A SHORT NOTE ON IDEMPOTENT RINGS

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Abstract. In this paper we introduce a new class of rings that we say idempotent rings. We call a ring $R$ is idempotent, if every ideal of $R$ is generated by an idempotent element. In this paper we prove some properties of this rings, where one of the important results is the following:

Let $t \geq 2$ be an integer number. Then the ring $\mathbb{Z}_t$ is an idempotent ring if and only if $t = p_1p_2 \ldots p_n$, where all of the $p_i$ are distinct prime numbers.

Keywords: idempotent, artinian ring, noetherian ring.

1. Introduction

Throughout this paper, all rings are commutative rings with identity and all modules are unital. Let $M$ be a submodule of the $R$-module $L$. We say that $L$ is an essential extension of $M$ precisely when $B \cap M \neq 0$ for every non-zero submodule $B$ of $L$. We say that $L$ is an injective envelope (or injective hull) of $M$ precisely when $L$ is an injective $R$-module which is also an essential extension of $M$. We denote by $E(M)$ the injective envelope of $M$. For any unexplained notation and terminology we refer the reader to [2] and [3].

2. Main results

Definition 2.1. Let $R$ be a ring. We say that $R$ is idempotent if every ideal of $R$ is generated by an idempotent element.

Lemma 2.2. Every idempotent ring is Artinian ring.

Proof. Suppose that $m \in \text{Max}(R)$. Then there exist an element $e \in R$ such that $m = (e)$ and $e^2 = e$. Now in local Noetherian ring $R_m$, we have $(mR_m)^2 = mR_m$, and so by Nakayama’s lemma, $mR_m = 0$. Hence $\dim R_m = 0$. Since $m$ is an arbitrary maximal ideal, it follows that $\dim R = 0$ and so $R$ is Artinian ring. □

Lemma 2.3. In idempotent ring, the Jacobson radical is zero.
Proof. We denote the Jacobson radical of $R$ by $J(R)$, so we show that $J(R) = 0$. There exists an element $e \in R$ such that $J(R) = \langle e \rangle$ and $e^2 = e$. Therefore $J(R) = J(R) \cdot J(R)$ and by Nakayama’s lemma, $J(R) = 0$. \qed

Theorem 2.4. Let $R$ be an idempotent ring and $m_1, \ldots, m_n$ be all of the maximal ideals of $R$. Then $R \cong \frac{R}{m_1} \oplus \cdots \oplus \frac{R}{m_n}$.

Proof. Let $\text{Max } R = \{m_1, \ldots, m_n\}$. By induction on $n$ we prove the theorem. If $n = 1$, then $J(R) = m_1 = 0$ and so $R \cong \frac{R}{m_1}$.

Now we suppose that $n \geq 0$ and consider the following exact sequence,

$$
0 \rightarrow \frac{R}{m_1 \cap m_2} \rightarrow \frac{R}{m_1} \oplus \frac{R}{m_2} \rightarrow \frac{R}{m_1 + m_2} \rightarrow 0
$$

Since $m_1 + m_2 = R$, it follow from the above exact sequence that $\frac{R}{m_1 \cap m_2} \cong \frac{R}{m_1} \oplus \frac{R}{m_2}$.

Now consider the following exact sequence

$$
0 \rightarrow \frac{R}{m_1 \cap m_2 \cap m_3} \rightarrow \frac{R}{m_1 \cap m_2} \oplus \frac{R}{m_3} \rightarrow \frac{R}{(m_1 \cap m_2) + m_3} \rightarrow 0.
$$

Again similar the above argument, since $(m_1 \cap m_2) + m_2 = R$, it follows that

$$
\frac{R}{m_1 \cap m_2 \cap m_3} \cong \frac{R}{m_1 \cap m_2} \oplus \frac{R}{m_3} \cong \frac{R}{m_1} \oplus \frac{R}{m_2} \oplus \frac{R}{m_3}.
$$

By repeating this argument we have

$$
R \approx \frac{R}{(0)} \approx \frac{R}{J(R)} = \frac{R}{\cap_{i=1}^n m_i} \approx \oplus_{i=1}^n \frac{R}{m_i}.
$$

\qed

Theorem 2.5. Every idempotent ring $R$ as an $R$-module is injective.

