ON HYPER BCH-ALGEBRA

M. Sajjad Ali Khan*

Khaista Rahman
Department of Mathematics
Hazara University
Mansehra, KPk
Pakistan
sajjadalimath@yahoo.com
khaista355@yahoo.com

S. Abdullah
Department of Mathematics
Abdul Wali Khan University
Mardan, KPk
Pakistan
saleemabdullah81@yahoo.com

Fawad Hussain
Department of Mathematics
Abbottabad University of Science and Technology
KPk, Pakistan
fawadhussain998@hotmail.com

Abstract. In this paper we initiate the concept of a hyper BCH-algebra which is a generalization of a BCH-algebra, and hyper BCK/BCI algebras and investigate some related properties. Moreover we introduce a hyper BCH-ideal, weak hyper BCH-ideal and strong hyper BCH-ideal in hyper BCH-algebras, and give a few relations among these hyper BCH-ideals. Finally we define homomorphism of hyper BCH-algebras.

Keywords: hyper BCH-algebra, hyper BCH-ideals, week hyper BCH-ideals, Strong hyper BCH-ideals, homomorphism.

1. Introduction

In (1966) the notion of BCK-algebra was first introduces by Y. Imai and K. Iseki [6]. The notion of BCK-algebra is a generalization of properties of the Set-difference. In (1975), the concept of ideal in BCK-algebra was first initiated by K. Iseki [7]. A remarkable feature of K. Iseki definition is that, its formulation is free from those of ring theoretical and lattice theoretical concepts. In same year K. Iseki initiated the concept of BCI-algebra [6, 8] which is the generalization of BCK-algebra. These algebras have been extensively studied

∗. Corresponding author

since their introduction. The concept of ideals has played an important role in the study of the theory of BCI-algebras, [9]. In a BCI-algebra X, an ideal I need not be subalgebra of X. If the ideal I is also a subalgebra of X, then it has better algebraic properties. In (1983), Q. P. Hu and X. Li, introduced the concept of BCH-algebra [3, 4] and prove some motivating results. In (1990) and (1991) certain other properties have been studied by W. A. Dudek and J. Thomys [2] and M. A. Chaudhry, [1], respectively. In [1], the author also defines ideals in BCH-algebras. Hyperstructure represent a natural extension of classical algebraic structures and they were introduced by the French mathematician F. Marty in (1934), [12]. Algebraic hyperstructures are a suitable generalization of classical algebraic structures. Hyperstructures have many applications to several sectors of both pure and applied sciences. In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure; the composition of two elements is a set. In (2000) Y. B. Jun et al applied the hyperoperation to BCK-algebras and introduced the concept of a hyper BCK-algebra [12] which is a generalization of a BCK-algebra, and investigated some related properties. Ideal theory of hyper BCK-algebra studied in [11]. Further in (2006), X.L. Xin initiated the concept of hyper BCI-algebras [13], which is basically a generalization of hyper BCK-algebras, and he proved that every hyper BCK-algebra is a hyper BCI-algebra. It should be pointed out that the research of hyper BCI-algebras seems to have been focused on the ideal theory. The author introduced the concepts of hyper BCI-ideals, weak hyper BCI-ideals, strong hyper BCI-ideals and reflexive hyper BCI-ideals in hyper BCI-algebras, and he gave the relations among these hyper BCI-ideals. In this paper we initiated the notion of hyper BCH-algebra which is a generalization of BCH-algebra and hyper BCI/BCK-algebras and studied some basic properties. Moreover we introduce a hyper BCH-ideal, weak hyper BCH-ideal and strong hyper BCH-ideal in hyper BCH-algebras, and give some relations among these hyper BCH-ideals. We define homomorphism in hyper BCH-algebra and then we investigate some related results.

2. Premilinaries

Let H be a non-empty set and "$\circ$" a function from $H \times H \to P(H) \setminus \{\phi\}$, where $P(H)$ denotes the power set of H. For any two non-empty subsets A and B of H, denote by $A \circ B$ the set $\bigcup_{a\in A, b\in B} a \circ b$. We will use $x \circ y$ instead of $x \circ \{y\}$, $\{x\} \circ y$ or $\{x\} \circ \{y\}$. Also we define $x \ll y$ by $0 \in x \circ y$ and for every $A, B \subseteq H$, $A \ll B$ is defined by for all $a \in A$, there exist $b \in B$ such that $a \ll b$.

Definition 2.1 ([10]). A non-empty set H endowed with a constant 0 and a hyperoperation is called hyper BCK-algebra if it satisfies the following axioms:

$HK1)$ $(x \circ y) \circ (y \circ z) \ll x \circ y$,
$HK2)$ $(x \circ y) \circ z = (x \circ z) \circ y$,
$HK3)$ $x \circ H \ll \{x\}$,
$HK4)$ $x \ll y$ and $y \ll x \Rightarrow x = y$.

for all $x, y, z \in H$.

