ON THE CONFORMAL CURVATURE TENSOR OF 
\( \epsilon \)-KENMOTSU MANIFOLDS

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Abstract. The conformal curvature tensor under certain curvature conditions has been studied for an \( \epsilon \)-Kenmotsu manifold with respect to the semi-symmetric non-metric connection. Finally, we give an example of a 3-dimensional \( \epsilon \)-Kenmotsu manifold with respect to the semi-symmetric non-metric connection.

Keywords: \( \epsilon \)-Kenmotsu manifolds, semi-symmetric non-metric connection, \( \eta \)-Einstein manifold, conformal curvature tensor.

1. Introduction

In 1972, K. Kenmotsu [14] studied a class of contact Riemannian manifolds satisfying some special conditions. We call it Kenmotsu manifold. Kenmotsu manifolds have been studied by various authors such as J. B. Jun et al. [13], G. Pathak and U. C. De [17], M. M. Tripathi and N. Nakkar [19], A. Yildiz et al. [23] and many others. In 1993, A. Bejancu and K. L. Duggal [4] introduced the concept of \( (\epsilon) \)-Sasakian manifolds, which later on showed by X. Xufeng and C. Xiaoli [20] that the manifolds are real hypersurfaces of indefinite Kahlerian manifolds. An \( (\epsilon) \)-almost paracontact manifolds were introduced by M. M. Tripathi et al. [18], while the concept of \( (\epsilon) \)-Kenmotsu manifolds was introduced

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by U. C. De and A. Sarkar [8] who showed that the existence of new structure on indefinite metrics influences the curvatures.

In 1924, the idea of semi-symmetric linear connection on a differentiable manifold was introduced by A. Friedmann and J. A. Schouten [9]. In 1930, E. Bartolotti [3] gave a geometrical meaning of such a connection. In 1932, H. A. Hayden [12] introduced semi-symmetric metric connection in a Riemannian manifold and this was studied systematically by K. Yano [21].

A linear connection $\nabla$ in a Riemannian manifold $M$ is said to be a semi-symmetric connection if the torsion tensor $T$ satisfies

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$$

Further, a semi-symmetric connection is called a semi-symmetric non-metric connection [1], if

$$(\nabla_X g)(Y, Z) = \nabla_X g(Y, Z) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z) - \eta(Y)g(X, Z) - \eta(Z)g(X, Y),$$

where $X, Y, Z \in \chi(M)$ and $\chi(M)$ is the set of all differentiable vector fields on $M$.

Let $M$ be an $n$-dimensional $\epsilon$-Kenmotsu manifold and $\nabla$ be the Levi-Civita connection on $M$, the semi-symmetric non-metric connection $\nabla$ on $M$ is given by [1]

$$\nabla_X Y = \nabla_X Y + \eta(Y)X.$$

The semi-symmetric non-metric connection have been studied by several authors such as N. S. Agashe and M. R. Chafie [1], L. S. Das et al. [5], U. C. De and S. C. Biswas [6], U. C. De and D. Kamilya [7], S. K. Pandey et al. [16], Ajit Barman [2] and many others. Motivated by the above studies, in this paper we study certain curvature conditions of an $\epsilon$-Kenmotsu manifold with respect to the semi-symmetric non-metric connection.

The paper is organized as follows: In Section 2, we give a brief introduction of an $\epsilon$-Kenmotsu manifold. In Section 3, we obtain the relation between the curvature tensor of an $\epsilon$-Kenmotsu manifold with respect to the semi-symmetric non-metric connection and the Levi-Civita connection. In Section 4, we study quasi-conformally flat, $\xi$-conformally flat, pseudoconformally flat and $\phi$-conformally flat $\epsilon$-Kenmotsu manifolds with respect to the semi-symmetric non-metric connection and it is shown that in each case the manifold is an $\eta$-Einstein manifold. Section 5 is devoted to study $\epsilon$-Kenmotsu manifolds with respect to the semi-symmetric non-metric connection satisfying the curvature condition $S \cdot C = 0$. 
2. Preliminaries

