SOME NEW PROPERTIES ON \( \lambda \)-COMMUTING OPERATORS

Abdeslam El Bakkali  
Chouaib Doukkali University  
Faculty of Sciences  
Department of Mathematics  
El Jadida  
Morocco  
aba0101q@yahoo.fr

Abdelaziz Tajmouati*  
Sidi Mohamed Ben Abdellah University  
Faculty of Sciences Dhar Al Mahraz  
Laboratory of Mathematical Analysis and Applications  
Fez  
Morocco  
abelaziz.tajmouati@usmba.ac.ma

Mohamed Ahmed Mohamed Baba  
Sidi Mohamed Ben Abdellah University  
Faculty of Sciences Dhar Al Mahraz  
Laboratory of Mathematical Analysis and Applications  
Fez  
Morocco  
bbaba2012@gmail.com

Abstract. In this paper, we study the operator equation \( AB = \lambda BA \) for a bounded linear operators \( A, B \) on a complex Hilbert space. We focus on algebraic relations between different operators that include normal, \( M \)-hyponormal, quasi \( \ast \)-paranormal and other classes.

Keywords: Hilbert space, \( \lambda \)-commute, binormal, \( M \)-hyponormal, isometry, \( k \)-paranormal, quasi \( \ast \)-paranormal.

1. Introduction

Throughout, we will denote by \( \mathcal{B}(\mathcal{H}) \) the complex Banach algebra of all bounded linear operators on a infinite dimensional complex Hilbert space \( \mathcal{H} \). We denote the range and the kernel of \( A \in \mathcal{B}(\mathcal{H}) \) by \( R(A) \) and \( N(A) \) respectively.

Recall that an operator \( A \in \mathcal{B}(\mathcal{H}) \) is said to be:

- positive if \( \langle Ax, x \rangle \geq 0 \) for all \( x \in \mathcal{H} \)
- self-adjoint if \( A = A^* \)

* Corresponding author
• isometry if $A^*A = I$, which equivalent to the condition $\|Ax\| = \|x\|$ for all $x \in \mathcal{H}$
• normal if $A^*A = AA^*$
• unitary $A^*A = AA^* = I$ (i.e. $A$ is an onto isometry)
• quasinormal if $A(A^*A) = (A^*A)A$
• binormal if $(A^*A)(AA^*) = (AA^*)(A^*A)$ [3]
• subnormal if $A$ has a normal extension
• hyponormal if $A^*A \geq AA^*$, which equivalent to the condition $\|A^*x\| \leq \|Ax\|$ for all $x \in \mathcal{H}$ [15]
• $M$-hyponormal if $A^*A \geq MA^*A^*$, where $M \in \mathbb{R}$ and $M \geq 1$ which equivalent to the condition $\|A^*x\| \leq M\|Ax\|$ for all $x \in \mathcal{H}$ [20]
• $p$-hyponormal if $(A^*A)^p \leq (AA^*)^p$, where $0 < p \leq 1$ [1]
• class $\mathcal{A}$ if $|A|^2 \leq |A|^2$, where $|A| = (A^*A)^{1/2}$
• paranormal if $\|Ax\|^2 \leq \|A^2x\|\|x\|$ for all $x \in \mathcal{H}$ [4]
• $k$-paranormal if $\|Ax\|^k \leq \|A^kx\|\|x\|^{k-1}$ for all $x \in \mathcal{H}$ and $k \geq 2$
• $*$-paranormal if $\|A^*x\|^2 \leq \|A^2x\|\|x\|$ for all $x \in \mathcal{H}$ [10]
• quasi $*$-paranormal if $\|A^*Ax\|^2 \leq \|A^3x\|\|Ax\|$ for all $x \in \mathcal{H}$ [12]
• log-hyponormal if $A$ invertible and satisfies $\log(A^*A) \geq \log(AA^*)$ [16]
• $p$-quasihyponormal if $A^*[A^*A]^p - (AA^*)^p]A \geq 0$, where $0 < p \leq 1$ [2]
• normoloid if $\|A\| = r(A)$
• quasinilpotent if $r(A) = 0$, where $r(A) = \lim \|A^n\|^\frac{1}{n}$.

We can notice that $A$ is hyponormal if $A$ is $p$-hyponormal with $p = 1$. By Löwner-Heinz inequality $p$-hyponormal is $q$-hyponormal for every $0 < q \leq p \leq 1$ [14]. Also we can notice that $A$ is paranormal if $A$ is $k$-paranormal with $k = 2$. It known that invertible $p$-hyponormal is log-hyponormal. We can consider log-hyponormal operator as $0$-hyponormal [16]. It is well known that for any operators $A, B$ and $C$ we have

$$A^*A - 2\lambda B^*B + \lambda^2 C^*C \geq 0 \forall \lambda > 0 \iff \|Bx\|^2 \leq \|Ax\|\|Cx\|$$

for all $x \in H$.