Definition 2.2 ([13])**.** A non-empty set H endowed with a constant 0 and a hyperoperation is called hyper BCI-algebra if it satisfies the following axioms:
$HI1)$ $(x \circ y) \circ (y \circ z) \ll x \circ y$,
$HI2)$ $(x \circ y) \circ z = (x \circ z) \circ y$,
$HI3)$ $x \circ H \ll \{x\}$,
$HI4)$ $x \ll y$ and $y \ll x \Rightarrow x = y$.
$HI5)$ $0 \circ (0 \circ x) \ll x$.
for all $x, y, z \in H$.

Definition 2.3 ([11])**.** Let I be a nonempty subset of a hyper BCK-algebra H and $0 \in I$. Then I is said to be a hyper BCK-ideal of H if $x \circ y \ll I$ and $y \in I$ implies $x \in I$ for all $x, y \in H$, reflexive if $x \circ x \subseteq I$ for all $x \in H$, strong hyper BCK-ideal of H if $(x \circ y) \cap I = \phi$ and $y \in I$ implies $x \in I$ for all $x, y \in H$, hyper subalgebra of H if $x \circ y \subseteq I$ for all $x, y \in I$.

Proposition 2.4 ([11])**.** *Let H be hyper BCK-algebra. Then,*
(i) *any strong hyper BCK-ideal of H is a hyper BCK-ideal of H.*
(ii) *if I is a hyper BCK-ideal of H and A is a nonempty subset of H. Then $A \ll I$ implies $A \subseteq I$.*
(iii) *if I is a reflexive hyper BCK-ideal of H and $(x \circ y) \cap I = \phi$, then $x \circ y \subseteq I$ for all $x, y \in H$.*
(iv) *H is a BCK-algebra if and only if $H = \{x \in H : x \circ x = \{0\}\}$.*

3. Hyper BCH-algebra

In this section we introduce a notion of hyper BCH-algebra and studied some of its basic properties.

Definition 3.1. Let H be a on-empty set with a constant "0" and "$\circ$" be a hyper operation defined on H. Then $(H, \circ, 0)$ is said to be a hyper BCH-algebra if the following axioms are satisfied:
$HCH1)$ $x \ll x$,
$HCH2)$ $(x \circ y) \circ z = (x \circ z) \circ y$,
$HCH3)$ $x \ll y$ and $y \ll x \Rightarrow x = y$
for all $x, y, z \in H$; where $x \ll y$ is defined by $0 \in x \circ y$ and for every $A, B \subseteq H$, $A \ll B$ is defined by for all $a \in A$, there exists $b \in B$ such that $a \ll b$. In such case, "$\ll$" is called a hyper order in H.

Example 3.2. Let $H = \{0, 1, 2\}$ and "$\circ$" be a hyperoperation defined on H in the following table:

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{1\}$
1	$\{1\}$	$\{0, 1\}$	$\{0, 1\}$
2	$\{2\}$	$\{0, 2\}$	$\{0, 1, 2\}$

Then $(H, \circ)$ is a hyper BCH-algebra.

Example 3.3. Let $H = \{0, 1, 2, 3\}$ and "$\circ$" be a hyperoperation defined on H in the following table:

$\circ$	0	1	2	3
0	$\{0\}$	$\{0\}$	$\{2\}$	$\{3\}$
1	$\{1\}$	$\{0,1\}$	$\{0,3\}$	$\{0,3\}$
2	$\{2\}$	$\{0,2\}$	$\{0,2\}$	$\{0,2\}$
3	$\{3\}$	$\{0,2\}$	$\{0,2\}$	$\{0,2\}$

Then $(H, \circ)$ is a hyper BCH-algebra.

Proposition 3.4. *Any hyper BCK/BCI- algebra is a hyper BCH-algebra.*

Proposition 3.5. *Let H be a hyper BCH-algebra, then for all $x, y, z \in H$ and $A \subseteq H$;the following holds.*

1) $x\circ\, y \ll z \Leftrightarrow x \circ z \ll y$
2) $x \circ y \ll x$
3) $0 \ll x$
4) $t \in 0 \circ 0 \Leftrightarrow t = 0$
5) $x \in x \circ 0$
6) $A \circ y \ll A$
7) $x \circ A \ll y \Leftrightarrow x \circ y \ll A$
8) $A \ll A \circ 0$
9) $x \circ x = \{x\} \Leftrightarrow x = 0.$

Proof. We only prove 1, 2, 5, 6, 7 and 9.

1) Let$x, y, z \in H$,be such that $x \circ y \ll z$.Then there exists $t \in x \circ y$ such that $t \ll z$.Thus $0 \in t \circ z \subseteq (x \circ y) \circ z = (x \circ z) \circ y$ and hence there exists $w \in x \circ z$ such that $0 \in w \circ y$ that is $w \ll y$.Therefore $x \circ z \ll y$.

Conversly, let $x, y, z \in H$ be such that $x \circ z \ll y$. Then there exists $w \in x \circ z$ such that $w \ll y$.Thus $0 \in w \circ y \subseteq (x \circ z) \circ y = (x \circ y) \circ z$ and hence there exists $t \in x \circ y$ such that $0 \in t \circ z$that is $t \ll z$.Therefore $x \circ y \ll z$.

2) Let $0 \in 0 \circ y \subseteq (x \circ x) \circ y = (x \circ y) \circ x$. Then there exists $t \in x \circ y$ such that $0 \in t \circ x \Rightarrow t \ll x \Rightarrow x \circ y \ll x$.

