ON HYPERIDEALS OF ORDERED SEMIHYPERGROUPS

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Abstract. Prime, weakly prime and semiprime hyperideals in ordered semihypergroups were studied by Kehayopulu. In this paper, we introduce the concepts of weakly semiprime and irreducible hyperideals in ordered semihypergroups. The relationship between the five classes of hyperideals is established. Finally, we characterize semisimple ordered semihypergroups and intra-regular ordered semihypergroups in terms of these hyperideals.

Keywords: ordered semihypergroups, hyperideals, semisimple ordered semihypergroups, intra-regular ordered semihypergroups.

1. Introduction and preliminaries

The algebraic hyperstructure theory was first introduced in 1934 by Marty [18]. Hyperstructures have many applications in several branches of both pure and applied sciences (see [4-6,8,9,11-13,23]). Recently, Heidari and Davvaz applied the hyperstructure theory to ordered semigroups and introduced the concept of ordered semihypergroups (see [16]), which is a generalization of the concept of ordered semigroups. Furthermore, the ordered semihypergroup theory was enriched by the work of many researchers, for example [2,3,10,14,15,22,23]. In particular, the hyperideal theory on semihypergroups and ordered semihypergroups can be seen in [1-3,17-19,22,23]. Recently, N. Kehayopulu (see [18-19]) introduced prime, weakly prime and semiprime hyperideals in ordered semihyergroups and studied ordered semihypergroups with these hyperideals. Motivated by the previous work on hyperideals of (ordered) semihypergroups, we attempt in the present paper to study hyperideals of ordered semihypergroups in detail. In this article, we introduce the notions of weakly semiprime and irreducible hyperideals in ordered semihypergroups, and moreover establish the relationship between the five classes of hyperideals. Finally, semisimple ordered semihypergroups and intra-regular ordered semihypergroups are characterized in terms of these hyperideals. Partial results which are consistent with the conclusions in [19] are reorganized and proved. We recall first some basic notions of ordered semihypergroup.

A *hypergroupoid* $(S, \circ)$ is a nonempty set S together with a *hyperoperation* or *hypercomposition*, that is a mapping $\circ : S \times S \to \mathcal{P}^*(S)$, where $\mathcal{P}^*(S)$ denotes the family of all nonempty subsets of S. If $x \in S$ and A, B are nonempty subsets of S, then we denote $A \circ B = \bigcup_{a \in A, b \in B} a \circ b, x \circ A = \{x\} \circ A$ and $A \circ x = A \circ \{x\}$. A hypergroupoid $(S, \circ)$ is called a *semihypergroup* if $\circ$ is associative, that is $x \circ (y \circ z) = (x \circ y) \circ z$ for every $x, y, z \in S$.

An *ordered semihypergroup* $(S, \circ, \leq)$ is a semihypergroup $(S, \circ)$ with an order relation $\leq$ which is *compatible* with the hyperoperation $\circ$, meaning that for any $a, b, x \in S$, $a \leq b$ implies that $a \circ x \leq b \circ x$ and $x \circ a \leq x \circ b$. Here, let $A, B \in P^*(S)$, then we say that $A \leq B$ if for every $a \in A$ there exists $b \in B$ such that $a \leq b$. Let S be an ordered semihypergroup and A be a nonempty subset of S. We say that A is a *hyperideal* of S if (1) $S \circ A \subseteq A, A \circ S \subseteq A$ and (2) $a \in A, b \in S$ and $b \leq a$ imply that $b \in A$. For $\emptyset \neq H \subseteq S$, we denote

$$(H] := \{t \in S \mid t \leq h \text{ for some } h \in H\}.$$

For convenience, we write $(a]$ instead of $(\{a\}]$ for $H = \{a\}$. We denote by $I(a)$ the hyperideal of S generated by a. On can easily obtain that

$$I(a) = (a \cup S \circ a \cup a \circ S \cup S \circ a \circ S].$$

2. Hyperideals of ordered semihypergroups

In this section, we recall prime, weakly prime and semiprime hyperideals and introduce the concepts of weakly semiprime and irreducible hyperideals. Moreover, we study the relationship between the five classes of hyperideals.

