mu_{A_3}^+(g(x_2)) = \mu_{A_3}^+(x_3), \\
\nu_{A_2}^{-1}(x_2) &= \nu_{A_3}^{-1}(g(x_2)) = \nu_{A_3}^{-1}(x_3), \\
\nu_{A_2}^+(x_2) &= \nu_{A_3}^+(g(x_2)) = \nu_{A_3}^+(x_3), \forall x_2 \in V_2.
\end{align*}
$$

$$
\begin{align*}
\mu_{B_2}^{-1}(x_2 y_2) &= \mu_{B_3}^{-1}(g(x_2) g(y_2)) = \mu_{B_3}^{-1}(x_3 y_3), \\
\mu_{B_2}^+(x_2 y_2) &= \mu_{B_3}^+(g(x_2) g(y_2)) = \mu_{B_3}^+(x_3 y_3), \\
\nu_{B_2}^{-1}(x_2 y_2) &= \nu_{B_3}^{-1}(g(x_2) g(y_2)) = \nu_{B_3}^{-1}(x_3 y_3), \\
\nu_{B_2}^+(x_2 y_2) &= \nu_{B_3}^+(g(x_2) g(y_2)) = \nu_{B_3}^+(x_3 y_3).
\end{align*}
$$
From 3.2 and 3.4 and $f(x_1) = x_2, x_1 \in V_1$, we have

\[
\begin{align*}
\mu_{A_1}(x_1) &= \mu_{A_2}(f(x_1)) = \mu_{A_3}(g(x_2)) = \mu_{A_4}(g(f(x_1))), \\
\mu_{A_1}^+(x_1) &= \mu_{A_2}^+(f(x_1)) = \mu_{A_3}^+(g(x_2)) = \mu_{A_4}^+(g(f(x_1))), \\
(3.6) \quad \nu_{A_1}(x_1) &= \nu_{A_2}(f(x_1)) = \nu_{A_3}(g(x_2)) = \nu_{A_4}(g(f(x_1))), \\
\nu_{A_1}^+(x_1) &= \nu_{A_2}^+(f(x_1)) = \nu_{A_3}^+(g(x_2)) = \nu_{A_4}^+(g(f(x_1))), \forall x_1 \in V_1.
\end{align*}
\]

From 3.3 and 3.5, we have

\[
\begin{align*}
\mu_{B_1}(x_1y_1) &= \mu_{B_2}(f(x_1)f(y_1)) = \mu_{B_3}(g(x_2)y_2) = \mu_{B_4}(f(x_1)g(y_1)) \\
&= \mu_{B_3}(g(f(x_1))g(y_1)), \\
\mu_{B_1}^+(x_1y_1) &= \mu_{B_2}^+(f(x_1)f(y_1)) = \mu_{B_3}^+(g(x_2)y_2) = \mu_{B_4}^+(f(x_1)g(y_1)) \\
&= \mu_{B_3}^+(g(f(x_1))g(y_1)), \\
(3.7) \quad \nu_{B_1}(x_1y_1) &= \nu_{B_2}(f(x_1)f(y_1)) = \nu_{B_3}(g(x_2)y_2) = \nu_{B_4}(f(x_1)g(y_1)) \\
&= \nu_{B_3}(g(f(x_1))g(y_1)), \\
\nu_{B_1}^+(x_1y_1) &= \nu_{B_2}^+(f(x_1)f(y_1)) = \nu_{B_3}^+(g(x_2)y_2) = \nu_{B_4}^+(f(x_1)g(y_1)) \\
&= \nu_{B_3}^+(g(f(x_1))g(y_1)), \forall x_1y_1 \in E_1.
\end{align*}
\]

Thus, we prove that $g \circ f$ is an isomorphism between $G_1$ and $G_3$.

Hence the proof. \hfill \Box

**Proposition 3.5.** Let $G_1, G_2, G_3$ be IVIFGs. Then the weak isomorphism between these IVIFGs is a partial order relation.