5) By (2) above we have $x \circ 0 \ll x$, so there exists $t \in x \circ 0$ such that t $\ll x$, since $t \in x \circ 0$, then $x \circ 0 \ll t$ and so by (1) $x \circ t \ll 0$. Thus there is $r \in x \circ t$ such that $r \ll 0$, so by (3) and $(HCH3)$ $r = 0$. so $0 \in x \circ t$, that is $x \ll t$ since $x \ll t$ and $t \ll x$; then by $(HCH3) \Rightarrow x = t$. Therefore $x \in x \circ 0$.

6) Let $a \in A$ be any element, then by (2) $a \circ y \ll a$ hence there is $b \in a \circ y \subseteq A \circ y$ such that $b \ll a$, that is $A \circ y \ll A$.

7) Since $x \circ A \ll y$ which implies that there exists $a \in A$ such that $x \circ a \ll y$. Hence by (1) $x \circ a \ll a \ll A$ implies that $x \circ y \ll A$. The proof of the converse is easy to prove.

9) $\{x\} = x \circ x \subseteq x \circ (x \circ 0)$. Hence by (5) $x \ll 0$; thus $x = 0$. The converse follows from (4). □

Proposition 3.6. *In any hyper BCH-algebra H, $x \circ 0 = \{x\}$ for all $x \in H$.*

Proof. We have from above proposition (5) $x \in x \circ 0$, now let $t \in x \circ 0$.Since $x \circ 0 \ll \{x\}$, we have $t \ll x$. So, $0 \in t \circ t \subseteq (x \circ 0) \circ t = (x \circ t) \circ 0$. Then there exists $a \in x \circ t$ such that $0 \in a \circ 0$.Thus $a \ll 0$.Then $a = 0$; Thus $x \ll t$. We have that $x = t$. Therefore, $x \circ 0 = \{x\}$. □

It is known that every hyper BCI-algebra is a hyper BCH-algebrs, but the following example show that the converse is not true.

Example 3.7. Let $H = \{0, 1, 2, 3\}$and "$\circ$" be a hyperoperation define on H in the following table:

$\circ$	0	1	2	3
0	$\{0\}$	$\{1\}$	$\{1\}$	$\{1\}$
1	$\{1\}$	$\{0\}$	$\{3\}$	$\{3\}$
2	$\{2\}$	$\{3\}$	$\{0\}$	$\{2\}$
3	$\{3\}$	$\{0\}$	$\{0\}$	$\{0\}$

Then $(H, \circ)$ is a hyper BCH-algebra, but it is not a hyper BCI-algebra. Because,

$$(2 \circ 3) \circ (2 \circ 1) = \{2\} \circ \{3\} = \{2, 3\}$$

and

$$(1 \circ 3) = \{3\}.(2 \circ 3) \circ (2 \circ 1) \neq (1 \circ 3)$$

Example 3.8. Let $H = \{0, 1, 2, 3, 4\}$ and "$\circ$" be a hyperoperation defined of H in the following table:

$\circ$	0	1	2	3	4
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{2\}$	$\{1\}$	$\{0, 4\}$
2	$\{2\}$	$\{2\}$	$\{0\}$	$\{2\}$	$\{0, 4\}$
3	$\{3\}$	$\{3\}$	$\{3\}$	$\{0\}$	$\{4\}$
4	$\{4\}$	$\{4\}$	$\{4\}$	$\{4\}$	$\{0\}$

Then $(H, \circ)$ is a hyper BCH-algebra, but it is not a hyper BCI-algebra. Because,

$$(1 \circ 3) \circ (1 \circ 2) = \{1\} \circ \{2\} = \{1, 2\}$$

and $(2 \circ 3) = \{2\}$ that is $\{1, 2\} \not\ll \{2\}$.

Definition 3.9. A hyper BCH-algebra H is called proper if it is not a hyper BCI-algebra.

In above examples the hyper BCH-algebras are proper hyper BCH-algebras.

Definition 3.10. Let $(H, \circ)$ be a hyper BCH-algebra, and X a non-empty subset of H containing "0". Then X is called hypersubalgebra of H if X is a hyper BCH-algebra under the same hyperoperation "$\circ$" on H.

Example 3.11. From the above Example 3.8 if we let $X = \{0, 1, 2\}$, then X is a hypersubalgebra of H as we in the following table:

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{0\}$
2	$\{2\}$	$\{0, 2\}$	$\{0\}$

Also, let $X = \{0, 1, 3\}$. Then X is a hypersubalgebra of H.