Proof. Let $I$ be an ideal of $R$ and consider the diagram

$$
\begin{array}{ccc}
0 & \rightarrow & I \\
 & \overset{i}{\rightarrow} & R \\
\downarrow & & \downarrow f \\
 & & R
\end{array}
$$

There exists an element $e \in R$ such that $I = \langle e \rangle$ and $e^2 = e$. Let $f(e) = x$. We define the function $g : R \rightarrow R$ with $g(r) = rx$. Then we have the following relations for all $r \in R$.

$$
\text{goi}(re) = g(re) = rex = ref(e) = rf(e^2) = rf(e) = f(re),
$$

and so $R$ is an injective $R$-module. \qed
Corollary 2.6. If $R$ is an idempotent ring, then every simple $R$-module is injective.

Proof. Let $N$ be a simple $R$-module. Then there exists $m \in \text{Max}(R)$, such that $N \approx \frac{R}{m}$. Therefore $N$ is a direct summand of an injective $R$-module $R$ and so $N$ is injective. \hfill \Box

Corollary 2.7. Let $R$ be an idempotent ring. Then for every $m \in \text{Max}(R)$, $E_R \left( \frac{R}{m} \right) \approx \frac{R}{m}$.

Proof. Follows from the above corollary. \hfill \Box

Lemma 2.8. Let $R$ be an idempotent ring and $M$ be a finitely generated $R$-module. Then $M$ is injective.

Proof. Since $R$ is Artinian ring, it follows that $l(M) < \infty$. We prove the assertion by induction on $l(M)$. If $l(M) = 1$, then $M$ is a simple and so the assertion follows from Corollary 2.6. Now suppose that $l(M) = n \geq 2$ and the assertion holds for $n-1$.

Since $M$ is Artinian $R$-module, it follows that $M$ has a simple submodule such as $N$. Consider the following exact sequence.

$$0 \rightarrow N \rightarrow M \rightarrow \frac{M}{N} \rightarrow 0$$

$l(N) = 1$ and so $N$ is injective. $l\left( \frac{M}{N} \right) = n - 1$ then $\frac{M}{N}$ is injective. Therefore $M$ is also injective. \hfill \Box

Lemma 2.9. Let $R$ be a Noetherian ring and $\{ E_i \}_{i \in A}$ be a family of injective $R$-modules. Then $\lim_{i \in A} E_i$ is injective.

Proof. Is simple. \hfill \Box

Theorem 2.10. Let $R$ be an idempotent ring. Then every $R$-module is injective.

Proof. Let $T$ be an $R$-module. Then $T$ is a direct limit of its finitely generated submodules. \hfill \Box

Theorem 2.11. Let $R$ be an idempotent ring. Then every $R$-module is projective and so is flat.

Proof. By the above theorem every $R$-module is injective. Let $T$ be an $R$-module. Then $T$ is injective. By Matlis theorem

$$T = \bigoplus_{p \in \text{Spec}(R)} E \left( \frac{R}{p} \right) .$$
On the other hand $\text{Max}(R) = \text{Spec}(R)$. Therefore,

$$T = \bigoplus_{m \in \text{Max}(R)} \frac{R}{m}.$$ 

Also for any $m \in \text{Max}(R)$, $\frac{R}{m}$ is a direct summand of $R$ and so is projective. Consequently $T$ is projective. \hfill \square

**Theorem 2.12.** Let $t \geq 2$ be an integer number. Then the ring $\mathbb{Z}_t$ is an idempotent ring if and only if $t = p_1p_2 \ldots p_n$, where all of the $p_i$ are distinct prime numbers.

**Proof.** Let $\mathbb{Z}_t$ be an idempotent ring. Suppose on the contrary that there exists a prime number $p$ such that $p^2 | t$.

In this case, we set $J = \langle \tilde{p} \rangle$. Then $J^2 = \langle \tilde{p}^2 \rangle \neq J$ and so $J$ is not an idempotent ideal of $\mathbb{Z}_t$ and therefore $\mathbb{Z}_t$ is not idempotent ring which is a contradiction.

Conversely, let $t = p_1 \ldots p_n$, where $p_i$ are distinct prime numbers. If $n = 1$, then all of the ideals of $\mathbb{Z}_t$ are $I = \langle 0 \rangle$ and $J = \mathbb{Z}_{p_1} = \mathbb{Z}_t = \langle 1 \rangle$ and so $\mathbb{Z}_t$ is idempotent ring.