An $n$-dimensional smooth manifold $(M, g)$ is said to be an $\epsilon$-almost contact metric manifold [8], if it admits a $(1, 1)$ tensor field $\phi$, a structure vector field $\xi$, a 1-form $\eta$ and an indefinite metric $g$ such that

\begin{align*}
(2.1) & \quad \phi^2 X = -X + \eta(X)\xi, \\
(2.2) & \quad \eta(\xi) = 1, \\
(2.3) & \quad g(\xi, \xi) = \epsilon, \\
(2.4) & \quad \eta(X) = \epsilon g(X, \xi), \\
(2.5) & \quad g(\phi X, \phi Y) = g(X, Y) - \epsilon \eta(X)\eta(Y)
\end{align*}

for all vector fields $X, Y$ on $M$, where $\epsilon$ is 1 or -1 according as $\xi$ is space like or time like vector field and rank $\phi$ is $(n - 1)$. If

\begin{equation}
(2.6) \quad d\eta(X, Y) = g(X, \phi Y)
\end{equation}

for every $X, Y \in \chi(M)$, then we say that $M(\phi, \xi, \eta, g, \epsilon)$ is an almost contact metric manifold. Also, we have

\begin{equation}
(2.7) \quad \phi \xi = 0, \quad \eta(\phi X) = 0.
\end{equation}

If an $\epsilon$-contact metric manifold satisfies

\begin{equation}
(2.8) \quad (\nabla X \phi)(Y) = -g(X, \phi Y)\xi - \epsilon \eta(Y)\phi X,
\end{equation}

where $\nabla$ denotes the Levi-Civita connection with respect to $g$, then $M$ is called an $\epsilon$-Kenmotsu manifold [8].

An $\epsilon$-almost contact metric manifold is an $\epsilon$-Kenmotsu manifold, if and only if

\begin{equation}
(2.9) \quad \nabla X \xi = \epsilon (X - \eta(X)\xi).
\end{equation}

Further, in an $\epsilon$-Kenmotsu manifold, the following relations hold [8, 10, 11]:

\begin{align*}
(2.10) & \quad (\nabla X \eta)Y = g(X, Y) - \epsilon \eta(X)\eta(Y), \\
(2.11) & \quad R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \\
(2.12) & \quad R(\xi, X)Y = \eta(Y)X - \epsilon g(X, Y)\xi,
\end{align*}
ON THE CONFORMAL CURVATURE TENSOR OF $\epsilon$-KENMOTSU MANIFOLDS

(2.13) $R(\xi, X)\xi = -R(X, \xi)\xi = X - \eta(X)\xi,$

(2.14) $\eta(R(X, Y)Z) = \epsilon[g(X, Z)\eta(Y) - g(Y, Z)\eta(X)],$

(2.15) $S(X, \xi) = -(n - 1)\eta(X),$

(2.16) $Q\xi = -\epsilon(n - 1)\xi,$

where $g(QX, Y) = S(X, Y).$

(2.17) $S(\phi X, \phi Y) = S(X, Y) + \epsilon(n - 1)\eta(X)\eta(Y).$

We note that if $\epsilon = 1$ and the structure vector field $\xi$ is space like, then an $\epsilon$-Kenmotsu manifold is usual Kenmotsu manifold.

An $\epsilon$-Kenmotsu manifold $M$ is said to be an $\eta$-Einstein manifold if its Ricci tensor $S$ is of the form [22]

(2.18) $S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y),$

where $a$ and $b$ are scalar functions of $\epsilon.$

3. Curvature tensor in an $\epsilon$-Kenmotsu manifold with respect to the
semi-symmetric non-metric connection

Let $M$ be an $n$-dimensional $\epsilon$-Kenmotsu manifold. The curvature tensor $\bar{R}$ with respect to the semi-symmetric non-metric connection $\bar{\nabla}$ is defined by

(3.1) $\bar{R}(X, Y)Z = \bar{\nabla}_X \bar{\nabla}_Y Z - \bar{\nabla}_Y \bar{\nabla}_X Z - \bar{\nabla}_{[X,Y]} Z.$

By virtue of (1.1) and (3.1), we have

(3.2) $\bar{R}(X, Y)Z = R(X, Y)Z + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y$

$+ ((\nabla_X \eta)Y) - ((\nabla_Y \eta)Z)X,$

where

$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$

is the Riemannian curvature tensor of the connection $\nabla.$ Using (2.10) in (3.2) we get

(3.3) $\bar{R}(X, Y)Z = R(X, Y)Z + g(X, Z)Y - g(Y, Z)X$

$+ (1 + \epsilon)[\eta(Y)X - \eta(X)Y]\eta(Z).$

Now contracting $X$ in (3.3), we get

(3.4) $\bar{S}(Y, Z) = S(Y, Z) + (1 - n)g(Y, Z) + (1 + \epsilon)(n - 1)\eta(Y)\eta(Z),$
where $\tilde{S}$ and $S$ are the Ricci tensors of the connections $\nabla$ and $\nabla$, respectively on $M$. This gives

\[ QY = QY + (1 - n)Y + (1 + \epsilon)(n - 1)\eta(Y)\xi. \]

Contracting again $Y$ and $Z$ in (3.4), it follows that

\[ \tilde{r} = r + n(1 - n) + (1 + \epsilon)(n - 1), \]

where $\tilde{r}$ and $r$ are the scalar curvatures of the connections $\nabla$ and $\nabla$, respectively on $M$.