Theorem 3.12. *Let X be a non-empty subset of a hyper BCH-algebra $(H, \circ)$. The X is a hypersubalgebra of H if and only if $x \circ y \subseteq X$ for all $x, y \in X$.*

Proof. Straghtfarword. $\square$

Theorem 3.13. *Let $(H, \circ)$ be a hyper BCH-algebra and $X(H) = \{x \in H \mid 0 \circ x \ll \{0\}\}$. Then $X(H)$ is a hypersubalgebra of H.*

Proof. Let $x, y \in X(H)$, then by definition $a = 0 \circ a \ll \{0\}$ and $b = 0 \circ b \ll \{0\}$. Now

$$a \circ b = (0 \circ a) \circ (0 \circ b) \ll \{0\} \circ \{0\} = \{0\}$$

Hence, $a \circ b \ll \{0\}$.Which implies that $a \circ b \ll X(H)$. Hence $X(H)$ is a hypersubalgebra of H. The set $X(H)$ is called the hyper BCA-part of the hyper BCH-algebra H. $\square$

4. Hyper BCH-Ideals

Definition 4.1. Let $(H, \circ)$ be a hyper BCH-algebra and I a subset of H. Then I is called a hyper BCH-ideal of H if:

i) $0 \in I$
ii) $x \circ y \ll I$ and $y \in I \Rightarrow x \in I$ for all $x, y \in I$.

Example 4.2. Let $H = \{0, 1, 2, 3, 4, 5\}$ and "$\circ$" be a hyperoperation defined on H in the following table:

$\circ$	0	1	2	3	4	5
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0, 4\}$	$\{0, 5\}$
1	$\{1\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{1\}$	$\{0, 5\}$
2	$\{2\}$	$\{0, 2\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0, 5\}$
3	$\{3\}$	$\{0, 3\}$	$\{0, 3\}$	$\{0\}$	$\{0\}$	$\{0, 5\}$
4	$\{4\}$	$\{0, 4\}$	$\{0, 4\}$	$\{0, 4\}$	$\{0\}$	$\{0\}$
5	$\{5\}$	$\{0, 5\}$	$\{0, 5\}$	$\{0, 5\}$	$\{0, 5\}$	$\{0\}$

Then $(H, \circ)$ is a hyper BCH-algebra. Let $I = \{0, 1, 2, 3\}$ is an ideal of H.

Example 4.3. Let $H = \{0, 1, 2, 3, 4\}$ and "$\circ$" be a hyperoperation defined on H in the following table:

$\circ$	0	1	2	3	4
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0,3\}$	$\{0,4\}$
1	$\{1\}$	$\{0\}$	$\{0,1\}$	$\{0,1\}$	$\{0,4\}$
2	$\{2\}$	$\{0,2\}$	$\{0\}$	$\{0,2\}$	$\{0,3\}$
3	$\{3\}$	$\{0,3\}$	$\{0,3\}$	$\{0\}$	$\{0,2\}$
4	$\{4\}$	$\{0,4\}$	$\{0,4\}$	$\{0,1\}$	$\{0\}$

Then $(H, \circ)$ is a hyper BCH-algebra.

Let $I_1 = \{0, 1, 2\}$, then I_1 is a hyper BCH-ideal of H.

Let $I_2 = \{0, 1, 3\}$, then I_2 is a hyper BCH-ideal of H.

Let $I_3 = \{0, 2, 3\}$, then I_3 is not a hyper BCH-ideal of H. Because $(3 \circ 4) = \{0, 2\} \ll I_3$ and $4 \in I_3$ but $3 \notin I_3$.

Theorem 4.4. *Let $(H, \circ)$ be a hyper BCH-algebra and $\{I_\lambda |\ \lambda \in \Lambda\}$ a family of hyper BCH-ideals of H, then $\bigcap_{\lambda\in\Lambda} I_\lambda$ is a hyper BCH-ideal of H.*

Proof. For any $\lambda \in \Lambda$; let I_λ be a hyper BCH-ideal of a hyper BCH-algebra H, then clearly $0 \in \bigcap_{\lambda\in\Lambda} I_\lambda$. Now let $x, y \in H$ be such that $x \circ y \ll I_\lambda$ and $y \in I_\lambda$ for every $\lambda \in \Lambda$. Since each I_λ for every $\lambda \in \Lambda$ is a hyper BCH-ideal of H. Therefore it implies that $x \circ\ y \ll I_\lambda$ for every $\lambda \in \Lambda$ and $y \in I_\lambda \Rightarrow x \in I_\lambda$. Hence $x \circ\ y \ll \bigcap_{\lambda\in\Lambda} I_\lambda$ and $y \in \bigcap_{\lambda\in\Lambda} I_\lambda \Rightarrow x \in \bigcap_{\lambda\in\Lambda} I_\lambda$. Thus $\bigcap_{\lambda\in\Lambda} I_\lambda$ is a hyper BCH-ideal of H. □

Remark 4.5. The union of two hyper BCH-ideals need not be hyper BCH-ideals. For this we have the following example.

Example 4.6. Let $H = \{0, 1, 2, 3, 4\}$ be a hyper BCH-algebra define in Example 4.3. Let $I_1 = \{0, 1, 3\}$ and $I_2 = \{0, 1, 4\}$ be hyper BCH-ideals of H. But, $(3 \circ 4) = \{0, 2\} \not\ll I_1 \cup I_2$, which show that union of two hyper BCH-ideals is not a hyper BCH-ideal.

Theorem 4.7. *Every hyper BCH-ideal of a hyper BCH-algebra is a hypersubalgebra.*

Proof. Let $(H, \circ)$ be a hyper BCH-algebra and I a hyper BCH-ideal of H. Let $x, y \in I$. Then since I is a hyper BCH-ideal of H, and so by definition it implies that, $x \circ y \ll I$; which shows that I is a hypersubalgebra of H. □

The convers of the above theorem is not true, that is a hypersubalgebra is not a hyper BCH-ideal. From the above example if we consider $I_3 = \{0, 2, 4\}$, then is a hypersubalgebra of H but not a hyper BCH-ideal of H.