Thus we have
A is quasi $*$-paranormal if and only if $A^*[\lambda(A^*)^2A^2 - 2\lambda A A^* + \lambda^2]A \geq 0$ for all $\lambda > 0$.

A is $*$-paranormal if and only if $(A^*)^2A^2 - 2\lambda AA^* + \lambda^2 \geq 0$ for all $\lambda > 0$.

We have also the following inclusions:

- quasinormal $\subseteq$ binormal
- class $\mathcal{A}$ $\subseteq$ paranormal
- hyponormal $\subseteq$ $*$-paranormal $\subseteq$ quasi $*$-paranormal
- invertible $p$-hyponormal $\subseteq$ log-hponormal $\subseteq$ paranormal.
- self-adjoint $\subseteq$ normal $\subseteq$ quasinormal $\subseteq$ subnormal $\subseteq$ hyponormal
- hyponormal $\subseteq$ $p$-hyponormal $\subseteq$ $p$-quasihyponormal $\subseteq$ class $\mathcal{A}$.

For a scalar $\lambda$, two operators $A$ and $B$ in $\mathcal{B}(\mathcal{H})$ are said to be $\lambda$-commute if $AB = \lambda BA$. Recently many authors have studied this equation for several classes of operators, for example:

- In [11] the authors have proved that if an operator in $\mathcal{B}(\mathcal{H})$ $\lambda$-commutes with a compact, then this operator has a non-trivial hyperinvariant subspace.
- In [8] Conway and Prajitura characterized the closure and the interior of the set of operators that $\lambda$-commute with a compact operator.
- In [19] Zhang, Ohawada and Cho have studied the properties of an operator $\lambda$-commutes with a paranormal.
- In [5] Brooke, Busch and Pearson showed that if $AB$ is not quasinilpotent, then $|\lambda| = 1$, and if $A$ or $B$ is self-adjoint then $\lambda \in \mathbb{R}$.
- In [18] Yang and Du gave a simple proofs and generalizations of this results, particulary if $AB$ is bounded below if and only if both $A$ and $B$ are bounded below.
- In [14] Schmeger generalized this results to hermitian or normal elements of a complex Banach algebra.

The aim of this paper is to study the situation for binormal, $M$-hyponormal, quasi $*$-paranormal operators. Again other related results are also given.
2. Main results

We begin with the following result.

Lemma 2.1. Let $A \in \mathcal{B}(\mathcal{H})$ be quasi $*$-paranormal. If $A$ is quasinilpotent, then $A = 0$.

**Proof.** Let $A \in \mathcal{B}(\mathcal{H})$ be quasi $*$-paranormal, then we have

$$
\|A^*Ax\| = \|A^3x\|^{\frac{2}{3}} \|Ax\|^{\frac{4}{3}} \text{ for all } x \in \mathcal{H}.
$$

Therefore

$$
\|Ax\|^4 = \langle A^*Ax, x \rangle^2 \leq \|A^*Ax\|^2 \|x\|^2 \leq \|A^3x\| \|Ax\| \|x\|^2.
$$

Thus

$$
\|Ax\|^3 \leq \|A^3x\| \|x\|^2 \text{ for all } x \in \mathcal{H}, \text{ whence } A \text{ is 3-paranormal. By [17, Lemma 1], then every } k \text{-paranormal is normaloid. Thus we conclude that } A \text{ is normaloid and hence } r(A) = \|A\|. \text{ On the other hand } A \text{ is quasinilpotent, then we obtain } \|A\| = r(A) = 0. \text{ Therefore } A = 0.
$$

Theorem 2.2. Let $A, B \in \mathcal{B}(\mathcal{H})$ and $\lambda \in \mathbb{C}$ such that $AB = \lambda BA \neq 0$, $A$ is quasinormal and $B$ is normal. If $|\lambda| = 1$, then $AB$ is quasinormal.

**Proof.** Assume that $AB = \lambda BA \neq 0$, then $B^*A^* = \bar{\lambda}A^*B^*$. Since $B$ and $\lambda B$ are normal operators and by Fuglede-Putnam Theorem, then $BA^* = \lambda A^*B$ and $AB^* = \lambda B^*A$. Moreover we have

$$
AB[(AB)^*AB] = [AB][B^*A^*AB]
$$

$$
= [\lambda BA]B^*A^*AB
$$

$$
= \lambda B[AB^*]A^*AB
$$

$$
= \lambda B[\lambda B^*A]A^*AB
$$

$$
= |\lambda|^2[BB^*][AA^*]AB
$$

$$
= [B^*B][A^*AA]B
$$

$$
= B^*[BA^*]AAB
$$

$$
= B^*[..]AABB
$$

$$
= B^*A^*[..][AB]A
$$

$$
= B^*A^*[..][AB]AB
$$

$$
= [..][AB]AB.
$$

Therefore $AB$ is quasinormal.
Proof. Since $B$ and $\lambda B$ are normal operators and by Fuglede-Putnam Theorem, then we have $BA^* = \lambda A^* B$ and $AB^* = \lambda B^* A$. Therefore we obtain