**Proof.** Reflexive property is obvious.

To prove the antisymmetry, let $f : V_1 \rightarrow V_2$ be a weak isomorphism of $G_1$ onto $G_2$. Then $f$ is a bijective map defined by $f(x_1) = x_2$, for all $x_1 \in V_1$ satisfying the following

\[
\begin{align*}
(i) \quad &\mu_{A_1}(x_1) = \mu_{A_2}(f(x_1)), \mu_{A_1}^+(x_1) = \mu_{A_2}^+(f(x_2)), \\
(ii) \quad &\nu_{A_1}(x_1) = \nu_{A_2}(f(x_1)), \nu_{A_1}^+(x_1) = \nu_{A_2}^+(f(x_2)), \\
(iii) \quad &\mu_{B_1}(x_2y_2) \leq \mu_{B_2}(f(x_1)f(y_1)), \mu_{B_1}^+(x_1y_1) \leq \mu_{B_2}^+(f(x_1)f(y_1)), \\
(iv) \quad &\nu_{B_1}(x_2y_2) \geq \nu_{B_2}(f(x_1)f(y_1)), \nu_{B_1}^+(x_1y_1) \\
&\geq \nu_{B_2}^+(f(x_1)f(y_1)), x_1 \in V_1, \forall x_1y_1 \in E_1.
\end{align*}
\]

Let $g : V_2 \rightarrow V_1$ be a weak isomorphism of $G_2$ onto $G_1$. Then $g$ is a bijective map defined by $g(x_2) = x_1$, for all satisfying

\[
\begin{align*}
\mu_{A_2}(x_2) &= \mu_{A_1}(g(x_2)), \mu_{A_2}^+(x_2) = \mu_{A_1}^+(g(x_2)), \\
\nu_{A_2}(x_2) &= \nu_{A_1}(g(x_2)), \nu_{A_2}^+(x_2) = \nu_{A_1}^+(g(x_2)), \forall x_1 \in V_2 \\
(3.9) \quad &\mu_{B_2}(x_2y_2) \leq \mu_{B_1}(g(x_2)y_2), \mu_{B_2}^+(x_2y_2) \leq \mu_{B_1}^+(g(x_2)y_2), \\
\nu_{B_2}(x_2y_2) &\geq \nu_{B_1}(g(x_2)y_2), \nu_{B_2}^+(x_2y_2) \geq \nu_{B_1}^+(g(x_2)y_2), \forall x_2y_2 \in E_2.
\end{align*}
\]
The inequalities \(3.8\) and \(3.9\) holds on the finite sets \(V_1\) and \(V_2\) only when \(G_1\) and \(G_2\) have the same number of edges and the corresponding edges have weight. Hence \(G_1\) and \(G_2\) are identical.

To prove the transitivity, let \(f : V_1 \to V_2\) and \(g : V_2 \to V_3\) be the isomorphisms of \(G_1\) onto \(G_2\) and \(G_2\) onto \(G_3\), respectively. Then \(g \circ f : V_1 \to V_3\) is a bijective map from \(V_1\) and \(V_3\), where \((g \circ f)(x_1) = g(f(x_1))\), for all \(x_1 \in V_1\). Since a map \(f : V_1 \to V_2\) defined by \(f(x_1) = x_2\), for all \(x_1 \in V_1\) is a weak isomorphism, so we have

\[
\begin{align*}
\mu^{-}_{A_1}(x_1) &= \mu^{-}_{A_2}(f(x_1)) = \mu^{-}_{A_2}(x_2), \\
\mu^{+}_{A_1}(x_1) &= \mu^{+}_{A_2}(f(x_1)) = \mu^{+}_{A_2}(x_2), \\
\nu^{-}_{A_1}(x_1) &= \nu^{-}_{A_2}(f(x_1)) = \nu^{-}_{A_2}(x_2), \\
\nu^{+}_{A_1}(x_1) &= \nu^{+}_{A_2}(f(x_1)) = \nu^{+}_{A_2}(x_2), \forall x_1 \in V_1.
\end{align*}
\]

\[
\begin{align*}
\mu^{-}_{B_1}(x_1y_1) &\leq \mu^{-}_{B_2}(f(x_1)f(y_1)) = \mu^{-}_{B_2}(x_2y_2), \\
\mu^{+}_{B_1}(x_1y_1) &\leq \mu^{+}_{B_2}(f(x_1)f(y_1)) = \mu^{+}_{B_2}(x_2y_2) \\
\nu^{-}_{B_1}(x_1y_1) &\geq \nu^{-}_{B_2}(f(x_1)f(y_1)) = \nu^{-}_{B_2}(x_2y_2) \\
\nu^{+}_{B_1}(x_1y_1) &\geq \nu^{+}_{B_2}(f(x_1)f(y_1)) = \nu^{+}_{B_2}(x_2y_2), \forall x_1y_1 \in E_1.
\end{align*}
\]