Proposition 4.8. *Let I be a hyper BCH-ideal and A a subset of a hyper BCH-algebra H such that $A \ll I$. Then $A \subseteq I$.*

Proof. Let I be a hyper BCH-ideal of H and A a subset of H. Let $A \ll I$ implies there exists $a \in A$ and $x \in I$ such that $a \ll x \Rightarrow 0 \in a \circ x \ll I$. Since I is a hyper BCH-ideal of H it implies that $a \in I$ and so $A \subseteq I$. □

Definition 4.9. Let I be a non-empty subset of a hyper BCH-algebra H. Then I is said to be a weak hyper BCH-ideal of H, if for all $x, y \in H$

(i) $0 \in I$

(ii) $x \circ y \subseteq I$ and $y \in I \Rightarrow x \in I$.

Theorem 4.10. *The intersection of any family of weak hyper BCH-ideal of a hyper BCH-algebra is a weak hyper BCH-ideal.*

Proof. For any $\lambda \in \Lambda$; let I_λ be a weak hyper BCH-ideal of a hyper BCH-algebra H. Then clearly $0 \in \bigcap_{\lambda\in\Lambda} I_\lambda$. Now let $x, y \in H$ be such that $x \circ y \subseteq I_\lambda$and $y \in I_\lambda$for every $\lambda \in \Lambda$. Since each I_λ for every $\lambda \in \Lambda$ is a weak hyper BCH-ideal of H. Therefore it implies that $x \circ\ y \subseteq I_\lambda$ for every $\lambda \in \Lambda$ and $y \in I_\lambda \Rightarrow x \in I_\lambda$ for every $\lambda \in \Lambda$. Hence $x \circ\ y \subseteq \bigcap_{\lambda\in\Lambda} I_\lambda$ and $y \in \bigcap_{\lambda\in\Lambda} I_\lambda \Rightarrow x \in \bigcap_{\lambda\in\Lambda} I_\lambda$. Thus $\bigcap_{\lambda\in\Lambda} I_\lambda$ is a weak hyper BCH-ideal of H. □

Proposition 4.11. *Every hyper BCH-ideal in hyper BCH-algebra H is a weak hyper BCH-ideal.*

Proof. Let I be a hyper BCH-ideal of a hyper BCH-algebra H. Let $x \circ y \subseteq I$ and $y \in I$ for some $x, y \in H$. Since $x \circ y \subseteq I$ which implies that $x \circ y \ll I$. Now since I is a hyper BCH-ideal of H, so it implies that $x \in I$. Hence I is a weak hyper BCH-ideal of H. □

Definition 4.12. Let I be a non-empty subset of a hyper BCH-algebra H. Then I is said to be a strong hyper BCH-ideal of H if for all $x, y \in H$

(i) $0 \in I$

(ii) $(x \circ y) \cap I \neq \phi$ and $y \in I \Rightarrow x \in I$.

Theorem 4.13. *The intersection of any family of strong hyper BCH-ideal of a hyper BCH-algebra is a stong hyper BCH-ideal.*

Proof. For any $\lambda \in \Lambda$; let I_λ be a strong hyper BCH-ideal of a hyper BCH-algebra H. Then clearly $0 \in \bigcap_{\lambda\in\Lambda} I_\lambda$. Now let $x, y \in H$ be such that $(x \circ y) \cap \bigcap_{\lambda\in\Lambda} I_\lambda \neq \phi$ and $y \in \bigcap_{\lambda\in\Lambda} I_\lambda$. Since each I_λ for every $\lambda \in \Lambda$ is a strong hyper BCH-ideal of H. Therefore it implies that $(x \circ y) \cap I_\lambda \neq \phi$ for every $\lambda \in \Lambda$ and $y \in I_\lambda \Rightarrow x \in I_\lambda$. Hence $(x \circ y) \cap \bigcap_{\lambda\in\Lambda} I_\lambda \neq \phi$ and $y \in \bigcap_{\lambda\in\Lambda} I_\lambda \Rightarrow x \in \bigcap_{\lambda\in\Lambda} I_\lambda$. Thus $\bigcap_{\lambda\in\Lambda} I_\lambda$ is a strong hyper BCH-ideal of H. □

Proposition 4.14. *Every strong hyper BCH-ideal in hyper BCH-algebra H is a hyper BCH-ideal.*

Proof. Let I be a strong hyper BCH-ideal of H. Let $x, y \in H$ be such that $x \circ y \ll I$ and $y \in I$. Then for $a \in x \circ y$ there exists $b \in I$ such that $a \ll b \Rightarrow 0 \in a \circ b$. It follows that $(a \circ b) \cap I \neq \phi \Rightarrow a \in I$. Thus $x \circ y \subseteq I$ and so $(x \circ y) \cap I \neq \phi$. Since I is a strong hyper BCH-ideal of H. It follows that $x \in I$. Hence I is a hyper BCH-ideal of H. □

5. Homomorphisms of hyper BCH-algebras

Definition 5.1. Let H_1 and H_2 be two hyper BCH-algebras. A mapping $\psi : H_1 \to H_2$ is called a homomorphism if

(i) $\psi(0) = 0$

(ii) $\psi(x \circ y) = \psi(x) \circ \psi(y)$; for all $x, y \in H_1$.