**Lemma 3.1.** Let $M$ be an $n$-dimensional $\epsilon$-Kenmotsu manifold with respect to the semi-symmetric non-metric connection, then

\[ \tilde{R}(X, Y)\xi = 0, \]

\[ \tilde{R}(\xi, X)Y = -(1 + \epsilon)[g(X, Y)\xi - \eta(X)\eta(Y)\xi], \]

\[ \tilde{S}(Y, \xi) = 0, \]

\[ \tilde{Q}\xi = 0. \]

**Proof.** By replacing $Z = \xi$ in (3.3) and using (2.4) and (2.11), we get (3.7). (3.8) follows from (3.3) and (2.12). Taking $Z = \xi$ in (3.4) and using (2.2), (2.4) and (2.15), we get (3.9). From (3.5), (3.16) and (2.2), we get (3.10). \qed

**Definition 3.2.** The conformal curvature tensor $\tilde{C}$ of type $(1,3)$ in an $n$-dimensional $\epsilon$-Kenmotsu manifold with respect to the semi-symmetric non-metric connection $\nabla$ is given by $([15], [22])$

\[ \tilde{C}(X, Y)Z = \tilde{R}(X, Y)Z - \frac{1}{(n - 2)}[\tilde{S}(Y, Z)X - \tilde{S}(X, Z)Y + g(Y, Z)\tilde{Q}X \]

\[ - g(X, Z)QY] + \frac{\tilde{r}}{(n - 1)(n - 2)}[g(Y, Z)X - g(X, Z)Y], \]

where $\tilde{R}$, $\tilde{S}$, $\tilde{Q}$ and $\tilde{r}$ are the Riemannain curvature tensor, the Ricci tensor, the Ricci operator and the scalar curvature with respect to the semi-symmetric non-metric connection, respectively. The Ricci tensor $\tilde{S}$ and the Ricci operator $\tilde{Q}$ are related by $g(\tilde{Q}X, Y) = \tilde{S}(X, Y)$. 
By using (3.3)-(3.6) in (3.11), we obtain
\[
\bar{C}(X,Y)Z = C(X,Y)Z + \frac{2n + 1 + \epsilon}{n - 2} (g(Y,Z)X - g(X,Z)Y) - \frac{1 + \epsilon}{n - 2} (\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y)
\]
\[
- \frac{(1 + \epsilon)(n - 1)}{n - 2} (g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi),
\]
where
\[
C(X,Y)Z = R(X,Y)Z - \frac{1}{(n - 2)} [S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] + \frac{r}{(n - 1)(n - 2)} [g(Y,Z)X - g(X,Z)Y]
\]
is the conformal curvature tensor with respect to the Levi-Civita connection $\nabla$.

The equation (3.12) is the relation between the conformal curvature tensors with respect to the semi-symmetric non-metric connection $\nabla$ and the Levi-Civita connection $\nabla$.

4. Flatness conditions in $\epsilon$-Kenmotsu manifolds with respect to the semi-symmetric non-metric connection

**Definition 4.1.** An $\epsilon$-Kenmotsu manifold $M$ is said to be:

(i) quasi-conformally flat with respect to the semi-symmetric non-metric connection, if

\[
g(\bar{C}(X,Y)Z, \phi W) = 0, \quad X,Y,Z,W \in \chi(M);
\]

(ii) $\xi$-conformally flat with respect to the semi-symmetric non-metric connection, if

\[
\bar{C}(X,Y)\xi = 0, \quad X,Y \in \chi(M);
\]

(iii) pseudoconformally flat with respect to the semi-symmetric non-metric connection, if

\[
g(\bar{C}(\phi X,Y)Z, \phi W) = 0, \quad X,Y,Z,W \in \chi(M); \quad \text{and}
\]

(iv) $\phi$-conformally flat with respect to the semi-symmetric non-metric connection, if

\[
\phi^2 \bar{C}(\phi X,\phi Y)\phi Z = 0, \quad X,Y,Z \in \chi(M).
\]

