ON HYPER BCH-ALGEBRA

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Abstract. In this paper we initiate the concept of a hyper BCH-algebra which is a
generalization of a BCH-algebra, and hyper BCK/BCI algebras and investigate some
related properties. Moreover we introduce a hyper BCH-ideal, weak hyper BCH-ideal
and strong hyper BCH-ideal in hyper BCH-algebras, and give a few relations among
these hyper BCH-ideals. Finally we define homomorphism of hyper BCH-algebras.

Keywords: hyper BCH-algebra, hyper BCH-ideals, week hyper BCH-ideals, Strong
hyper BCH-ideals, homomorphism.

1. Introduction

In (1966) the notion of BCK-algebra was first introduces by Y. Imai and K.
Iseki [6]. The notion of BCK-algebra is a generalization of properties of the
Set-difference. In (1975), the concept of ideal in BCK-algebra was first initiated
by K. Iseki [7]. A remarkable feature of K. Iseki definition is that, its for-
mulation is free from those of ring theoretical and lattice theoretical concepts.
In same year K. Iseki initiated the concept of BCI-algebra [6, 8] which is the
generalization of BCK-algebra. These algebras have been extensively studied

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since their introduction. The concept of ideals has played an important role in the study of the theory of BCI-algebras, [9]. In a BCI-algebra $X$, an ideal $I$ need not be subalgebra of $X$. If the ideal $I$ is also a subalgebra of $X$, then it has better algebraic properties. In (1983), Q. P. Hu and X. Li, introduced the concept of BCH-algebra [3, 4] and prove some motivating results. In (1990) and (1991) certain other properties have been studied by W. A. Dudek and J. Thomys [2] and M. A. Chaudhry, [1], respectively. In [1], the author also defines ideals in BCH-algebras. Hyperstructure represent a natural extension of classical algebraic structures and they were introduced by the French mathematician F. Marty in (1934), [12]. Algebraic hyperstructures are a suitable generalization of classical algebraic structures. Hyperstructures have many applications to several sectors of both pure and applied sciences. In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure; the composition of two elements is a set. In (2000) Y. B. Jun et al applied the hyperoperation to BCK-algebras and introduced the concept of a hyper BCK-algebra [12] which is a generalization of a BCK-algebra, and investigated some related properties. Ideal theory of hyper BCK-algebra studied in [11]. Further in (2006), X.L. Xin initiated the concept of hyper BCI-algebras [13], which is basically a generalization of hyper BCK-algebras, and he proved that every hyper BCK-algebra is a hyper BCI-algebra. It should be pointed out that the research of hyper BCI-algebras seems to have been focused on the ideal theory. The author introduced the concepts of hyper BCI-ideals, strong hyper BCI-ideals and reflexive hyper BCI-ideals in hyper BCI-algebras, and he gave the relations among these hyper BCI-ideals. In this paper we initiated the notion of hyper BCH-algebra which is a generalization of BCH-algebra and hyper BCI/BCK-algebras and studied some basic properties. Moreover we introduce a hyper BCH-ideal, weak hyper BCH-ideal and strong hyper BCH-ideal in hyper BCH-algebras, and give some relations among these hyper BCH-ideals. We define homomorphism in hyper BCH-algebra and then we investigate some related results.

2. Preliminaries

Let $H$ be a non-empty set and "$\circ$" a function from $H \times H \rightarrow P(H) \setminus \{\emptyset\}$, where $P(H)$ denotes the power set of $H$. For any two non-empty subsets $A$ and $B$ of $H$, denote by $A \circ B$ the set $\bigcup_{a \in A, b \in B} a \circ b$. We will use $x \circ y$ instead of $x \circ \{y\}$, $\{x\} \circ y$ or $\{x\} \circ \{y\}$. Also we define $x \ll y$ by $0 \in x \circ y$ and for every $A, B \subseteq H$, $A \ll B$ is defined by for all $a \in A$, there exist $b \in B$ such that $a \ll b$.

**Definition 2.1 ([10]).** A non-empty set $H$ endowed with a constant $0$ and a hyperoperation is called hyper BCK-algebra if it satisfies the following axioms:

- **HK1** $(x \circ y) \circ (y \circ z) \ll x \circ y$,
- **HK2** $(x \circ y) \circ z = (x \circ z) \circ y$,
- **HK3** $x \circ H \ll \{x\}$,
- **HK4** $x \ll y$ and $y \ll x \Rightarrow x = y$.
for all $x, y, z \in H$.

**Definition 2.2** ([13]). A non-empty set $H$ endowed with a constant 0 and a hyperoperation is called hyper BCI-algebra if it satisfies the following axioms:
- $H1_1$ $(x \circ y) \circ (y \circ z) \ll x \circ y$,
- $H1_2$ $(x \circ y) \circ z = (x \circ z) \circ y$,
- $H1_3$ $x \circ H \ll \{x\}$,
- $H1_4$ $x \ll y$ and $y \ll x \Rightarrow x = y$.

for all $x, y, z \in H$.