If ψ is $1-1$ (or onto) we say that ψ is a monomorphism (or epimorphism). And if ψ is both $1-1$ and onto, we say that ψ is an isomorphism.

Theorem 5.2. *Let $\psi : H_1 \to H_2$ be a homomorphism of hyper BCH-algebras. Then*

(i) *If S is a hyper BCH-subalgebra of H_1, then $\psi(S)$ is a hyper BCH-subalgebra of H_2,*

(ii) *$\psi(H_1)$ is a hyper BCH-subalgebra of H_2,*

(iii) *If S is a hyper BCH-subalgebra of H_2, then $\psi^{-1}(S)$ is a hyper BCH-subalgebra of H_1,*

(iv) *If I is a (weak) hyper BCH-ideal of H_2, then $\psi^{-1}(I)$ is a (weak) hyper BCH-ideal of H_1,*

(v) *$Ker\psi = \{x \in H_1 | \psi(x) = 0\}$ is a hyper BCH-ideal and hence a weak hyper BCH-ideal of H_1,*

(vi) *If ψ is onto and I is a hyper BCH-ideal of H_1 which contains $Ker\psi$, then $\psi(I)$ is a hyper BCH-ideal of H_2.*

Proof. (i) Let $x, y \in \psi(S)$. Then there exist $a, b \in S$ such that $\psi(a) = x$ and $\psi(b) = y$. It follows from Theorem 3.12 that $x \circ y = \psi(a) \circ \psi(b) = \psi(a \circ b) \subseteq \psi(S)$ so that $\psi(S)$ is a hyper BCH-subalgebra of H_2.

(ii) Proof of this is same as (i).

(iii) Since $0 \in S$, we have $\psi^{-1}(0) \subseteq \psi^{-1}(S)$. Since $\psi(0) = 0$, so $0 \in \psi^{-1}(0) \subseteq \psi^{-1}(S)$. Therefore $\psi^{-1}(S)$ is non-empty. Now let $x, y \in \psi^{-1}(S)$. Then $\psi^{-1}(x), \psi^{-1}(y) \in S$. Thus $\psi(x \circ y) = \psi(x) \circ \psi(y) \subseteq S$ and so $x \circ y \subseteq \psi^{-1}(S)$, which implies that $\psi^{-1}(S)$ is a hyper BCH-subalgebra of H_1.

(iv) Let I be a weak hyper BCH-ideal of H_2. Clearly $0 \in \psi^{-1}(I)$. Let $x, y \in H_1$ such that $x \circ y \subseteq \psi^{-1}(I)$ and $y \in \psi^{-1}(I)$. Then $\psi(x) \circ \psi(y) = \psi(x \circ y) \subseteq I$ and $\psi(y) \in I$. Since I is a weak hyper BCH-ideal, it follows from (Id2) that $\psi(x) \in I$, i.e., $x \in \psi^{-1}(I)$. Hence $\psi^{-1}(I)$ is a weak hyper BCH-ideal of H_1. Now let I be a hyper BCH-ideal of H_2. Obviously $0 \in \psi^{-1}(I)$. Let $x, y \in H_1$ such that $x \circ y \ll \psi^{-1}(I)$ and $y \in \psi^{-1}(I)$. Then there exist $t \in x \circ y$ and $z \in \psi^{-1}(I)$ such that $t \ll z$, that is $0 \in t \circ z$. Since $\psi(z) \in I$ and $0 \in t \circ z \subseteq (x \circ y) \circ z$, it follows that $0 = \psi(0) \in \psi((x \circ y) \circ z) = \psi(x \circ y) \circ \psi(z) \subseteq \psi(x \circ y) \circ I$ so that $\psi(x) \circ \psi(y) = \psi(x \circ y) \ll I$. As $\psi(y) \in I$ and I is hyper BCH-ideal, by using (Id3) we have $\psi(x) \in I$, that is $x \in \psi^{-1}(I)$. Hence $\psi^{-1}(I)$ is a hyper BCH-ideal of H_1.

(v) First we show that $\{0\} \subseteq H_2$ is a hyper BCH-ideal. To do this, let $x, y \in H_2$ be such that $x \circ y \ll \{0\}$ and $y \in \{0\}$. Then $y = 0$ and so $x \circ 0 = x \circ y \ll \{0\}$. Therefore there exists $t \in x \circ 0$ such that $t \ll 0$. Thus $t = 0$, and consequently

$0 \in x \circ 0$, that is $x \ll 0$, which implies that $x = 0$. This shows that $\{0\}$ is a hyper BCH-ideal of H_2. Now by (iv), $Kerf = \psi^{-1}(\{0\})$ is a hyper BCH-ideal of H_1.