Let $(S, \circ, \leq)$ be an ordered semihypergroup and I be a hyperideal of S. I is called *prime* if for any $A, B \subseteq S$, $A \circ B \subseteq I$ implies $A \subseteq I$ or $B \subseteq I$ (equivalent to for all $a, b \in S$, $a \circ b \subseteq I$ implies $a \in I$ or $b \in I$); I is called *semiprime* if for any $A \subseteq S$, $A \circ A \subseteq I$ implies $A \subseteq I$ (equivalent to for any $a \in S$, $a \circ a \subseteq I$ implies $a \in I$); I is called *weakly prime* if for all hyperideals A, B of S, $A \circ B \subseteq I$ implies $A \subseteq I$ or $B \subseteq I$;

Definition 2.1. Let $(S, \circ, \leq)$ be an ordered semihypergroup and I be a hyperideal of S. I is called weakly semiprime if for any hyperideal A of S, $A \circ A \subseteq I$ implies $A \subseteq I$; I is called irreducible if for all hyperideals I_1, I_2 of S, $I_1 \cap I_2 = I$ implies $I_1 = I$ or $I_2 = I$.

Remark 2.2. It is easy to see that a prime hyperideal is semiprime and weakly prime, and a semiprime hyperideal is weakly semiprime.

For further studying the relationship between the five classes of hyperideals, we need the following result which can be easily obtained.

Lemma 2.3. *Let S be an ordered semihypergroup. Then*

(1) $A \subseteq (A], ((A]] = (A]$ *for all* $A \subseteq S$;

(2) *If* $A \subseteq B \subseteq S$, *then* $(A] \subseteq (B]$;

(3) $(A] \circ (B] \subseteq (A \circ B], ((A] \circ (B]] = (A \circ B]$ *for all* $A, B \subseteq S$;

(4) $(T] = T$ *for every hyperideal* T *of* S;

(5) *If* A, B *are hyperideals of* S, *then* $(A \circ B], A \cap B$ *and* $A \cup B$ *are hyperideals of* S;

(6) $(S \circ A \circ S]$ *is a hyperideal of* S *for all* $A \subseteq S$, *in particular, we write* $(S \circ \{a\} \circ S]$ *as* $(S \circ a \circ S]$.

Lemma 2.4. *Let* S *be an ordered semihypergroup and* I *a hyperideal of* S. *Then* I *is the intersection of all irreducible hyperideals of* S *containing* I.

Proof. Let $\{I_\alpha \mid \alpha \in \Gamma\}$ be the set of all irreducible hyperideals of S containing I. Obviously, $I \subseteq \bigcap_{\alpha\in\Gamma} I_\alpha$. It suffices to prove that $\bigcap_{\alpha\in\Gamma} I_\alpha - I = \emptyset$. Suppose that there exists $a \in \bigcap_{\alpha\in\Gamma} I_\alpha - I$. Let $\Omega = \{H \text{ is a hyperideal} \mid I \subseteq H, a \notin H\}$. Then $\Omega \neq \emptyset$ from $I \in \Omega$, and Ω is partially ordered under inclusion. By Zorn's Lemma, there exists a maximal element $M \in \Omega$. Next we show that M is irreducible. Let A, B be two hyperideals such that $A \cap B = M$. Then $A \in \Omega$ or $B \in \Omega$. Thus $A = M$ or $B = M$. Otherwise, $M \subsetneqq A$ and $M \subsetneqq B$ which contradicts the maximality of M. Hence $I = \bigcap_{\alpha\in\Gamma} I_\alpha$. □

Theorem 2.5. *Let* S *be an ordered semihypergroup and* I *a hyperideal of* S. *Then* I *is prime if and only if it is semiprime and weakly prime. In particular, if* S *is commutative, then the prime and weakly prime hyperideals concide.*

Proof. ($\Rightarrow$) It is obtained from Remark 2.2.