**Definition 2.3** ([11]). Let $I$ be a nonempty subset of a hyper BCK-algebra $H$ and $0 \in I$. Then $I$ is said to be a hyper BCK-ideal of $H$ if $x \circ y \ll I$ and $y \in I$ implies $x \in I$ for all $x, y \in H$, reflexive if $x \circ x \subseteq I$ for all $x \in H$, strong hyper BCK-ideal of $H$ if $(x \circ y) \cap I = \emptyset$ and $y \in I$ implies $x \in I$ for all $x, y \in H$, hyper subalgebra of $H$ if $x \circ y \subseteq I$ for all $x, y \in I$.

**Proposition 2.4** ([11]). Let $H$ be hyper BCK-algebra. Then,
- (i) any strong hyper BCK-ideal of $H$ is a hyper BCK-ideal of $H$.
- (ii) if $I$ is a hyper BCK-ideal of $H$ and $A$ is a nonempty subset of $H$. Then $A \ll I$ implies $A \subseteq I$.
- (iii) if $I$ is a reflexive hyper BCK-ideal of $H$ and $(x \circ y) \cap I = \emptyset$, then $x \circ y \subseteq I$ for all $x, y \in H$.
- (iv) $H$ is a BCK-algebra if and only if $H = \{x \in H : x \circ x = \{0\}\}$.

### 3. Hyper BCH-algebra

In this section we introduce a notion of hyper BCH-algebra and studied some of its basic properties.

**Definition 3.1.** Let $H$ be a on-empty set with a constant "0" and "$\circ$" be a hyper operation defined on $H$. Then $(H, \circ, 0)$ is said to be a hyper BCH-algebra if the following axioms are satisfied:
- $HCH1$ $x \ll x$,
- $HCH2$ $(x \circ y) \circ z = (x \circ z) \circ y$,
- $HCH3$ $x \ll y$ and $y \ll x \Rightarrow x = y$

for all $x, y, z \in H$: where $x \ll y$ is defined by $0 \in x \circ y$ and for every $A, B \subseteq H$, $A \ll B$ is defined by for all $a \in A$, there exists $b \in B$ such that $a \ll b$. In such case, "$\ll$" is called a hyper order in $H$.

**Example 3.2.** Let $H = \{0, 1, 2\}$ and "$\circ$" be a hyperoperation defined on $H$ in the following table:

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<tr>
<th>$\circ$</th>
<th>0</th>
<th>1</th>
<th>2</th>
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<tbody>
<tr>
<td>0</td>
<td>${0}$</td>
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<td>${1}$</td>
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<td>1</td>
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<td>${0, 1}$</td>
<td>${0, 1}$</td>
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<tr>
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<td>${2}$</td>
<td>${0, 2}$</td>
<td>${0, 1, 2}$</td>
</tr>
</tbody>
</table>

Then $(H, \circ)$ is a hyper BCH-algebra.
Example 3.3. Let \( H = \{0, 1, 2, 3\} \) and "\( \circ \)" be a hyperoperation defined on \( H \) in the following table:

<table>
<thead>
<tr>
<th>( \circ )</th>
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<td>{0, 1}</td>
<td>{0, 3}</td>
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<td>{0, 2}</td>
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</tbody>
</table>

Then \((H, \circ)\) is a hyper BCH-algebra.

Proposition 3.4. Any hyper BCK/BCI- algebra is a hyper BCH-algebra.

Proposition 3.5. Let \( H \) be a hyper BCH-algebra, then for all \( x, y, z \in H \) and \( A \subseteq H \); the following holds.

1. \( x \circ y \ll z \iff x \circ z \ll y \)
2. \( x \circ y \ll x \)
3. \( 0 \ll x \)
4. \( t \in 0 \circ 0 \iff t = 0 \)
5. \( x \in x \circ 0 \)
6. \( A \circ y \ll A \)
7. \( x \circ A \ll y \iff x \circ y \ll A \)
8. \( A \ll A \circ 0 \)
9. \( x \circ x = \{x\} \iff x = 0 \).

Proof. We only prove 1, 2, 5, 6, 7 and 9.

1) Let \( x, y, z \in H, \) be such that \( x \circ y \ll z. \) Then there exists \( t \in x \circ y \) such that \( t \ll z. \) Thus \( 0 \in t \circ z \subseteq (x \circ y) \circ z = (x \circ z) \circ y \) and hence there exists \( w \in x \circ z \) such that \( 0 \in w \circ y \) that is \( w \ll y. \) Therefore \( x \circ z \ll y. \)

Conversely, let \( x, y, z \in H \) be such that \( x \circ z \ll y. \) Then there exists \( w \in x \circ z \) such that \( w \ll y. \) Thus \( 0 \in w \circ y \subseteq (x \circ z) \circ y = (x \circ y) \circ z \) and hence there exists \( t \in x \circ y \) such that \( 0 \in t \circ z \) that is \( t \ll z. \) Therefore \( x \circ y \ll z. \)