(vii) Since $0 \in I$, we have $0 = \psi(0) \in \psi(I)$. Let x and y be arbitrary elements in H_2 such that $x \circ y \ll f(I)$ and $y \in \psi(I)$. Since $y \in \psi(I)$ and ψ is onto, there are $y_1 \in I$ and $x_1 \in H_1$ such that $y = \psi(y_1)$ and $x = \psi(x_1)$. Thus $\psi(x_1 \circ y_1) = \psi(x_1) \circ \psi(y_1) = x \circ \circ y \ll \psi(I)$. Therefore there are $a \in x_1 \circ y_1$ and $b \in I$ such that $\psi(a) \ll \psi(b)$. So $0 \in \psi(a) \circ \psi(b) = \psi(a \circ b)$, which implies that $\psi(c) = 0$ for some $c \in a \circ b$. It follows that $c \in Ker\psi \subseteq I$ so that $a \circ b \ll I$. Now since I is a hyper BCH-ideal of H_1 and $b \in I$, we get $a \in I$. Thus $x_1 \circ y_1 \ll I$, which implies that $x_1 \in I$. Thus $x = \psi(x_1) \in \psi(I)$, and so $\psi(I)$ is a hyper BCH-ideal of H_2. □

Theorem 5.3. *Let $\psi : H_1 \to H_2$ be an epimorphism of hyper BCH-algebras. Then there is a one to one correspondence between the set of all hyper BCH-ideals of H_1 containing $Ker\psi$ and the set of all hyper BCH-ideals of H_2.*

Theorem 5.4. *Let $\psi : H_1 \to H_2$ and $\pi : H_1 \to H_3$ be two homomorphisms of hyper BCH- algebras such that ψ is onto and $Ker\psi \subseteq Ker\pi$. Then there exists a homomorphism $\tau : H_2 \to H_3$ such that $\tau \circ \psi = \pi$.*

Proof. Let $y \in H_2$ be arbitrary. Since ψ is onto, there exists $x \in H_1$ such that $y = \psi(x)$. Define $\tau : H_2 \to H_3$ by $\tau(y) = \pi(x)$, for all $y \in H_2$. Now we show that τ is well-defined. Let $y_1; y_2 \in H_2$ and $y_1 = y_2$. Since ψ is onto, there are $x_1; x_2 \in H_1$ such that $y_1 = \psi(x_1)$ and $y_2 = \psi(x_2)$. Therefore $\psi(x_1) = \psi(x_2)$ and thus $0 \in \psi(x_1) \circ \psi(x_2) = \psi(x_1 \circ x_2)$. It follows that there exists $t \in x_1 \circ x_2$ such that $\psi(t) = 0$. Thus $t \in Ker\psi \subseteq Ker\pi$ and so $\pi(t) = 0$. Since $t \in x_1 \circ x_2$ we conclude that $0 = \pi(t) \in \pi(x_1 \circ x_2) = \pi(x_1) \circ \pi(x_2)$ which implies that $\pi(x_1) \ll \pi(x_2)$. On the other hand since $0 \in \psi(x_2) \circ \psi(x_1) = \psi(x_2 \circ x_1)$, similarly we can conclude that $0 \in \pi(x_2) \circ \pi(x_1)$, that is $\pi(x_2) \ll \pi(x_1)$. Thus $\pi(x_1) = \pi(x_2)$, which shows that τ is well-defined. Clearly $\tau \circ \psi = \pi$. Finally we show that τ is a homomorphism. Let $y_1; y_2 \in H_2$ be arbitrary. Since ψ is onto there are $x_1, x_2 \in H_1$ such that $y_1 = \psi(x_1)$ and $y_2 = \psi(x_2)$. Then

$$\begin{aligned}
\tau(y_1 \circ y_2) &= \tau(\psi(x_1) \circ \psi(x_2)) \\
&= \tau(\psi(x_1 \circ x_2)) \\
&= (\tau \circ \psi)(x_1 \circ x_2) \\
&= \pi(x_1 \circ x_2) \\
&= \pi(x_1) \circ \pi(x_2) \\
&= (\tau \circ \psi)(x_1) \circ (\tau \circ \psi)(x_2) \\
&= \tau(\psi(x_1)) \circ \tau(\psi(x_2)) \\
&= \tau(y_1) \circ \tau(y_2)
\end{aligned}$$

Moreover since $\psi(0) = 0$ and $\pi(0) = 0$, we conclude that $\tau(0) = \tau(\psi(0)) = (\tau \circ \psi)(0) = \pi(0) = 0$. Thus τ is a homomorphism. □

Theorem 5.5. *Let $\psi : H_1 \to H_2$ be a homomorphism of hyper BCH-algebras. If I is a strong hyper BCH-ideal of H_2, then $\psi^{-1}(I)$ is a strong hyper BCH-ideal of H_1.*

Proof. Suppose I is a strong hyper BCH-ideal, then clearly $0 \in \psi^{-1}(I)$. Let $a, b \in H_1$ be such that $(a \circ b) \cap \psi^{-1}(I) \neq \phi$ and $b \in \psi^{-1}(I)$. Then we have $\phi \neq \psi(a \circ b) \cap \psi^{-1}(I)) \subseteq \psi(a \circ b) \cap \psi\psi^{-1}(I) \subseteq \psi(a) \circ \psi(b) \cap I$ and so $(\psi(a) \circ \psi(b)) \cap I \neq \phi$ and $\psi(a) \in \psi(\psi^{-1}(I)) \subseteq I$. Since I is a strong hyper BCH-ideal of H_2, we have $\psi(a) \in I$ and so $x \in \psi^{-1}(I)$. Therefore $\psi^{-1}(I)$ is a strong hyper BCH-ideal of H_1. □