Firstly, we consider quasi-conformally flat $\epsilon$-Kenmotsu manifolds with respect to the semi-symmetric non-metric connection. From the equations (3.11)
and (4.1), we have
\[ g(\bar{R}(X, Y) Z, \phi W) = \frac{1}{(n-2)} [\bar{S}(Y, Z)g(X, \phi W) - \bar{S}(X, Z)g(Y, \phi W) + g(Y, Z)g(\varphi X, \phi W)] - \frac{\bar{r}}{(n-1)(n-2)} [g(Y, Z)g(X, \phi W) - g(X, Z)g(Y, \phi W)] \]

which by taking \( Y = Z = \xi \) and using (2.4), (3.7), (3.9) and (3.10) reduces to

\[ \bar{S}(X, \phi W) = \frac{\bar{r}}{(n-1)} g(X, \phi W). \]

Replacing \( W \) by \( \phi W \) and using (2.1), (4.6) yields

\[ \bar{S}(X, W) = \frac{\bar{r}}{(n-1)} g(X, W) - \frac{\bar{r}}{(n-1)} \eta(X)\eta(W). \]

In view of (3.4) and (3.6), (4.7) takes the form

\[ S(X, W) = (\epsilon + \frac{r}{n-1}) g(X, W) - (n + \frac{\epsilon r}{n-1}) \eta(X)\eta(W). \]

Thus we have the following theorem:

**Theorem 4.2.** An \( n \)-dimensional quasi-conformally flat \( \epsilon \)-Kenmotsu manifold with respect to the semi-symmetric non-metric connection is an \( \eta \)-Einstein manifold with respect to the Levi-Civita connection.

Secondly, we consider \( \xi \)-conformally flat \( \epsilon \)-Kenmotsu manifolds with respect to the semi-symmetric non-metric connection. From (3.11) and (4.2), we can write

\[ g[\bar{R}(X, Y) \xi - \frac{1}{(n-2)} (\bar{S}(Y, \xi) X - \bar{S}(X, \xi) Y + g(Y, \xi) \varphi X - g(X, \xi) \varphi Y) + \frac{\bar{r}}{(n-1)(n-2)} (g(Y, \xi) X - g(X, \xi) Y), W] = 0 \]

which by using (2.4), (3.7) and (3.9) reduces to

\[ \eta(X) \bar{S}(Y, W) - \eta(Y) \bar{S}(X, W) + \frac{\bar{r}}{(n-1)} (\eta(Y) g(X, W) - \eta(X) g(Y, W)) = 0. \]

Putting \( Y = \xi \) and then using (2.2), (2.4) and (3.9) in (4.10), we get

\[ \bar{S}(X, W) = \frac{\bar{r}}{(n-1)} g(X, W) - \frac{\epsilon \bar{r}}{n-1} \eta(X)\eta(W). \]

In view of (3.4) and (3.6), (4.11) takes the form

\[ S(X, W) = (\epsilon + \frac{r}{n-1}) g(X, W) - (n + \frac{\epsilon r}{n-1}) \eta(X)\eta(W). \]

Thus we have the following theorem:
Theorem 4.3. An n-dimensional $\xi$-conformally flat $\epsilon$-Kenmotsu manifold with respect to the semi-symmetric non-metric connection is an $\eta$-Einstein manifold with respect to the Levi-Civita connection.

Next, taking $Z = \xi$ in (3.12) and using (2.2) and (2.4), we have

$$C(X, Y)\xi = C(X, Y)\xi + \frac{2n\epsilon}{n - 2}(\eta(Y)X - \eta(X)Y).$$

Since $\eta(X)Y - \eta(Y)X = R(X, Y)\xi \neq 0$, in an $\epsilon$-Kenmotsu manifold, in general, then we have the following theorem:

Theorem 4.4. In an $\epsilon$-Kenmotsu manifold $\xi$-conformally flatness with respect to the semi-symmetric non-metric connection and the Levi-Civita connection are not equivalent.