2) Let \( 0 \in 0 \circ 0 \subseteq (x \circ x) \circ y = (x \circ y) \circ x. \) Then there exists \( t \in x \circ y \) such that \( 0 \in t \circ x \Rightarrow t \ll t \Rightarrow x \circ y \ll x. \)

5) By (2) above we have \( x \circ 0 \ll x, \) so there exists \( t \in x \circ 0 \) such that \( t \ll x, \) since \( t \in x \circ 0, \) then \( x \circ 0 \ll t \) and so by (1) \( x \circ t \ll 0. \) Thus there is \( r \in x \circ t \) such that \( r \ll 0, \) so by (3) and \((HCH3)\) \( r = 0. \) so \( 0 \in x \circ t, \) that is \( x \ll t \) since \( x \ll t \) and \( t \ll x; \) then by \((HCH3)\) \( x = t. \) Therefore \( x \in x \circ 0. \)

6) Let \( a \in A \) be any element, then by (2) \( a \circ y \ll a \) hence there is \( b \in a \circ y \subseteq A \circ y \) such that \( b \ll a, \) that is \( A \circ y \ll A. \)

7) Since \( x \circ A \ll y \) which implies that there exists \( a \in A \) such that \( x \circ a \ll y. \) Hence by (1) \( x \circ a \ll a \ll A \) implies that \( x \circ y \ll A. \) The proof of the converse is easy to prove.

9) \( \{x\} = x \circ x \subseteq x \circ (x \circ 0). \) Hence by (5) \( x \ll 0; \) thus \( x = 0. \) The converse follows from (4). \( \square \)
Proposition 3.6. In any hyper BCH-algebra \( H \), \( x \circ 0 = \{ x \} \) for all \( x \in H \).

**Proof.** We have from above proposition (5) \( x \in x \circ 0 \), now let \( t \in x \circ 0 \). Since \( x \circ 0 \ll \{ x \} \), we have \( t \ll x \). So, \( 0 \in t \circ t \subseteq (x \circ 0) \circ t = (x \circ t) \circ 0 \). Then there exists \( a \in x \circ t \) such that \( 0 \in a \circ 0 \). Thus \( a \ll 0 \). Then \( a = 0 \); Thus \( x \ll t \). We have that \( x = t \). Therefore, \( x \circ 0 = \{ x \} \). \( \square \)

It is known that every hyper BCI-algebra is a hyper BCH-algebra, but the following example show that the converse is not true.

**Example 3.7.** Let \( H = \{0, 1, 2, 3\} \) and \( \circ \) be a hyperoperation define on \( H \) in the following table:

<table>
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<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
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</tbody>
</table>

Then \( (H, \circ) \) is a hyper BCH-algebra, but it is not a hyper BCI-algebra. Because,

\[
(2 \circ 3) \circ (2 \circ 1) = \{2\} \circ \{3\} = \{2, 3\}
\]

and

\[
(1 \circ 3) = \{3\}, (2 \circ 3) \circ (2 \circ 1) \neq (1 \circ 3)
\]

**Example 3.8.** Let \( H = \{0, 1, 2, 3, 4\} \) and \( \circ \) be a hyperoperation defined of \( H \) in the following table:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
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</thead>
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<tr>
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</tbody>
</table>

Then \( (H, \circ) \) is a hyper BCH-algebra, but it is not a hyper BCI-algebra. Because,

\[
(1 \circ 3) \circ (1 \circ 2) = \{1\} \circ \{2\} = \{1, 2\}
\]

and \( (2 \circ 3) = \{2\} \) that is \( \{1, 2\} \not\ll \{2\} \).

**Definition 3.9.** A hyper BCH-algebra \( H \) is called proper if it is not a hyper BCI-algebra.

In above examples the hyper BCH-algebras are proper hyper BCH-algebras.

**Definition 3.10.** Let \( (H, \circ) \) be a hyper BCH-algebra, and \( X \) a non-empty subset of \( H \) containing \( \{0\} \). Then \( X \) is called hypersubalgebra of \( H \) if \( X \) is a hyper BCH-algebra under the same hyperoperation \( \circ \) on \( H \).
Example 3.11. From the above Example 3.8 if we let $X = \{0, 1, 2\}$, then $X$ is a hypersubalgebra of $H$ as we in the following table:

<table>
<thead>
<tr>
<th>$\circ$</th>
<th>0</th>
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<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
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Also, let $X = \{0, 1, 3\}$. Then $X$ is a hypersubalgebra of $H$.

Theorem 3.12. Let $X$ be a non-empty subset of a hyper BCH-algebra $(H, \circ)$. The $X$ is a hypersubalgebra of $H$ if and only if $x \circ y \subseteq X$ for all $x, y \in X$.

Proof. Straightforward.

Theorem 3.13. Let $(H, \circ)$ be a hyper BCH-algebra and $X(H) = \{x \in H \mid 0 \circ x \ll \{0\}\}$. Then $X(H)$ is a hypersubalgebra of $H$.