Theorem 5.6. *Let $\psi : H_1 \to H_2$ be a homomorphism of hyper BCH-algebras. Then $ker\psi = \{x \in H_1 | \psi(x) = 0\}$ is a strong hyper BCH-ideal of H_1.*

Proof. To prove this first we show that $\{0\}$ is a strong hyper BCH-ideal of H_2. For this, let $a, b \in H_1$ be such that $(a \circ b) \cap \{0\} \neq \phi$ and $b \in \{0\}$. Then $b = 0$ and so $0 \in a \circ 0$ since $(a \circ 0) \cap \{0\} \neq \phi$. Thus we have $a \ll 0$. By (HCH3) and 3.5 3, we get $a = 0 \in \{0\}$. This shows that $\{0\}$ is a strong hyper BCH-ideal of H_2. It follows fromTheorem 5.5 that $ker\psi = \psi^{-1}(\{0\})$ is a strong hyper BCH-ideal of H_1. □

Theorem 5.7. *Let $\psi : H_1 \to H_2$ be a homomorphism of hyper K-algebras. If $\psi$is onto and I is a strong hyper BCH-ideal of H_1 which contains $ker\psi$, then $\psi(I)$ is a strong hyper BCH-ideal of H_2.*

Proof. Suppose I is a strong hyper BCH-ideal of H_1. Clearly $0 \in \psi(I)$. Let $x, y \in H_2$ be such that $(x \circ y) \cap \psi(I) \neq \phi$ and $y \in \psi(I)$. Since $y \in \psi(I)$ and ψ is onto, there are $y_1 \in I$ and $x_1 \in H_1$ such that $y = \psi(y_1)$ and $x = \psi(x_1)$. Thus $\phi \neq (x \circ y) \cap \psi(I) = \psi(x_1 \circ y_1) \cap \psi(I)$ and so there exists $a \in H_2$ such that $a \in \psi(x_1 \circ y_1)$ and $a \in \psi(I)$. It follows that there are $a_1 \in x_1 \circ y_1$ and $b_1 \in I$ such that $a = \psi(a_1)$ and $a = \psi(b_1)$ so that $0 \in a \circ a = \psi a_1 \circ \psi b_1 = \psi\,(a_1 \circ b_1)$ which implies that $\psi(c) = 0$ for some $c \in a_1 \circ b_1$. Hence $c \in ker\psi \subseteq I$ and so $(a_1 \circ b_1) \cap I \neq \phi$. Now since I is a strong hyper BCH-ideal of H_1 and $b_1 \in I$, we get $a_1 \in I$. Thus $(x_1 \circ y_1) \cap I \neq \phi$, which implies that $x_1 \in I$. Thereby $x = \psi(x_1) \in \psi(I)$, and so $\psi(I)$ is a strong hyper BCH-ideal of H_2. □

References

[1] M.A. Chaudhry, *On BCH-algebras*, Math. Japonica, 36 (1991), 665-676.

[2] W.A. Dudek, J. Thomys, *On decompositions of BCH-algebras*, Math. Japon. 35 (1990) no. 6, 1131–1138.

[3] Q.P. Hu, X. Li, *On BCH-algebras*, Math. Seminar Notes, 11 (1983), 313-320.

[4] Q.P. Hu, X. Li, *On proper BCH-algebras*, Math. Japonica, 30 (1985), 659-661.

[5] Y. Imai, K. Iseki, *On axiom systems of propositional calculi*, XIV, Proc.Japan Academy., 4 (1966), 219-22.

[6] K. Iseki, *An Algebra related with a propositional calculus*, Proc. Japan Acad., 42 (1966), 351-366.

[7] K. Iseki, *On some ideals in BCK-algebras*, Seminar Notes., 1975, 365-70.

[8] K. Iseki, *On BCI-algebras*, Math. Seminar Notes, Kobe University 8125-130.

[9] K. Iseki, S. Tanaka, *Ideal theory of BCI-algebras*, Math. Japon. 21 (1976), 351-366.

[10] Y.B. Jun, M.M. Zahedi, X. L. Xin, R. A. Borzooei, *On hyper BCKalgebras,* Italian J. Pure and Appl. Math. 10 (2000), 127-136.

[11] Y.B. Jun, X.L. Xin, E.H. Roh, M.M. Zahedi, *Strong hyper BCK-ideals of hyper BCK-algebra*, Mathematicae Japonicae, Vol. 51, No. 3 (2000), 493-498.

[12] F. Marty, *Sur une generalization de la notion de groupe*, 8th Congress Math. Scandinavas, Stockholm, 45-49, 2000.

[13] X.L. Xin, *Hyper BCI-algebras*, Discussions Mathematicae General Algebra and Applications, 26 (2006), 5-19.

Accepted: 20.02.2017