Since a map \(g : V_2 \to V_3\) defined by \(g(x_2) = x_3\) for \(x_2 \in V_2\) is a weak isomorphism, We have

\[
\begin{align*}
\mu^{-}_{A_2}(x_2) &= \mu^{-}_{A_3}(g(x_2)) = \mu^{-}_{A_3}(x_3), \\
\mu^{+}_{A_2}(x_2) &= \mu^{+}_{A_3}(g(x_2)) = \mu^{+}_{A_3}(x_3), \\
\nu^{-}_{A_2}(x_2) &= \nu^{-}_{A_3}(g(x_2)) = \nu^{-}_{A_3}(x_3), \\
\nu^{+}_{A_2}(x_2) &= \nu^{+}_{A_3}(g(x_2)) = \nu^{+}_{A_3}(x_3), \forall x_2 \in V_2.
\end{align*}
\]

\[
\begin{align*}
\mu^{-}_{B_2}(x_2y_2) &\leq \mu^{-}_{B_3}(g(x_2)g(y_2)) = \mu^{-}_{B_3}(x_3y_3), \\
\mu^{+}_{B_2}(x_2y_2) &\leq \mu^{+}_{B_3}(g(x_2)g(y_2)) = \mu^{+}_{B_3}(x_3y_3), \\
\nu^{-}_{B_2}(x_2y_2) &\geq \nu^{-}_{B_3}(g(x_2)g(y_2)) = \nu^{-}_{B_3}(x_3y_3), \\
\nu^{+}_{B_2}(x_2y_2) &\geq \nu^{+}_{B_3}(g(x_2)g(y_2)) = \nu^{+}_{B_3}(x_3y_3), \forall x_1y_1 \in E_1.
\end{align*}
\]

From \(3.10\) and \(3.12\) and \(f(x_1) = x_2, x_1 \in V_1\), we have

\[
\begin{align*}
\mu^{-}_{A_1}(x_1) &= \mu^{-}_{A_2}(f(x_1)) = \mu^{-}_{A_3}(g(x_2)) = \mu_{A_3}(g(f(x_1))), \\
\mu^{+}_{A_1}(x_1) &= \mu^{+}_{A_2}(f(x_1)) = \mu^{+}_{A_3}(g(x_2)) = \mu^{+}_{A_3}(g(f(x_1))), \\
\nu^{-}_{A_1}(x_1) &= \nu^{-}_{A_2}(f(x_1)) = \nu^{-}_{A_3}(g(x_2)) = \nu_{A_3}(g(f(x_1))), \\
\nu^{+}_{A_1}(x_1) &= \nu^{+}_{A_2}(f(x_1)) = \nu^{+}_{A_3}(g(x_2)) = \nu^{+}_{A_3}(g(f(x_1))), \forall x_1 \in V_1.
\end{align*}
\]
From 3.11 and 3.13, we have
\[
\mu_{B_1}(x_1 y_1) \leq \mu_{B_2}(f(x_1) f(y_1)) = \mu_{B_2}(x_2 y_2) = \mu_{B_3}(x_2 g(y_2))
\]
\[
= \mu_{B_3}(g(f(x_1)) g(f(y_1))),
\]
\[
\mu_{B_1}(x_1 y_1) \leq \mu_{B_2}^+(f(x_1) f(y_1)) = \mu_{B_2}^+(x_2 y_2) = \mu_{B_3}^+(x_2 g(y_2))
\]
\[
= \mu_{B_3}^+(g(f(x_1)) g(f(y_1))),(3.15)
\]
\[
\nu_{B_1}^-(x_1 y_1) \geq \nu_{B_2}^- f(x_1) f(y_1)) = \nu_{B_2}^- (x_2 y_2)
\]
\[
= \nu_{B_3}^-(g(x_2) g(y_2))
\]
\[
= \nu_{B_3}^-(g(f(x_1)) g(f(y_1))),
\]
\[
\nu_{B_1}^+(x_1 y_1) \geq \nu_{B_2}^+ f(x_1) f(y_1)) = \nu_{B_2}^+(x_2 y_2) = \nu_{B_3}^+(x_2 g(y_2))
\]
\[
= \nu_{B_3}^+(g(f(x_1)) g(f(y_1))), \forall x_1 y_1 \in E_1.
\]
Thus, we prove that \(g \circ f\) is a weak isomorphism between \(G_1\) and \(G_3\). Hence the proof.

4. Conclusion

Interval-valued intuitionistic fuzzy graph have numerous application in the real life systems and real life applications where the level of information inherited in the system varies with respect to time and have different level of precision. Most of the actions in real life situations are time dependent and also ambiguous in partial information, symbolic models in expert system are more effective than traditional methods to identify the upper and lower bounds of the true and false membership values in an interval. In this paper, we introduced the concept of automorphism on IVIFG. Also we investigate the properties of morphism on IVIFG.

References


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