($\Leftarrow$) Let $a, b \in S$ and $a \circ b \subseteq I$. Then $(b \circ S \circ a] \circ (b \circ S \circ a] \subseteq (b \circ S \circ a \circ b \circ S \circ a] \subseteq (S \circ (a \circ b) \circ S] \subseteq (S \circ I \circ S] \subseteq (I] = I$. Since I is semiprime, $(b \circ S \circ a] \subseteq I$. Thus $(S \circ b \circ S] \circ (S \circ a \circ S] \subseteq (S \circ b \circ S \circ S \circ a \circ S] \subseteq (S \circ (b \circ S \circ a) \circ S] \subseteq (S \circ I \circ S] \subseteq I$. Since I is weakly prime, we have $(S \circ b \circ S] \subseteq I$ or $(S \circ a \circ S] \subseteq I$. Suppose that $(S \circ a \circ S] \subseteq I$. Then $(I^2(a)] \circ I(a) = (I^2(a)] \circ (I(a)] \subseteq (I^3(a)] = ((a \cup S \circ a \cup a \circ S \cup S \circ a \circ S]^3] \subseteq (((a \cup S \circ a \cup a \circ S \cup S \circ a \circ S)^2] \circ (a \cup S \circ a \cup a \circ S \cup S \circ a \circ S]] \subseteq ((S \circ a \cup S \circ a \circ S] \circ (a \cup S \circ a \cup a \circ S \cup S \circ a \circ S]] \subseteq (((S \circ a \cup S \circ a \circ S) \circ (a \cup S \circ a \cup a \circ S \cup S \circ a \circ S)]] \subseteq ((S \circ a \circ S]] \subseteq (I] = I$. Since I is weakly prime, we have $(I^2(a)] \subseteq I$ or $I(a) \subseteq I$. If $I(a) \subseteq I$, then $a \in I(a) \subseteq I$. If $(I^2(a)] \subseteq I$, then $I^2(a) \subseteq I$. Since I is weakly prime, $I(a) \subseteq I$ and so $a \in I$. By symmetries, we obtain $b \in I$ if $(S \circ b \circ S] \subseteq I$. Therefore, I is a prime hyperideal of S.

Let S be commutative, I weakly prime and $a, b \in S$ such that $a \circ b \subseteq I$. Then $I(a) \circ I(b) = (a \cup S \circ a] \circ (b \cup S \circ b] \subseteq (a \circ b \cup S \circ a \circ b] \subseteq (I \cup S \circ I] \subseteq (I] = I$. Since I is weakly prime, we have $a \in I(a) \subseteq I$ or $b \in I(b) \subseteq I$. □

Theorem 2.6. *Let* S *be an ordered semihypergroup and* I *a hyperideal of* S. *Then* I *is weakly prime if and only if it is weakly semiprime and irreducible.*

Proof. ($\Rightarrow$) From Remark 2.2, we know that I is weakly semiprime. Next we show that I is irreducible. Let I_1, I_2 be two hyperideals such that $I_1 \cap I_2 = I$. Then $I \subseteq I_1$ and $I \subseteq I_2$. Moreover, $I_1 \circ I_2 \subseteq I_1 \cap I_2 = I$. Since I is weakly prime, $I_1 \subseteq I$ or $I_2 \subseteq I$. Hence $I_1 = I$ or $I_2 = I$.

($\Leftarrow$) Let A, B be two hyperideals such that $A \circ B \subseteq I$. Then $(A \cap B) \circ (A \cap B) \subseteq A \circ B \subseteq I$. Since I is weakly semiprime, $A \cap B \subseteq I$. Thus $I = I \cup (A \cap B) = (I \cup A) \cap (I \cup B)$. Since I is irreducible, we have $I \cup A = I$ or $I \cup B = I$. Hence $A \subseteq I$ or $B \subseteq I$. Consequently, I is weakly prime. $\square$

Combining Theorem 2.5 and 2.6, we have the following result.

Corollary 2.7. *Let S be an ordered semihypergroup and I a hyperideal of S. Then I is prime if and only if it is semiprime and irreducible.*

3. Semisimple and intra-regular ordered semihypergroups

In this section, we mainly characterize semisimple ordered semihypergroups and intra-regular semihypergroups in terms of the five classes hyperideals introduced in the previous section.

Let $(S, \circ, \leq)$ be an ordered semihypergroup. S is called *semisimple* if $a \in (S \circ a \circ S \circ a \circ S]$ for every $a \in S$; S is called *intra-regular* if $a \in (S \circ a \circ a \circ S]$ for every $a \in S$.

Theorem 3.1. *Let S be an ordered semihypergroup. Then the following statements are equivalent:*
(1) *S is semisimple;*
(2) *$A \cap B = (A \circ B]$ for all hyperideals A, B of S;*
(3) *$(A^2] = A$ for every hyperideal A of S;*
(4) *Every hyperideal of S is weakly semiprime.*

Proof. (1)$\Rightarrow$(2) Let A, B be hyperideals of S. If $a \in A \cap B$, then $a \in (S \circ a \circ S \circ a \circ S] = ((S \circ a \circ S) \circ (a \circ S)] \subseteq (A \circ B]$. Thus $A \cap B \subseteq (A \circ B]$. On the other hand, $(A \circ B] \subseteq (A] \cap (B] = A \cap B$. Hence $A \cap B = (A \circ B]$.

(2)$\Rightarrow$(3) It is obvious.

(3)$\Rightarrow$(4) Let A and I be hyperideals of S such that $A^2 \subseteq I$. Then $(A^2] \subseteq (I] = I$. By hypothesis, $A = (A^2] \subseteq I$.