Now let $n \geq 2$ and we set $m_i = \langle \tilde{p}_i \rangle$. Then $m_1, \ldots, m_n$ are all the maximal ideals of $R$. We claim that each $m_i$ is generated by an idempotent element. It is enough to show that for $m_1$. Since $(p_1, p_2 p_3 \ldots p_n) = 1$, then there exist $r, s \in \mathbb{Z}_t$ such that $rp_1 + sp_2 \ldots p_n = 1$. Therefore $p_2 \ldots p_n | rp_1 - 1$ and $p_1 | rp_1$. Hence $p_1 \ldots p_n | rp_1(rp_1 - 1) = (rp_1)^2 - rp_1$ and so in the ring $\mathbb{Z}_t$, we have $\overline{r}p_1^2 = \overline{r}p_1$. Set $e = \overline{r}p_1$ and we claim that $m_1 = \langle e \rangle$. $rp_1 + sp_2 \ldots p_n = 1$ implies that $r\overline{p}_1^2 + sp_1p_2 \ldots p_n = p_1$. Hence in the ring $\mathbb{Z}_t$, $\overline{r}p_1^2 = \overline{p}_1$ and so $\overline{p}_1 e = \overline{p}_1$ and $m_1 = \langle \overline{p}_1 \rangle = \langle \overline{p}_1 e \rangle = \langle e \rangle \subseteq m_1$. Therefore $m_1 = \langle e \rangle$ and every maximal ideal of $\mathbb{Z}_t$ is generated by an idempotent element. Now let $I$ be an arbitrary ideal of $\mathbb{Z}_t$. Then there exists an element $p_{i_1}p_{i_2} \ldots p_{i_k}$ such that $I = \langle p_{i_1}p_{i_2} \ldots p_{i_k} \rangle$, where $p_{i_1}, \ldots, p_{i_k}$ are different elements of the set $\{p_1, \ldots, p_n\}$. Also we have,

$$I = m_{i_1}m_{i_2} \ldots m_{i_k} = \langle e_{i_1} \rangle \ldots \langle e_{i_k} \rangle = \langle e_{i_1} \ldots e_{i_k} \rangle$$

where all of the $e_{i_j}$ are idempotent and so the element $e_{i_1}e_{i_2} \ldots e_{i_k}$ is also idempotent and the assertion follows. \hfill \square

**Remark 2.13.** It is well known, in a Noetherian ring $R$, for any ideal $I$ of $R$ and any injective $R$-module $E$, $0 :_E (0 :_R I) = IE$.

**Theorem 2.14.** Let $R$ be a Noetherian ring and every $R$-module be an injective $R$-module. Then $R$ is idempotent.

**Proof.** Let $I$ be an ideal of $R$. Then $I$ is injective $R$-module and by Remark 2.13, we have

$$I \subseteq 0 : (0 :_R I) = II = I^2 \subseteq I$$

Hence $I = I^2$ and by [1, Corollary 2.5], there exists $a \in I$ such that $(1 - a)I = 0$ and so $I = \langle a \rangle$ and $a^2 = a$. \hfill \square
Corollary 2.15. The Noetherian ring $R$ is idempotent iff every $R$-module is an injective $R$-module.

Corollary 2.16. If $p_1, \ldots, p_n$ are distinct prime numbers and $R = \mathbb{Z}_{p_1 \cdots p_n}$, then every $R$-module is injective and projective.

Corollary 2.17. Let $R$ be an idempotent ring and $M$ be an $R$-module. Then the following are equivalent:

i) There exists an exact sequence $0 \rightarrow R \rightarrow M$

ii) $\text{Ann } M = 0$

Proof. $i \rightarrow ii$) is clear.

$ii \rightarrow i$) Since $M$ is injective, it follows by Matlis theorem $M = \bigoplus_{\gamma \in A} E \left( \frac{R}{m_{\gamma}} \right) = \bigoplus_{\gamma \in A} \frac{R}{m_{\gamma}}$. Set $T = \{ m_{\gamma} \mid \gamma \in A \}$ and we prove that $T = \text{Max}(R)$. Suppose on the contrary that $T \neq \text{Max}(R)$. Let $m \in \text{Max}(R) \setminus T$.

$$0 = \text{Ann } M = \bigcap_{\gamma \in A} m_{\gamma} \Rightarrow \bigcap_{\gamma \in A} m_{\gamma} = 0 \subseteq m$$

and so there exist $\gamma \in A$ such that $m_{\gamma} \subseteq m$ which implies that $m = m_{\gamma} \in T$, which is a contradiction. Therefore $M = \bigoplus_{m_{\gamma} \in \text{Max} R} \frac{R}{m_{\gamma}} \cong R$ and so the sequence $0 \rightarrow R \rightarrow M$ is exact.

References


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