Proof. Let $x, y \in X(H)$, then by definition $a = 0 \circ a \ll \{0\}$ and $b = 0 \circ b \ll \{0\}$. Now

$$a \circ b = (0 \circ a) \circ (0 \circ b) \ll \{0\} \circ \{0\} = \{0\}$$

Hence, $a \circ b \ll \{0\}$. Which implies that $a \circ b \ll X(H)$. Hence $X(H)$ is a hypersubalgebra of $H$. The set $X(H)$ is called the hyper BCA-part of the hyper BCH-algebra $H$.

4. Hyper BCH-Ideals

Definition 4.1. Let $(H, \circ)$ be a hyper BCH-algebra and $I$ a subset of $H$. Then $I$ is called a hyper BCH-ideal of $H$ if:

i) $0 \in I$

ii) $x \circ y \ll I$ and $y \in I \Rightarrow x \in I$ for all $x, y \in I$.

Example 4.2. Let $H = \{0, 1, 2, 3, 4, 5\}$ and "$\circ$" be a hyperoperation defined on $H$ in the following table:

<table>
<thead>
<tr>
<th>$\circ$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
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</table>

Then $(H, \circ)$ is a hyper BCH-algebra. Let $I = \{0, 1, 2, 3\}$ is an ideal of $H$. 

Example 4.3. Let \( H = \{0, 1, 2, 3, 4\} \) and "\( \circ \)" be a hyperoperation defined on \( H \) in the following table:

<table>
<thead>
<tr>
<th>( \circ )</th>
<th>0</th>
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</table>

Then \((H, \circ)\) is a hyper BCH-algebra.

Let \( I_1 = \{0, 1, 2\} \), then \( I_1 \) is a hyper BCH-ideal of \( H \).

Let \( I_2 = \{0, 1, 3\} \), then \( I_2 \) is a hyper BCH-ideal of \( H \).

Let \( I_3 = \{0, 2, 3\} \), then \( I_3 \) is not a hyper BCH-ideal of \( H \). Because \((3 \circ 4) = \{0, 2\} \not\subseteq I_3 \) and \( 4 \in I_3 \) but \( 3 \not\in I_3 \).

Theorem 4.4. Let \((H, \circ)\) be a hyper BCH-algebra and \( \{I_\lambda | \lambda \in \Lambda\} \) a family of hyper BCH-ideals of \( H \), then \( \bigcap_{\lambda \in \Lambda} I_\lambda \) is a hyper BCH-ideal of \( H \).

Proof. For any \( \lambda \in \Lambda \); let \( I_\lambda \) be a hyper BCH-ideal of a hyper BCH-algebra \( H \), then clearly \( 0 \in \bigcap_{\lambda \in \Lambda} I_\lambda \). Now let \( x, y \in H \) be such that \( x \circ y \ll I_\lambda \) and \( y \in I_\lambda \) for every \( \lambda \in \Lambda \). Since each \( I_\lambda \) for each \( \lambda \in \Lambda \) is a hyper BCH-ideal of \( H \). Therefore it implies that \( x \circ y \ll I_\lambda \) for every \( \lambda \in \Lambda \) and \( y \in I_\lambda \Rightarrow x \in I_\lambda \). Hence \( x \circ y \ll \bigcap_{\lambda \in \Lambda} I_\lambda \) and \( y \in \bigcap_{\lambda \in \Lambda} I_\lambda \Rightarrow x \in \bigcap_{\lambda \in \Lambda} I_\lambda \). Thus \( \bigcap_{\lambda \in \Lambda} I_\lambda \) is a hyper BCH-ideal of \( H \).

Remark 4.5. The union of two hyper BCH-ideals need not be hyper BCH-ideals. For this we have the following example.

Example 4.6. Let \( H = \{0, 1, 2, 3, 4\} \) be a hyper BCH-algebra define in Example 4.3. Let \( I_1 = \{0, 1, 3\} \) and \( I_2 = \{0, 1, 4\} \) be hyper BCH-ideals of \( H \). But, \((3 \circ 4) = \{0, 2\} \not\subseteq I_1 \cup I_2 \), which show that union of two hyper BCH-ideals is not a hyper BCH-ideal.

Theorem 4.7. Every hyper BCH-ideal of a hyper BCH-algebra is a hypersubalgebra.

Proof. Let \((H, \circ)\) be a hyper BCH-algebra and \( I \) a hyper BCH-ideal of \( H \). Let \( x, y \in I \). Then since \( I \) is a hyper BCH-ideal of \( H \), and so by definition it implies that, \( x \circ y \ll I \); which shows that \( I \) is a hypersubalgebra of \( H \).

The convers of the above theorem is not true, that is a hypersubalgebra is not a hyper BCH-ideal. From the above example if we consider \( I_3 = \{0, 2, 4\} \), then is a hypersubalgebra of \( H \) but not a hyper BCH-ideal of \( H \).

