THE FORM OF AUTOMORPHISMS OF AN ABELIAN GROUP HAVING THE WEAK EXTENSION PROPERTY

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Abstract. Let $A$ be an abelian group and let $\alpha$ be an automorphism of $A$. In this paper we show that if the restriction of $\alpha$ to any $p$-component $A_p$ of $A$ is of the form: $\alpha_{A_p} = \pi id_{A_p} + \rho$, where $p$ is a prime number, $\pi$ a $p$-adic invertible number and $\rho \in Hom(A_p, A^1)$ with $A^1$ is the first subgroup Ulm of the group $A$. Then $\alpha$ satisfies the weak extension property.

Keywords: abelian groups, $p$-groups, torsion groups, automorphism group.

1. Introduction

The study of the characterization of automorphisms having the property of extension had begun by P. E Schupp showed, in [12], that the extension property in the category of groups, characterizes the inner automorphisms. M. R. Pettet gives in, [10], a simpler proof of Schupp’s result and shows that the inner automorphisms of a group are also characterized by the lifting property in the category of groups. In [8] M. Dugas and R. Gobel gave another simpler proof of Schupp’s result, using only the elementary theory of groups. In [2] L. Ben Yakoub shows that the result of Schupp is not valid in general for algebras on a

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commutative ring. It is not yet known whether this result holds true for algebras (of finite dimensions) on a commutative field. In [3] L. Ben Yakoub and M. P. Malliavin show that the property of extension also characterizes derivations in associative algebras for some algebras quantum properties. In this article, we will define the property of the weak extension by:

An automorphism $\alpha$ of an abelian group $A$ has the weak extension property if for all abelian group $B$ for all monomorphism $\lambda : A \rightarrow B$ and if there exists an element $m \in \mathbb{N}^*$ such that the restriction of $\lambda$ to $mA$ is an isomorphism from $mA$ to $mB$, then there exists $\tilde{\alpha} \in \text{Aut}(B)$ such that the following diagram is commutative:

$$
\begin{array}{c}
A & \xrightarrow{\lambda} & B \\
\alpha \downarrow & & \downarrow \tilde{\alpha} \\
A & \xrightarrow{\lambda} & B 
\end{array}
$$

In other words: $\tilde{\alpha}\lambda = \lambda\alpha$. By way of example, any automorphism of an abelian group without torsion possesses the property of the weak extension (see [1], [16]).

2. Main result

**Theorem 2.1.** Let $A$ be an abelian group and let $\alpha$ be an automorphism of $A$.

If the restriction of $\alpha$ to any $p$-component $A_p$ of $A$ is of the form: $\alpha|_{A_p} = \pi id_{A_p} + \rho$, where $p$ is a prime number, $\pi$ a $p$-adic invertible number and $\rho \in \text{Hom}(A_p, A^1)$ with $A^1$ is the first subgroup Ulm of $A$. Then $\alpha$ satisfies the weak extension property.

Before giving proof of this theorem, we will need certain results.

**Lemma 2.2.** Let $T_A$ be the torsion part of $A$. If $\alpha_1$ is the restriction of $\alpha$ to $T_A$ then $\alpha_1$ is an automorphism of $T_A$. Moreover, $\alpha_1$ satisfies the weak extension property.

**Proof.** Let $x \in T_A$. There exists $n \in \mathbb{N}^*$ such that $nx = 0$, whence $n\alpha(x) = \alpha(nx) = 0$, we deduce that $\alpha_1(T_A) \subseteq T_A$.

On the other hand, $\forall y \in T_A$, there exists $x \in T_A$ such that $y = \alpha(x)$; If $0 = my = m\alpha(x)$ for some $m \in \mathbb{N}^*$, then $\alpha(mx) = 0$ implies $mx = 0$, then $x \in T_A$. We conclude that $\alpha(T_A) = T_A$, consequently $\alpha_1 = \alpha|_{T_A}$ is an automorphism of $T_A$. Let $(T_A)_p$ be the $p$-component of the torsion group $T_A$. From the above assumptions, $\alpha_1|_{(T_A)_p} = \pi id_{(T_A)_p} + \rho$ where $\pi$ is an invertible $p$-adic number and $\rho \in \text{Hom}((T_A)_p, (T_A)_p^1)$ with $(T_A)_p^1$ the first subgroup Ulm of the group $(T_A)_p$. Therefore, according to the characterization of the automorphisms possessing the weak extension property in the category of torsion abelian groups see, [15], we deduce that $\alpha_1 = \alpha|_{T_A}$ satisfies the weak extension property.

**Proposition 2.3.** Let $\lambda : A \rightarrow A'$ be a monomorphism of abelian groups and let $\lambda|_{m_0A}$ be the restriction of $\lambda$ to $m_0A$ such that $\lambda|_{m_0A} \in \text{Isom}(m_0A; m_0A')$
where \( m_0 \in \mathbb{N}^* \). If \( T_A \) and \( T_{A'} \) are respectively the torsion parts of \( A \) and \( A' \). So:

(i) \( \lambda_1 : T_A \to T_{A'} \) is a monomorphism.