(4)$\Rightarrow$(1) By the proof of Theorem 2.5, we have $(I^3(a)] \subseteq (S \circ a \circ S]$. Following the method, we can obtain $(I^5(a)] \subseteq (S \circ a \circ S \circ a \circ S]$. Moreover, $((I^2(a)]^2] \circ ((I^2(a)]^2] \subseteq (I^4(a)] \circ (I^4(a)] \subseteq (I^4(a)] \circ (I(a)] \subseteq (I^5(a)]$ and $((I^2(a)]^2]$ is a hyperideal. Since $(S \circ a \circ S \circ a \circ S]$ is weakly semiprime, $((I^2(a)]^2] \subseteq (S \circ a \circ S \circ a \circ S]$. Continuing the process twice, we have $a \in I(a) \subseteq (S \circ a \circ S \circ a \circ S]$. $\square$

Theorem 3.2. *Let S be an ordered semihypergroup. Then S is intra-regular if and only if every hyperideal of S is semiprime.*

Proof. ($\Rightarrow$) Let I be a hyperideal of S and $a \in S$ such that $a^2 \subseteq I$. Then $a \in (S \circ a^2 \circ S] \subseteq (S \circ I \circ S] \subseteq (I] = I$.

($\Leftarrow$) Let $a \in S$. Then $a^2 \circ a^2 = a \circ a^2 \circ a \subseteq S \circ a^2 \circ S \subseteq (S \circ a^2 \circ S]$. Since $(S \circ a^2 \circ S]$ is a hyperideal of S, by hypothesis, $(S \circ a^2 \circ S]$ is semiprime. Thus $a^2 \subseteq (S \circ a^2 \circ S]$. In the same way, we have $a \in (S \circ a^2 \circ S]$. $\square$

Theorem 3.3. *Let S be an ordered semihypergroup and Θ be the set of all hyperideals of S. Then I is weakly prime for every $I \in \Theta$ if and only if S is semisimple and Θ is a chain.*

Proof. ($\Rightarrow$) From Theorem 3.1, we know that S is semisimple. Let $A, B \in \Theta$. Since $(A \circ B] \in \Theta$ and $A \circ B \subseteq (A \circ B]$, by hypothesis, we have $A \subseteq (A \circ B]$ or $B \subseteq (A \circ B]$. Moreover, $(A \circ B] = A \cap B$ from Theorem 3.1. Hence $A \subseteq B$ or $B \subset A$. Consequently, Θ is a chain under inclusion.

($\Leftarrow$) From Theorem 2.6 and 3.1, it suffice to verify that I is irreducible for any $I \in \Theta$. Let $A, B \in \Theta$ and $A \cap B = I$. Since Θ is a chain, $A \subset B$ or $B \subset A$. Thus $A = I$ or $B = I$. Hence I is irreducible. □

From Theorem 3.1, 3.2 and 3.3, we can easily obtain the following result.

Theorem 3.4. *Let S be an ordered semihypergroup and Θ be the set of all hyperideals of S. Then I is prime for every $I \in \Theta$ if and only if S is intra-regular and Θ is a chain.*

We end this paper with an example by illustrating the previous results.

Example 3.5. Let $S = \{a, b, c, d, e\}$ with the hyperoperation $\circ$ and the order relation $\leq$ below:

$\circ$	a	b	c	d	e
a	{b,c}	{a}	{a}	{a}	{a}
b	{a}	{b,c}	{b,c}	{b,c}	{b,c}
c	{a}	{b,c}	{b,c}	{b,c}	{b,c}
d	{a}	{b,c}	{b,d}	{d,e}	{d,e}
e	{a}	{b,c}	{c}	{d,e}	{e}

$$\leq := \{(a, a), (b, b), (c, c), (c, b), (d, d), (e, e), (e, d)\}.$$

One can check that $(S, \circ, \leq)$ is an ordered semihypergoup (see [19]). The hyperideals of S are S and $I = \{a, b, c\}$. Obviously, the set of all hyperideals forms a chain. One can also check that S is intra-regular. Thus, by Theorem 3.4, the hyperideals of S are prime. We give an independent proof as following: indeed, $S \backslash I = \{d, e\}$ and $d \circ d = d \circ e = e \circ e = \{d, e\} \subsetneqq I$, $e \circ e = \{e\} \subsetneqq I$. By Theorem 2.5 and 2.6, we know that the ideals of S are weakly prime, semiprime, weakly semiprime and irreducible. Moreover, by Theorem 3.3, S is semisimple.

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