Thirdly, we consider pseudoconformally flat $\epsilon$-Kenmotsu manifolds with respect to the semi-symmetric non-metric connection. From (3.11) and (4.3), we can write

$$g(R(\phi X, Y)Z, \phi W) = \frac{1}{(n - 2)}[S(Y, Z)g(\phi X, \phi W)$$

$$- 2(n - 1)g(Y, Z)g(\phi X, \phi W) + (1 + \epsilon)\eta(Y)\eta(Z)g(\phi X, \phi W)$$

$$- \frac{r - (n - 1)(2n - 3 - \epsilon)}{(n - 1)(n - 2)}[g(Y, Z)g(\phi X, \phi W) - g(\phi X, Z)g(Y, \phi W)].$$

In view of (3.3), (3.4) and (3.6), (4.14) takes the form

$$g(R(\phi X, Y)Z, \phi W) = \frac{1}{(n - 2)}[S(Y, Z)g(\phi X, \phi W)$$

$$- 2(n - 1)g(Y, Z)g(\phi X, \phi W) + (1 + \epsilon)\eta(Y)\eta(Z)g(\phi X, \phi W)$$

$$+ g(Y, Z)S(\phi X, \phi W) - g(\phi X, Z)S(Y, \phi W)]$$

$$- \frac{r - (n - 1)(2n - 3 - \epsilon)}{(n - 1)(n - 2)}[g(Y, Z)g(\phi X, \phi W) - g(\phi X, Z)g(Y, \phi W)].$$

Let $\{e_1, e_2, \ldots, e_{n-1}, \xi\}$ be a local orthonormal basis of the vector fields in $M$. Using that $\{\phi e_1, \phi e_2, \ldots, \phi e_{n-1}, \xi\}$ is also a local orthonormal basis, if we put $X = W = e_i$ in (4.15) and sum up with respect to $i$, then we have

$$\sum_{i=1}^{n-1} g(R(\phi e_i, Y)Z, \phi e_i) = \frac{1}{(n - 2)} \sum_{i=1}^{n-1} [S(Y, Z)g(\phi e_i, \phi e_i)$$

$$- 2(n - 1)g(Y, Z)g(\phi e_i, \phi e_i) + (1 + \epsilon)\eta(Y)\eta(Z)g(\phi e_i, \phi e_i)$$

$$- \frac{r - (n - 1)(2n - 3 - \epsilon)}{(n - 1)(n - 2)} \sum_{i=1}^{n-1} [g(Y, Z)g(\phi e_i, \phi e_i) - g(\phi e_i, Z)g(Y, \phi e_i)].$$
It is easy to verify that
\[
\sum_{i=1}^{n-1} g(R(\phi e_i, Y)Z, \phi e_i) = S(Y, Z) + g(Y, Z) - \epsilon(Y)\eta(Z), \tag{4.17}
\]
\[
\sum_{i=1}^{n-1} g(R(\phi e_i, \phi Y)\phi Z, \phi e_i) = S(\phi Y, \phi Z) + g(\phi Y, \phi Z), \tag{4.18}
\]
\[
\sum_{i=1}^{n-1} S(\phi e_i, \phi e_i) = r + n - 1, \tag{4.19}
\]
\[
\sum_{i=1}^{n-1} g(\phi e_i, Z)S(Y, \phi e_i) = S(Y, Z) + \epsilon(n - 1)\eta(Y)\eta(Z), \tag{4.20}
\]
\[
\sum_{i=1}^{n-1} g(\phi e_i, \phi Z)S(\phi Y, \phi e_i) = S(\phi Y, \phi Z), \tag{4.21}
\]
\[
\sum_{i=1}^{n-1} g(\phi e_i, \phi e_i) = n - 1, \tag{4.22}
\]
\[
\sum_{i=1}^{n-1} g(\phi e_i, Z)g(Y, \phi e_i) = g(Y, Z) - \eta(Y)\eta(Z), \tag{4.23}
\]
\[
\sum_{i=1}^{n-1} g(\phi e_i, \phi Z)g(\phi Y, \phi e_i) = g(\phi Y, \phi Z). \tag{4.24}
\]

By virtue of (4.17), (4.19), (4.20), (4.22) and (4.23), (4.16) yields
\[
S(Y, Z) = \left[\frac{r}{n-1} - n + 3 - \epsilon(n - 2)\right]g(Y, Z) - \left[\frac{r}{n-1} - n + 2(1 + \epsilon)\right]\eta(Y)\eta(Z). \tag{4.25}
\]

Thus we have the following theorem:

**Theorem 4.5.** An \(n\)-dimensional pseudoconformally flat \(\epsilon\)-Kenmotsu manifold with respect to the semi-symmetric non-metric connection is an \(\eta\)-Einstein manifold with respect to the Levi-Civita connection.