$$AB(AB)^*(AB)^*AB = A[BB^*]A^*B^*A^*AB$$
$$= [AB^*BA^*[\lambda A^* B^*]AB$$
$$= (\lambda)^2 B^*[AB]A^*A^*[1/\lambda AB^*]B$$
$$= |\lambda|^2 B^*B[AA^*A^*A^*B^*B^*B$$
$$= B^*[BA^*]AA[A^*B^*]B$$
$$= B^*[^\lambda A^* B]AA[1/\lambda B^* A^*]B$$
$$= \frac{\lambda}{\lambda} B^* A^*[BA]AB^*[A^*B]$$
$$= \lambda^2 B^* A^*[1/\lambda AB][AB^*[1/\lambda BA^*]$$
$$= B^* A^* ABA[B^*B]A^*$$
$$= B^* A^* ABA[BB^*]A^*$$
$$= (AB)^* AB AB(AB)^*,$$

then $AB$ is binormal. \qed

**Theorem 2.3.** Let $A, B \in B(\mathcal{H})$ and $\lambda \in \mathbb{C}$ such that $AB = \lambda BA \neq 0$.

Suppose that $A$ is $k$-paranormal and $B$ is isometry, then the following statements are equivalent:

1. $AB$ is $k$-paranormal
2. $\sigma(AB) \neq \{0\}$
3. $|\lambda| = 1$.

**Proof.** Suppose that $A$ is $k$-paranormal and $B$ is isometry with $AB = \lambda BA \neq 0$.

We first show that (1) $\Rightarrow$ (2). Suppose that $AB$ is $k$-paranormal.

If $AB$ is quasinilpotent ($\sigma(AB) = \{0\}$). Since every $k$-paranormal is isometry, then we obtain $\|AB\| = r(AB) = 0$ and hence $AB = 0$ and this is a contradiction with $AB \neq 0$. Therefore $AB$ is not quasinilpotent and hence $\sigma(AB) \neq \{0\}$.

We prove that (2) $\Rightarrow$ (3). Suppose that $\sigma(AB) \neq \{0\}$, then

$$r(AB) \neq 0.$$  

(1)
Since $AB = \lambda BA \neq 0$ and by [5, Proposition 1], then $\sigma(AB) = \sigma(BA) = \lambda \sigma(AB)$. Hence

$$r(AB) = |\lambda| r(AB).$$

Therefore by (1) and (2) we obtain $|\lambda| = 1$. Finally we show that $(3) \Rightarrow (1)$. Suppose that $|\lambda| = 1$, for any unit vector $x \in \mathcal{H}$ we have

\[
\| (AB)x \|^k = \| A(Bx) \|^k \\
\leq \| A^k (Bx) \| \| Bx \|^{k-1} \quad (A \text{ is } k \text{- paranormal}) \\
\leq \| A^k Bx \| \quad (B \text{ is isometry}).
\]

Hence

$$\| (AB)x \|^k \leq \| A^k Bx \|. \quad (3)$$

On the other hand by induction we show that $(AB)^k = \lambda^{\frac{k(k-1)}{2}} B^{k-1} A^k B$ for every $k \in \mathbb{N}^*$. For $k = 1$ we have $(AB)^1 = \lambda^{\frac{(1)(1)}{2}} B^{1-1} A^1 B$. Assume that $(AB)^k = \lambda^{\frac{k(k-1)}{2}} B^{k-1} A^k B$ for $k \geq 2$. Finally we have

\[
(AB)^{k+1} = AB(AB)^k = (\lambda BA)(\lambda^{\frac{k(k-1)}{2}} B^{k-1} A^k B) \\
= \lambda^{\frac{k(k-1)}{2}+1} BAB^{k-1} A^k B \\
= \lambda^{\frac{k(k-1)}{2}+1} B(AB)B^{k-2} A^k B \\
= \lambda^{\frac{k(k-1)}{2}+1} B(\lambda BA)B^{k-2} A^k B \\
= \lambda^{\frac{k(k-1)}{2}+2} B^2 AB^{k-2} A^k B \\
= \lambda^{\frac{k(k-1)}{2}+2} B^2 AB^{k-2} A^k B
\]