(ii) There exists an automorphism \( \alpha' \) of \( T_{A'} \) which makes switch the following diagram:

\[
\begin{array}{ccc}
T_A & \xrightarrow{\lambda_1} & T_{A'} \\
\alpha_1 \downarrow & & \downarrow \alpha' \\
T_A & \xrightarrow{\lambda_1} & T_{A'}
\end{array}
\]

**Proof.** (i) It suffices to prove that: \( \lambda(T_A) \subseteq T_{A'} \).

Let \( a \in T_A \); There exists \( m \in \mathbb{N}^* \) such that \( ma = 0 \), hence \( m\lambda(a) = 0 \), we deduce that \( \lambda(T_A) \subseteq T_{A'} \).

(ii) From the assumptions we have: \( m_0A \simeq m_0A' \) where \( m_0 \in \mathbb{N}^* \).

The proposition 8.37 (see [18], p: 295) shows that for \( m_0 \in \mathbb{N}^* \): \( T(m_0A) \simeq T(m_0A') \), hence \( m_0T_A \simeq m_0T_{A'} \), so for some \( m_0 \in \mathbb{N}^* \): \( \lambda_1|_{m_0T_A} \in Isom(m_0T_A; m_0T_{A'}) \) and since the automorphism \( \alpha_1 \in Aut(T_A) \) satisfies the weak extension property, then there exists an automorphism \( \alpha' \in T_{A'} \) such that \( \alpha' \lambda_1 = \lambda_1 \alpha_1 \).

**Lemma 2.4.** Let \( A' \) be an abelian group and \( \lambda : A \to A' \) an monomorphism. If \( \lambda(A) = A_1 \) and if \( \alpha_2 = \lambda \alpha \lambda^{-1} \), then, \( \alpha_2 \) is an automorphism of \( A_1 \) which makes switch the following diagram:

\[
\begin{array}{ccc}
A & \xrightarrow{\lambda} & A_1 \\
\alpha \downarrow & & \downarrow \alpha_2 \\
A & \xrightarrow{\lambda} & A_1
\end{array}
\]

**Proof.** Since \( \lambda : A \to A' \) is a monomorphism and \( \lambda(A) = A_1 \), then, \( \lambda : A \to A_1 \) is an isomorphism, consequently, \( \alpha_2 = \lambda \alpha \lambda^{-1} \) is an automorphism of \( A_1 \) and we have: \( \alpha_2 \lambda = \lambda \alpha \).

**Proposition 2.5.** Let \( \lambda : A \to A' \) be a monomorphism of abelian groups and let \( \lambda|_{m_0A} \) be the restriction of \( \lambda \) to \( m_0A \) such that \( \lambda|_{m_0A} \in Isom(m_0A; m_0A') \) where \( m_0 \in \mathbb{N} \). If \( T_A \) and \( T_{A'} \) are respectively the torsion parts of \( A \) and \( A' \); So:

1. \( A' = A_1 + T_{A'} \);
2. \( A_1 \cap T_{A'} = T_{A_1} \);
3. \( \lambda(T_A) = T_{A_1} \).

**Proof.** 1) Since \( \lambda : A \to A' \) is a monomorphism and \( \lambda(A) = A_1 \), then \( \lambda : A \to A_1 \) is an isomorphism.

Hence \( \lambda(m_0A) = m_0 \lambda(A) = m_0A_1 \). And since \( \lambda|_{m_0A} \in Isom(m_0A; m_0A') \). So \( \lambda(m_0A) = m_0A' = m_0A_1 \). Let \( x \in A' \), hence \( m_0x \in m_0A' = m_0A_1 \) which implies that there exists \( a_1 \in A_1 \subseteq A' \) such that \( m_0x = m_0a_1 \). Therefore
if we have second case.

2) Since $A_1 \subset A'$, then $T_{A_1} \subset T_{A'}$. of plus $T_{A_1} \subset A_1$, we deduce that $T_{A_1} \subset A_1 \cap T_{A'}$. On the other hand, for all $x \in A_1 \cap T_{A'}$, then $x \in T_{A'}$. There exists $n \in \mathbb{N}^*$ such that $nx = 0$. Since $x \in A_1$, therefore $x \in T_{A_1}$, consequently, $A_1 \cap T_{A'} = T_{A_1}$.

3) Let $x \in T_A$; There exists $m \in \mathbb{N}^*$ such that $mx = 0$, whence $0 = \lambda(mx) = m\lambda(x)$, so $\lambda(x) \in T_{A_1}$, we deduce that $\lambda(T_A) \subset T_{A_1}$. Now either $x \in T_{A_1}$: There exists $m \in \mathbb{N}^*$ such that $mx = 0$. Since $T_{A_1} \subset A_1 = \lambda(A)$, then $x = \lambda(a)$ where $a \in A$. So $0 = mx = m\lambda(a) = \lambda(ma)$. Thus $ma = 0$, hence $a \in T_A$ which implies that $x \in \lambda(T_A)$. It is concluded that $\lambda(T_A) = T_{A_1}$.