Lastly, we consider \(\phi\)-conformally flat \(\epsilon\)-Kenmotsu manifolds with respect to the semi-symmetric non-metric connection. From (4.4), we have
\[
g(C(\phi X, \phi Y)\phi Z, \phi W) = 0. \tag{4.26}
\]

Using (3.11), (4.26) can be written as
\[
g(\bar{R}(\phi X, \phi Y)\phi Z, \phi W) = \frac{1}{(n-2)}[\bar{S}(\phi Y, \phi Z)g(\phi X, \phi W)
\]
\[
- \frac{r}{(n-1)(n-2)}[g(\phi Y, \phi Z)g(\phi X, \phi W) - g(\phi X, \phi Z)g(\phi Y, \phi W)]. \tag{4.27}
\]
In view of (3.3), (3.4) and (3.6), (4.27) takes the form
\[ g(R(\phi X, \phi Y)\phi Z, \phi W) = \frac{1}{(n-2)}[S(\phi Y, \phi Z)g(\phi X, \phi W)] \]

(4.28)
\[ -S(\phi X, \phi Z)g(\phi Y, \phi W) + S(\phi X, \phi W)g(\phi Y, \phi Z) - S(\phi Y, \phi W)g(\phi X, \phi Z) \]
\[ - \frac{r + (n - 1)(1 + \epsilon)}{(n-1)(n-2)} [g(\phi Y, \phi Z)g(\phi X, \phi W) - g(\phi X, \phi Z)g(\phi Y, \phi W)]. \]

Let \( \{e_1, e_2, \ldots, e_{n-1}, \xi\} \) be a local orthonormal basis of the vector fields in \( M \).
Using that \( \{\phi e_1, \phi e_2, \ldots, \phi e_{n-1}, \xi\} \) is also a local orthonormal basis, if we put
\( X = W = e_i \) in (4.28) and sum up with respect to \( i \), then we have
\[ \sum_{i=1}^{n-1} g(R(\phi e_i, \phi Y)\phi Z, \phi e_i) = \frac{1}{(n-2)} \sum_{i=1}^{n-1} [S(\phi Y, \phi Z)g(\phi e_i, \phi e_i)] \]

(4.29)
\[ -S(\phi e_i, \phi Z)g(\phi Y, \phi e_i) + S(\phi e_i, \phi e_i)g(\phi Y, \phi Z) - S(\phi Y, \phi e_i)g(\phi e_i, \phi Z) \]
\[ - \frac{r + (n - 1)(1 + \epsilon)}{(n-1)(n-2)} \sum_{i=1}^{n-1} [g(\phi Y, \phi Z)g(\phi e_i, \phi e_i) - g(\phi e_i, \phi Z)g(\phi Y, \phi e_i)]. \]

By virtue of (4.18), (4.19), (4.21), (4.22) and (4.24), (4.29) can be written as
\[ S(\phi Y, \phi Z) + g(\phi Y, \phi Z) = \frac{1}{(n-2)}[(n - 3)S(\phi Y, \phi Z)] \]
\[ + (r + n - 1)g(\phi Y, \phi Z)] - \frac{r + (n - 1)(1 + \epsilon)}{(n-1)} g(\phi Y, \phi Z) \]

(4.30)
from which it follows that
\[ S(\phi Y, \phi Z) = (n - 2)[\frac{r + n - 1}{n-2} - \frac{r + (n - 1)(1 + \epsilon)}{n-1} - 1]g(\phi Y, \phi Z). \]

Using (2.5) and (2.17), (4.31) yields
\[ S(Y, Z) = \frac{r}{n-1} - (n - 2)\epsilon - n + 3]g(Y, Z) \]
\[ - \epsilon[\frac{r}{n-1} - (n - 2)\epsilon + 2]g(Y, \xi). \]

Thus we have the following theorem:

**Theorem 4.6.** An \( n \)-dimensional \( \phi \)-conformally flat \( \epsilon \)-Kenmotsu manifold with respect to the semi-symmetric non-metric connection is an \( \eta \)-Einstein manifold with respect to the Levi-Civita connection.