We conclude that $(AB)^k = \lambda^{\frac{k(k-1)}{2}} B^{k-1} A^k B$, for every $k \in \mathbb{N}^*$. Then for every unit vector $x \in \mathcal{H}$ we obtain

\[
\| (AB)^k x \| = \| \lambda^{\frac{k(k-1)}{2}} B^{k-1} A^k Bx \| \\
= \| \lambda^{\frac{k(k-1)}{2}} B^{k-1} A^k Bx \| \\
= \| A^k Bx \| \quad (B^{k-1} \text{ is isometry and } |\lambda| = 1).
\]

Hence

$$\| (AB)^k x \| = \| A^k Bx \| \text{ for any unit vector } x. \quad (4)$$

Finally by (3) and (4) we conclude that $\| (AB)x \|^k \leq \| A^k Bx \| = \| (AB)^k x \|$, for any unit vector $x$. Therefore $AB$ is $k$-paranormal. \qed
**Theorem 2.4.** Let $A, B \in B(H)$ and $\lambda \in \mathbb{C}$ such that $AB = \lambda BA \neq 0$. Then

1. if $A^*$ is $M_1$-hyponormal and $B$ is $M_2$-hyponormal, then $|\lambda| \leq (M_1M_2)^{\frac{1}{2}}$

2. if $A$ is $M_1$-hyponormal and $B^*$ is $M_2$-hyponormal, then $|\lambda| \geq (M_1M_2)^{-\frac{1}{2}}$.

**Proof.** Let $A, B \in B(H)$ and $\lambda \in \mathbb{C}$ such that $AB = \lambda BA \neq 0$.

1. Since we have
   \[ |\lambda||BA| = ||\lambda BA|| \]
   \[ = ||AB|| \]
   \[ = ||B^*A^*AB||^{\frac{1}{2}} (||T|| = ||TT^*||^{\frac{1}{2}}) \]
   \[ \leq M_1^{\frac{1}{2}}||B^*AA^*B||^{\frac{1}{2}} (A^* \text{ is } M_1 \text{- hyponormal: } A^*A \leq M_1AA^*) \]
   \[ \leq M_2^{\frac{1}{2}}||A^*B|| (||T^*T||^{\frac{1}{2}} = ||T||) \]
   \[ \leq M_1^{\frac{1}{2}}||A^*BB^*A||^{\frac{1}{2}} (||T^*|| = ||TT^*||^{\frac{1}{2}}) \]
   \[ \leq (M_1M_2)^{\frac{1}{2}}||A^*B^*BA||^{\frac{1}{2}} (B \text{ is } M_2 \text{- hyponormal: } BB^* \leq M_2B^*B) \]
   \[ \leq (M_1M_2)^{\frac{1}{2}}||BA|| (||T^*T||^{\frac{1}{2}} = ||T||). \]

Therefore $|\lambda||BA| \leq (M_1M_2)^{\frac{1}{2}}||BA||$ Hence $|\lambda| \leq (M_1M_2)^{\frac{1}{2}}$.

2. Since $AB = \lambda BA$ and $\lambda \neq 0$, then $BA = \lambda^{-1}AB$ and by first implication we obtain $|\lambda^{-1}| \leq (M_2M_1)^{\frac{1}{2}}$ and hence $|\lambda| \geq (M_2M_1)^{-\frac{1}{2}}$.

\[ \square \]

**Corollary 2.2.** Let $A, B \in B(H)$ and $\lambda \in \mathbb{C}$ such that $AB = \lambda BA \neq 0$. Then

1. if $A^*$ and $B$ are hyponormal, then $|\lambda| \leq 1$

2. if $A$ and $B^*$ are hyponormal, then $|\lambda| \geq 1$.

**Proof.** By Theorem 2.4 and we take $M_1 = M_2 = 1$. \( \square \)

**Theorem 2.5.** Let $A, B \in B(H)$ and $\lambda \in \mathbb{C}$ such that $AB = \lambda BA \neq 0$.

If $A^*$ is $M_1$-hyponormal and $B$ is $M_2$-hyponormal, then $A^*B$ and $BA^*$ are $M_1M_2|\lambda|^2$-hyponormal.

**Proof.** Let $A, B \in B(H)$ and $\lambda \in \mathbb{C}$ such that $AB = \lambda BA \neq 0$. Then

\[ (A^*B)^*A^*B = B^*AA^*B \]
\[ \geq M_1B^*A^*AB \]
\[ \geq M_1\bar{\lambda}A^*B^*\lambda BA \]
\[ \geq M_1|\lambda|^2A^*B^*BA \]
\[ \geq M_1|\lambda|^2A^*M_2BB^*A \]
\[ \geq M_1M_2|\lambda|^2(B^*A)^*B^*A. \]
Therefore $A^*B$ is $M_1M_2|\lambda|^2$-hyponormal.

In the same way we obtain $BA^*$ is $M_1M_2|\lambda|^2$-hyponormal. □

References


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