Proposition 4.8. Let \( I \) be a hyper BCH-ideal and \( A \) a subset of a hyper BCH-algebra \( H \) such that \( A \ll I \). Then \( A \subseteq I \).
Let $I$ be a hyper BCH-ideal of $H$ and $A$ a subset of $H$. Let $A \ll I$ implies there exists $a \in A$ and $x \in I$ such that $a \ll x \Rightarrow 0 \in a \circ x \ll I$. Since $I$ is a hyper BCH-ideal of $H$ it implies that $a \in I$ and so $A \subseteq I$.

**Definition 4.9.** Let $I$ be a non-empty subset of a hyper BCH-algebra $H$. Then $I$ is said to be a weak hyper BCH-ideal of $H$, if for all $x, y \in H$

1. $0 \in I$
2. $x \circ y \subseteq I$ and $y \in I \Rightarrow x \in I$.

**Theorem 4.10.** The intersection of any family of weak hyper BCH-ideal of a hyper BCH-algebra is a weak hyper BCH-ideal.

**Proof.** For any $\lambda \in \Lambda$; let $I_\lambda$ be a weak hyper BCH-ideal of a hyper BCH-algebra $H$. Then clearly $0 \in \bigcap_{\lambda \in \Lambda} I_\lambda$. Now let $x, y \in H$ be such that $x \circ y \subseteq I_\lambda$ and $y \in I_\lambda$ for every $\lambda \in \Lambda$. Since each $I_\lambda$ for every $\lambda \in \Lambda$ is a weak hyper BCH-ideal of $H$. Therefore it implies that $x \circ y \subseteq I_\lambda$ for every $\lambda \in \Lambda$ and $y \in I_\lambda \Rightarrow x \in I_\lambda$ for every $\lambda \in \Lambda$. Hence $x \circ y \subseteq \bigcap_{\lambda \in \Lambda} I_\lambda$ and $y \in \bigcap_{\lambda \in \Lambda} I_\lambda \Rightarrow x \in \bigcap_{\lambda \in \Lambda} I_\lambda$. Thus $\bigcap_{\lambda \in \Lambda} I_\lambda$ is a weak hyper BCH-ideal of $H$.

**Proposition 4.11.** Every hyper BCH-ideal in hyper BCH-algebra $H$ is a weak hyper BCH-ideal.

**Proof.** Let $I$ be a hyper BCH-ideal of a hyper BCH-algebra $H$. Let $x \circ y \subseteq I$ and $y \in I$ for some $x, y \in H$. Since $x \circ y \subseteq I$ which implies that $x \circ y \ll I$. Now since $I$ is a hyper BCH-ideal of $H$, so it implies that $x \in I$. Hence $I$ is a weak hyper BCH-ideal of $H$.

**Definition 4.12.** Let $I$ be a non-empty subset of a hyper BCH-algebra $H$. Then $I$ is said to be a strong hyper BCH-ideal of $H$ if for all $x, y \in H$

1. $0 \in I$
2. $(x \circ y) \cap I \neq \emptyset$ and $y \in I \Rightarrow x \in I$.

**Theorem 4.13.** The intersection of any family of strong hyper BCH-ideal of a hyper BCH-algebra is a strong hyper BCH-ideal.

**Proof.** For any $\lambda \in \Lambda$; let $I_\lambda$ be a strong hyper BCH-ideal of a hyper BCH-algebra $H$. Then clearly $0 \in \bigcap_{\lambda \in \Lambda} I_\lambda$. Now let $x, y \in H$ be such that $(x \circ y) \cap \bigcap_{\lambda \in \Lambda} I_\lambda \neq \emptyset$ and $y \in \bigcap_{\lambda \in \Lambda} I_\lambda$. Since each $I_\lambda$ for every $\lambda \in \Lambda$ is a strong hyper BCH-ideal of $H$. Therefore it implies that $(x \circ y) \cap I_\lambda \neq \emptyset$ for every $\lambda \in \Lambda$ and $y \in I_\lambda \Rightarrow x \in I_\lambda$. Hence $(x \circ y) \cap \bigcap_{\lambda \in \Lambda} I_\lambda \neq \emptyset$ and $y \in \bigcap_{\lambda \in \Lambda} I_\lambda \Rightarrow x \in \bigcap_{\lambda \in \Lambda} I_\lambda$. Thus $\bigcap_{\lambda \in \Lambda} I_\lambda$ is a strong hyper BCH-ideal of $H$.