3. The proof of theorem 2.1

Let $A$ be an abelian group and let $\alpha$ be an automorphism of $A$.

Let $A'$ be an abelian group and $\lambda : A \to A'$ a monomorphism.

$T_A$, $T_{A_1}$, and $T_{A'}$ are respectively the torsion parts of $A$, $A_1$ and $A'$.

We define the endomorphism $\alpha_3$ of the group $A'$ by: $(\alpha_3)|_{A_1} = \alpha_2$ and $(\alpha_3)|_{T_{A'}} = \alpha'$.

The endomorphism $\alpha_3$ is well defined. Indeed, if $a_1 + b_1 = a_2 + b_2$ where $a_1, a_2 \in A_1$ and $b_1, b_2 \in T_{A'}$ then $a_1 - a_2 = b_2 - b_1 \in A_1 \cap T_{A'} = T_{A_1} = \lambda(T_A)$. This implies that there exists $a \in T_A$ such that $a_1 - a_2 = b_2 - b_1 = \lambda(a)$. Hence,

$$\begin{cases}
\alpha_2(a_1 - a_2) = \alpha_2\lambda(a) = \lambda\alpha(a), \\
\alpha'(b_2 - b_1) = \alpha'\lambda(a) = \lambda\alpha_3(a) = \lambda\alpha(a).
\end{cases}$$

Therefore, $\alpha_2(a_1 - a_2) = \alpha'(b_2 - b_1)$, i.e. $\alpha_2(a_1) + \alpha'(b_1) = \alpha_2(a_2) + \alpha'(b_2)$. Hence $\alpha_3(a_1 + b_1) = \alpha_3(a_2 + b_2)$. Moreover $\alpha_3$ is an automorphism of $A'$. Indeed:

**Injection.** Let $a' = a_1 + b_1 \in A'$ where $a_1 \in A_1$ and $b_1 \in T_{A'}$ such that $\alpha' \in \text{Ker}(\alpha_3)$.

**First case.** If we have $a_1 \in T_{A'}$. Then $a_1 + b_1 \in T_{A'}$, consequently $0 = \alpha_3(a_1 + b_1) = \alpha'(a_1 + b_1)$. And since $\alpha' \in \text{Aut}(T_{A'})$ then $a' = a_1 + b_1 = 0$.

**Second case.** If we have $a_1 \notin T_{A'}$. Then since $b_1 \in T_{A'}$, Whence there exists $m \in \mathbb{N}^*$ such that $mb_1 = 0$, so $0 = ma_3(a') = \alpha_3(ma_1 + mb_1) = \alpha_3(ma_1) = \alpha_2(ma_1)$ which implies that $ma_1 = 0$ because $\alpha_2 \in \text{Aut}(A_1)$, consequently $a_1 \in T_{A_1} \subset T_{A'}$ which is absurd.

**Surjection.** Let $a' = a_2 + b_2 \in A'$ where $a_2 \in A_1$ and $b_2 \in T_{A'}$, since $\alpha_2 \in \text{Aut}(A_1)$ and $\alpha' \in \text{Aut}(T_{A'})$, then

$$\begin{cases}
\exists a_1 \in A_1 : \alpha_2^{-1}(a_1) = a_2, \\
\exists b_1 \in T_A : \alpha'^{-1}(b_1) = b_2.
\end{cases}$$
So $\alpha_3(a_2 + b_2) = \alpha_2(a_2) + \alpha'(b_2) = \alpha_2^{-1}(a_1) + \alpha'\alpha'^{-1}(b_1) = a_1 + b_1$. Therefore $\alpha_3 \in \text{Aut}(A')$. In addition the automorphism $\alpha_3$ of group $A'$ commutes the following diagram:

$$
\begin{array}{c}
A \\ \alpha \downarrow \\
A' \\
A' \\
\end{array}
\overset{\lambda}{\longrightarrow}
\begin{array}{c}
A' \\
\alpha_3 \\
A' \\
\end{array}
$$

Indeed, according to proposition 0.3 and lemma 0.4 we have: $\alpha_2\lambda = \lambda\alpha$ and $\alpha'\lambda = \lambda\alpha$, so

$$
\alpha_3\lambda(a) = \alpha_3\lambda(a) = (a_2 + \alpha')\lambda(a) = \alpha_2\lambda(a) + \alpha'\lambda(a) = \lambda\alpha_2(a) + \lambda\alpha'(a)
$$

$$
= \lambda(\alpha_2 + \alpha')(a) = \lambda\alpha_3(a).
$$

We conclude that $\alpha$ satisfies the weak extension property.

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**References**


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