5. \( \epsilon \)-Kenmotsu manifolds with respect to the semi-symmetric non-metric connection satisfying the condition \( S \cdot C = 0 \)

In this section we investigate \( \epsilon \)-Kenmotsu manifolds with respect to the semi-symmetric non-metric connection satisfying the condition \( S \cdot C = 0 \), where \( S \) is
the Ricci tensor with respect to the semi-symmetric non-metric connection of type $(0,2)$. Let the manifold satisfies the condition

\[(5.1) \quad (\tilde{S}(X,Y) \cdot \tilde{C})(U,V) W = 0,\]

where $X,Y,U,V,W \in \chi(M)$. The above equation implies

\[(5.2) \quad (X_\Lambda S Y)\tilde{C}(U,V) W + \tilde{C}((X_\Lambda S Y)U,V) W + \tilde{C}((X_\Lambda S Y)V,U) W + \tilde{C}(U,V)(X_\Lambda S Y) W = 0,\]

where the endomorphism $X_\Lambda S Y$ is defined by

\[(5.3) \quad (X_\Lambda S Y) Z = S(Y,Z) X - S(X,Z) Y.\]

Therefore in view of (5.3), (5.2) takes the form

\[(5.4) \quad S(Y,\tilde{C}(U,V) W) X - S(X,\tilde{C}(U,V) W) Y + S(Y,U) \tilde{C}(X,V) W + S(Y,V) \tilde{C}(U,X) W - S(X,V) \tilde{C}(U,Y) W + \tilde{S}((Y,W) \tilde{C}(U,V) X - \tilde{S}(X,W) \tilde{C}(U,V) Y = 0.\]

Taking $X = \xi$ in (5.4) and using (3.9), we obtain

\[(5.5) \quad S(Y,\tilde{C}(U,V) W) \xi + S(Y,U) \tilde{C}(\xi,V) W + S(Y,V) \tilde{C}(U,\xi) W + \tilde{S}((Y,W) \tilde{C}(U,V) \xi + \tilde{S}(Y,V) \tilde{C}(\xi,\xi) \xi = 0.\]

which by taking $U = W = \xi$ and then using (3.9) reduces to

\[(5.6) \quad S(Y,\tilde{C}(\xi,V) \xi) + S(Y,V) \tilde{C}(\xi,\xi) \xi = 0.\]

Using (3.11) in (5.6), we obtain

\[(5.7) \quad S^2(Y,V) = \frac{r}{n-1} \tilde{S}(Y,V).\]

In view of (3.4)-(3.6) and (3.9), (5.7) yields

\[(5.8) \quad S^2(Y,V) = \left(\frac{r}{n-1} + n - 1 + \epsilon\right) S(X,Y) - (r + n \epsilon - \epsilon) g(Y,V) + (1 + \epsilon)(r + n^2 - n) \eta(Y) \eta(V).\]

Thus we can state the following theorem:

**Theorem 5.1.** If an $n$-dimensional $\epsilon$-Kenmotsu manifold with respect to the semi-symmetric non-metric connection satisfying the condition $\tilde{S} \cdot \tilde{C} = 0$. Then

\[S^2(Y,V) = \left(\frac{r}{n-1} + n - 1 + \epsilon\right) S(X,Y) - (r + n \epsilon - \epsilon) g(Y,V) + (1 + \epsilon)(r + n^2 - n) \eta(Y) \eta(V).\]
Example. We consider the 3-dimensional manifold $M = \{(x, y, z) \in \mathbb{R}^3 : z \neq 0\}$, where $(x, y, z)$ are standard coordinates of $\mathbb{R}^3$. Let $e_1$, $e_2$ and $e_3$ be the vector fields on $M$ given by

$$e_1 = \epsilon z \frac{\partial}{\partial x}, \quad e_2 = \epsilon z \frac{\partial}{\partial y}, \quad e_3 = -\epsilon z \frac{\partial}{\partial z} = \xi$$

which are linearly independent at each point of $M$ and hence form a basis of $M$. Define an indefinite metric $g$ on $M^3$ as

$$g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = \epsilon,$$

$$g(e_1, e_2) = g(e_1, e_3) = g(e_2, e_3) = 0.$$ Let $\eta$ be the 1-form on $M$ defined as $\eta(X) = \epsilon g(X, e_3)$ for all $X \in \chi(M)$, and let $\phi$ be the $(1, 1)$ tensor field on $M$ defined as

$$\phi e_1 = -e_2, \quad \phi e_2 = -e_1, \quad \phi e_3 = 0.$$ By applying linearity of $\phi$ and $g$, we have

$$\eta(\xi) = 1, \quad \phi^2 X = -X + \eta(X)\xi, \quad \eta(\phi X) = 0,$$

$$g(X, \xi) = \epsilon \eta(X), \quad g(\phi X, \phi Y) = g(X, Y) - \epsilon \eta(X)\eta(Y)$$

for all $X, Y \in \chi(M)$. Then for $\xi = e_3$, the structure $(\phi, \xi, \eta, g, \epsilon)$ defines an indefinite almost contact metric structure on $M$.