**Proposition 4.14.** Every strong hyper BCH-ideal in hyper BCH-algebra $H$ is a hyper BCH-ideal.

**Proof.** Let $I$ be a strong hyper BCH-ideal of $H$. Let $x, y \in H$ be such that $x \circ y \ll I$ and $y \in I$. Then for $a \in x \circ y$ there exists $b \in I$ such that $a \ll b \Rightarrow 0 \in a \circ b$. It follows that $(a \circ b) \cap I \neq \emptyset \Rightarrow a \in I$. Thus $x \circ y \subseteq I$ and so $(x \circ y) \cap I \neq \emptyset$. Since $I$ is a strong hyper BCH-ideal of $H$. It follows that $x \in I$. Hence $I$ is a hyper BCH-ideal of $H$. $\square$
5. Homomorphisms of hyper BCH-algebras

Definition 5.1. Let $H_1$ and $H_2$ be two hyper BCH-algebras. A mapping $\psi : H_1 \to H_2$ is called a homomorphism if

(i) $\psi(0) = 0$

(ii) $\psi(x \circ y) = \psi(x) \circ \psi(y)$; for all $x, y \in H_1$.

If $\psi$ is $1 - 1$ (or onto) we say that $\psi$ is a monomorphism (or epimorphism). And if $\psi$ is both $1 - 1$ and onto, we say that $\psi$ is an isomorphism.

Theorem 5.2. Let $\psi : H_1 \to H_2$ be a homomorphism of hyper BCH-algebras. Then

(i) If $S$ is a hyper BCH-subalgebra of $H_1$, then $\psi(S)$ is a hyper BCH-subalgebra of $H_2$.

(ii) $\psi(H_1)$ is a hyper BCH-subalgebra of $H_2$.

(iii) If $S$ is a hyper BCH-subalgebra of $H_2$, then $\psi^{-1}(S)$ is a hyper BCH-subalgebra of $H_1$.

(iv) If $I$ is a (weak) hyper BCH-ideal of $H_2$, then $\psi^{-1}(I)$ is a (weak) hyper BCH-ideal of $H_1$.

(v) $\ker\psi = \{x \in H_1 | \psi(x) = 0\}$ is a hyper BCH-ideal and hence a weak hyper BCH-ideal of $H_1$.

(vi) If $\psi$ is onto and $I$ is a hyper BCH-ideal of $H_1$ which contains $\ker\psi$, then $\psi(I)$ is a hyper BCH-ideal of $H_2$.

Proof. (i) Let $x, y \in \psi(S)$. Then there exist $a, b \in S$ such that $\psi(a) = x$ and $\psi(b) = y$. It follows from Theorem 3.12 that $x \circ y = \psi(a) \circ \psi(b) = \psi(a \circ b) \subseteq \psi(S)$ so that $\psi(S)$ is a hyper BCH-subalgebra of $H_2$.

(ii) Proof of this is same as (i).

(iii) Since $0 \in S$, we have $\psi^{-1}(0) \subseteq \psi^{-1}(S)$. Since $\psi(0) = 0$, so $0 \in \psi^{-1}(0) \subseteq \psi^{-1}(S)$. Therefore $\psi^{-1}(S)$ is non-empty. Now let $x, y \in \psi^{-1}(S)$. Then $\psi^{-1}(x), \psi^{-1}(y) \in S$. Thus $\psi(x \circ y) = \psi(x) \circ \psi(y) \subseteq S$ and so $x \circ y \subseteq \psi^{-1}(S)$, which implies that $\psi^{-1}(S)$ is a hyper BCH-subalgebra of $H_1$.

(iv) Let $I$ be a weak hyper BCH-ideal of $H_2$. Clearly $0 \in \psi^{-1}(I)$. Let $x, y \in H_1$ such that $x \circ y \in \psi^{-1}(I)$ and $y \in \psi^{-1}(I)$ and $\psi(y) \in I$. Since $I$ is a weak hyper BCH-ideal, it follows from (Id2) that $\psi(x) \in I$, i.e., $x \in \psi^{-1}(I)$. Hence $\psi^{-1}(I)$ is a weak hyper BCH-ideal of $H_1$. Now let $I$ be a hyper BCH-ideal of $H_2$. Obviously $0 \in \psi^{-1}(I)$. Let $x, y \in H_1$ such that $x \circ y \in \psi^{-1}(I)$ and $y \in \psi^{-1}(I)$. Then there exist $t \in x \circ y$ and $z \in \psi^{-1}(I)$ such that $t \ll z$, that is $0 \in t \circ z$. Since $\psi(z) \in I$ and $0 \in t \circ z \subseteq (x \circ y) \circ z$, it follows that $0 = \psi(0) = \psi((x \circ y) \circ z) = \psi(x \circ y) \circ \psi(z) \subseteq \psi(x \circ y) \circ I$ so that $\psi(x) \circ \psi(y) = \psi(x \circ y) \ll I$. As $\psi(y) \in I$ and $I$ is hyper BCH-ideal, by using (Id3) we have $\psi(x) \in I$, that is $x \in \psi^{-1}(I)$. Hence $\psi^{-1}(I)$ is a hyper BCH-ideal of $H_1$.