Let $\nabla$ be the Levi-Civita connection with respect to the indefinite metric $g$. Then we have

$$[e_1, e_2] = 0, \quad [e_2, e_3] = \epsilon e_2, \quad [e_1, e_3] = \epsilon e_1.$$ The Riemannian connection $\nabla$ of the indefinite metric $g$ is given by

$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g(X, [Y, Z])$$

$$+ g(Y, [Z, X]) + g(Z, [X, Y]),$$

which is known as Koszul’s formula. Using Koszul’s formula, we can easily calculate

$$(5.9) \quad \nabla_{e_1} e_1 = -\epsilon e_3, \quad \nabla_{e_1} e_2 = 0, \quad \nabla_{e_1} e_3 = \epsilon e_1, \quad \nabla_{e_2} e_1 = 0,$$

$$\nabla_{e_2} e_2 = -\epsilon e_3, \quad \nabla_{e_2} e_3 = \epsilon e_2, \quad \nabla_{e_3} e_1 = 0, \quad \nabla_{e_3} e_2 = 0, \quad \nabla_{e_3} e_3 = 0.$$ Thus from (5.9), it follows that the manifold satisfies $\nabla_X \xi = \epsilon (X - \eta(X)\xi)$ for $\xi = e_3$. Hence the manifold is an indefinite Kenmotsu manifold.

From the equation (5.9) and the expression of curvature tensor $R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z$, it can be easily verified that

$$(5.10) \quad R(e_1, e_2)e_1 = e_2, \quad R(e_1, e_3)e_1 = e_3, \quad R(e_2, e_3)e_1 = 0,$$
By using (1.3) in (5.9), we obtain
\begin{align*}
\nabla_{e_1} e_1 &= -e_3, \quad \nabla_{e_2} e_1 = 0, \quad \nabla_{e_3} e_1 = 0, \quad \nabla_{e_1} e_2 = 0, \quad \nabla_{e_2} e_2 = -e_3, \\
\nabla_{e_3} e_2 &= 0, \quad \nabla_{e_1} e_3 = (1+\epsilon)e_1, \quad \nabla_{e_2} e_3 = (1+\epsilon)e_2, \quad \nabla_{e_3} e_3 = e_3.
\end{align*}

From (3.3) and (5.10), we can easily obtain the following components of the curvature tensor with respect to the semi-symmetric non-metric connection as
\begin{align*}
(5.12) \quad R(e_1, e_2)e_1 &= (1+\epsilon)e_2, \quad R(e_1, e_3)e_1 = (1+\epsilon)e_3, \quad R(e_2, e_3)e_1 = 0, \\
R(e_1, e_2)e_2 &= -(1+\epsilon)e_1, \quad R(e_1, e_3)e_2 = 0, \quad R(e_2, e_3)e_2 = (1+\epsilon)e_3, \\
R(e_1, e_2)e_3 &= 0, \quad R(e_1, e_3)e_3 = 0, \quad R(e_2, e_3)e_3 = 0.
\end{align*}

By using the above expressions, we get the Ricci tensors and the scalar curvatures as follows:
\begin{align*}
S(e_1, e_1) &= S(e_2, e_2) = S(e_3, e_3) = -2\epsilon, \\
\tilde{S}(e_1, e_1) &= \tilde{S}(e_2, e_2) = -(1+\epsilon), \quad \tilde{S}(e_3, e_3) = -2(1+\epsilon), \\
r &= -6\epsilon, \quad \tilde{r} = -4(1+\epsilon).
\end{align*}

By using the equations (1.1) and (5.11), the torsion tensors are given by
\begin{align*}
T(e_1, e_1) &= 0, \quad T(e_1, e_3) = e_1, \quad T(e_1, e_2) = 0, \\
T(e_2, e_2) &= 0, \quad T(e_2, e_3) = e_2, \quad T(e_3, e_3) = 0.
\end{align*}

In view of (1.2), we find
\begin{align*}
(\nabla_{e_1} g)(e_1, e_3) &= -\epsilon, \quad (\nabla_{e_2} g)(e_2, e_3) = -\epsilon, \quad (\nabla_{e_3} g)(e_1, e_2) = 0.
\end{align*}

Hence $M$ is a 3-dimensional $\epsilon$-Kenmotsu manifold with respect to the semi symmetric non-metric connection.

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References


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