(v) First we show that $\{0\} \subseteq H_2$ is a hyper BCH-ideal. To do this, let $x, y \in H_2$ be such that $x \circ y \ll \{0\}$ and $y \in \{0\}$. Then $y = 0$ and so $x \circ 0 = x \circ y \ll \{0\}$. Therefore there exists $t \in x \circ 0$ such that $t \ll 0$. Thus $t = 0$, and consequently
0 ∈ x ⊙ 0, that is x ≪ 0, which implies that x = 0. This shows that \{0\} is a hyper BCH-ideal of H₂. Now by (iv), Kerf = ψ⁻¹(\{0\}) is a hyper BCH-ideal of H₁.

(vii) Since 0 ∈ I, we have 0 = ψ(0) ∈ ψ(I). Let x and y be arbitrary elements in H₂ such that x ⊙ y ≪ f(I) and y ∈ ψ(I). Since y ∈ ψ(I) and ψ is onto, there are y₁ ∈ I and x₁ ∈ H₁ such that y = ψ(y₁) and x = ψ(x₁). Thus ψ(x₁ ⊙ y₁) = ψ(x₁) ⊙ ψ(y₁) = x ⊙ ψ(y₁) ≪ ψ(I). Therefore there are a ∈ x₁ ⊙ y₁ and b ∈ I such that ψ(a) ≪ ψ(b). So 0 ∈ ψ(a) ⊙ ψ(b) = ψ(a ⊙ b), which implies that ψ(c) = 0 for some c ∈ a ⊙ b. It follows that c ∈ Kerψ ⊆ I so that a ⊙ b ≪ I. Now since I is a hyper BCH-ideal of H₁ and b ∈ I, we get a ∈ I. Thus x₁ ⊙ y₁ ≪ I, which implies that x₁ ∈ I. Thus x = ψ(x₁) ∈ ψ(I), and so ψ(I) is a hyper BCH-ideal of H₂.

\[ \text{Theorem 5.3. Let } \psi : H₁ → H₂ \text{ be an epimorphism of hyper BCH-algebras. Then there is a one to one correspondence between the set of all hyper BCH-ideals of } H₁ \text{ containing Ker}ψ \text{ and the set of all hyper BCH-ideals of } H₂. \]

\[ \text{Theorem 5.4. Let } \psi : H₁ → H₂ \text{ and } \pi : H₁ → H₃ \text{ be two homomorphisms of hyper BCH-algebras such that } \psi \text{ is onto and Ker}ψ \subseteq Kerπ. \text{ Then there exists a homomorphism } \tau : H₂ → H₃ \text{ such that } \tau ∘ ψ = \pi. \]

\[ \text{Proof. Let } y ∈ H₂ \text{ be arbitrary. Since } ψ \text{ is onto, there exists } x ∈ H₁ \text{ such that } y = ψ(x). \text{ Define } \tau : H₂ → H₃ \text{ by } \tau(y) = \pi(x), \text{ for all } y ∈ H₂. \text{ Now we show that } \tau \text{ is well-defined. Let } y₁, y₂ ∈ H₂ \text{ and } y₁ = y₂. \text{ Since } ψ \text{ is onto, there are } x₁, x₂ ∈ H₁ \text{ such that } y₁ = ψ(x₁) \text{ and } y₂ = ψ(x₂). \text{ Therefore } ψ(x₁) = ψ(x₂) \text{ and thus } 0 ∈ ψ(x₁) ⊙ ψ(x₂) = ψ(x₁ ⊙ x₂). \text{ It follows that there exists } t ∈ x₁ ⊙ x₂ \text{ such that } \tau(t) = 0. \text{ Thus } t ∈ Kerψ \subseteq Kerπ \text{ and so } \pi(t) = 0. \text{ Since } t ∈ x₁ ⊙ x₂ \text{ we conclude that } 0 = \pi(t) = \pi(x₁ ⊙ x₂) = \pi(x₁) ⊙ \pi(x₂) \text{ which implies that } \pi(x₁) ≪ \pi(x₂). \text{ On the other hand since } 0 ∈ ψ(x₂) ⊙ ψ(x₁) = ψ(x₂ ⊙ x₁), \text{ similarly we can conclude that } 0 ∈ \pi(x₂) ⊙ \pi(x₁), \text{ that is } \pi(x₂) ≪ \pi(x₁). \text{ Thus } \pi(x₁) = \pi(x₂), \text{ which shows that } \tau \text{ is well-defined. Clearly } \tau ∘ ψ = \pi. \text{ Finally we show that } \tau \text{ is a homomorphism. Let } y₁, y₂ ∈ H₂ \text{ be arbitrary. Since } ψ \text{ is onto there are } x₁, x₂ ∈ H₁ \text{ such that } y₁ = ψ(x₁) \text{ and } y₂ = ψ(x₂). \text{ Then } \]

\[ \tau(y₁ ⊙ y₂) = \tau(ψ(x₁) ⊙ ψ(x₂)) = \tau(ψ(x₁ ⊙ x₂)) = (τ ∘ ψ)(x₁ ⊙ x₂) = π(x₁ ⊙ x₂) = π(x₁) ⊙ π(x₂) = (τ ∘ ψ)(x₁) ∘ (τ ∘ ψ)(x₂) = τ(ψ(x₁)) ∘ τ(ψ(x₂)) = τ(y₁) ∘ τ(y₂) \]

Moreover since ψ(0) = 0 and π(0) = 0, we conclude that \( \tau(0) = τ(ψ(0)) = (τ ∘ ψ)(0) = π(0) = 0. \) Thus \( \tau \) is a homomorphism. \[ \square \]
Theorem 5.5. Let \( \psi : H_1 \rightarrow H_2 \) be a homomorphism of hyper BCH-algebras. If \( I \) is a strong hyper BCH-ideal of \( H_2 \), then \( \psi^{-1}(I) \) is a strong hyper BCH-ideal of \( H_1 \).

Proof. Suppose \( I \) is a strong hyper BCH-ideal, then clearly \( 0 \in \psi^{-1}(I) \). Let \( a, b \in H_1 \) be such that \( (a \circ b) \cap \psi^{-1}(I) \neq \phi \) and \( b \in \psi^{-1}(I) \). Then we have \( \psi((a \circ b) \cap \psi^{-1}(I)) \subseteq \psi((a \circ b) \cap \psi^{-1}(I)) \subseteq \psi((a \circ b) \cap I) \) and so \( (\psi(a) \circ \psi(b)) \cap I \neq \phi \) and \( \psi(a) \in \psi^{-1}(I) \). Since \( I \) is a strong hyper BCH-ideal of \( H_2 \), we have \( \psi(a) \in I \) and so \( x \in \psi^{-1}(I) \). Therefore \( \psi^{-1}(I) \) is a strong hyper BCH-ideal of \( H_1 \).

\[ \square \]

Theorem 5.6. Let \( \psi : H_1 \rightarrow H_2 \) be a homomorphism of hyper BCH-algebras. Then \( k \ker \psi = \{ x \in H_1 \mid \psi(x) = 0 \} \) is a strong hyper BCH-ideal of \( H_1 \).

Proof. To prove this first we show that \( \{ 0 \} \) is a strong hyper BCH-ideal of \( H_2 \). For this, let \( a, b \in H_1 \) be such that \( (a \circ b) \cap \{ 0 \} \neq \phi \) and \( b \in \{ 0 \} \). Then \( b = 0 \) and so \( 0 \in a \circ 0 \) since \( (a \circ 0) \cap \{ 0 \} \neq \phi \). Thus we have \( a \lessdot 0 \). By (HCH3) and 3.5, we get \( a = 0 = 0 \in \{ 0 \} \). This shows that \( \{ 0 \} \) is a strong hyper BCH-ideal of \( H_2 \).

It follows from Theorem 5.5 that \( \ker \psi = \psi^{-1}(\{ 0 \}) \) is a strong hyper BCH-ideal of \( H_1 \).

\[ \square \]

Theorem 5.7. Let \( \psi : H_1 \rightarrow H_2 \) be a homomorphism of hyper K-algebras. If \( \psi \) is onto and \( I \) is a strong hyper BCH-ideal of \( H_1 \) which contains \( k \ker \psi \), then \( \psi(I) \) is a strong hyper BCH-ideal of \( H_2 \).

Proof. Suppose \( I \) is a strong hyper BCH-ideal of \( H_1 \). Clearly \( 0 \in \psi(I) \). Let \( x, y \in H_2 \) be such that \( (x \circ y) \cap \psi(I) \neq \phi \) and \( y \in \psi(I) \). Since \( y \in \psi(I) \) and \( \psi \) is onto, there are \( y_1 \in I \) and \( x_1 \in H_1 \) such that \( y = \psi(y_1) \) and \( x = \psi(x_1) \).

Thus \( \phi \neq (x \circ y) \cap \psi(I) = \psi(x_1 \circ y_1) \cap \psi(I) \) and so there exists \( a \in H_2 \) such that \( a \in \psi(x_1 \circ y_1) \) and \( a \in \psi(I) \). It follows that there are \( a_1 \in x_1 \circ y_1 \) and \( b_1 \in I \) such that \( a = \psi(a_1) \) and \( a = \psi(b_1) \) so that \( 0 \in a \circ a = \psi(a_1 \circ b_1) = \psi(a_1 \circ b_1) \) which implies that \( \psi(c) = 0 \) for some \( c \in a_1 \circ b_1 \). Hence \( c \in \ker \psi \subseteq I \) and so \( (a_1 \circ b_1) \cap I \neq \phi \).

Now since \( I \) is a strong hyper BCH-ideal of \( H_1 \) and \( b_1 \in I \), we get \( a_1 \in I \). Thus \( (x_1 \circ y_1) \cap I \neq \phi \), which implies that \( x_1 \in I \). Therefore \( x = \psi(x_1) \in \psi(I) \), and so \( \psi(I) \) is a strong hyper BCH-ideal of \( H_2 \).

\[ \square \]

References


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