FINITE GROUPS WHOSE ALL PROPER SUBGROUPS ARE GPST-GROUPS

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Abstract. A set $W = \{W_1, \ldots, W_t\}$ of nilpotent Hall subgroups of $G$ is a complete Wielandt set if $(|W_i|, |W_j|) = 1$ for all $i, j$. A finite group $G$ is called a GPST-group if $G$ has a complete Wielandt set $W$ such that every member in $W$ permutes all maximal subgroups of any non-cyclic subgroup $S$ in $W$. In this paper, we give a complete classification of those groups which are not GPST-groups but all of whose proper subgroups are GPST-groups, i.e., they are precisely minimal non-PST-groups.

Keywords: Wielandt set, GPST-group, supersoluble group, power automorphism, permutable subgroup.

1. Introduction

All groups considered in this paper are finite and our notation is standard.

Let $\Sigma$ be an abstract group theoretical property, for example, nilpotency, supersolubility, solubility, etc. If all proper subgroups of a group $G$ have the property $\Sigma$ but $G$ does not have it, then $G$ is called a minimal non-$\Sigma$-group.

The structures of minimal non-$\Sigma$-groups have been studied for various classes of groups $\Sigma$, and many classical results about this topic have been obtained. For instance, Miller and Moreno [8], Schmidt [12], and Doerk [5] analyzed the structures of minimal non-abelian groups, minimal non-nilpotent groups, and minimal non-supersoluble groups, respectively. However, the complete classifications of minimal non-nilpotent groups and minimal non-supersoluble groups were given by Ballester-Bolinches, Esteban-Romero and Robinson [4], Ballester-Bolinches and Esteban-Romero [2], respectively.

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On the other hand, Robinson [10] characterized minimal non-T-groups (T-groups are groups in which normality is a transitive relation, i. e., if the normality of $H$ in $K$ and of $K$ in $G$ always imply that $H$ is normal in $G$). A subgroup $H$ of $G$ is said to be $s$-permutable in $G$ if $H$ permutes all Sylow subgroups of $G$. Agrawal [1] studied PST-groups, i. e., groups in which Sylow permutability is a transitive relation. A group is a soluble PST-group if and only if it has an abelian Hall subgroup $L$ of odd order such that $G/L$ is nilpotent, and every element of $G$ induces a power automorphism in $L$. Robinson [11] also gave a complete classification of minimal non-PST-groups.

The aim in the present work is to determine the structure of a kind of minimal non-$\Sigma$-groups. Guo and Skiba [6] called a set $W = \{W_1, \ldots, W_i\}$ of nilpotent Hall subgroups of $G$ a complete Wielandt set if $(|W_i|, |W_j|) = 1$ for all $i, j$, and characterized the structure of a group $G$ which has a complete Wielandt set $W$ such that every member in $W$ permutes all maximal subgroups of any non-cyclic subgroup $S$ in $W$. The specific result is as follows.

**Theorem A** [6, Theorem A]. A group $G$ has a complete Wielandt set of subgroups $W$ such that every member in $W$ permutes all maximal subgroups of any non-cyclic subgroup $S$ in $W$ if and only if $G = D \rtimes M$ is a supersoluble group where $D = G^N$ is a nilpotent Hall subgroup of $G$ of odd order whose maximal subgroups are normal in $G$.

In view of the structure of a group described in Theorem A is very close to soluble PST-group but weaker than soluble PST-group, so Guo and Skiba [6] called this group a generalized PST-group or GPST-group for short.

In this paper, we give a complete classification of those groups which are not GPST-groups but all of whose proper subgroups are GPST-groups. Our main result is as follows:

**Theorem 1.1.** Let $p$ and $q$ be distinct prime divisors of the order of a group $G$. Then $G$ is a minimal non-GPST-group if and only if $G$ is one of the following types:

1. $G = P \rtimes Q$, where $P = \langle a, b \rangle$ is an elementary abelian $p$-group of order $p^2$, and $Q = \langle x \rangle$ is cyclic of order $q'$. Define $a^x = a^i$, $b^x = b^j$, $p \equiv 1 (\text{mod } q')$, and $r \geq 1$, where $i$ is the least positive primitive $q'$-th root of unity modulo $p$, $j = 1 + kq^{f-1}$, with $0 < k < q$ and $r \geq f$;

2. $G = P \rtimes Q$, where $Q = \langle x \rangle$ is cyclic of order $q' > 1$, with $q \nmid p - 1$, and $P$ is an irreducible $Q$-module over the field of $p$ elements with kernel $\langle x^q \rangle$ in $Q$;

3. $G = P \rtimes Q$, where $P$ is a non-abelian special $p$-group of rank $2m$, the order of $p$ modulo $q$ being $2m$, $Q = \langle x \rangle$ is cyclic of order $q' > 1$, $x$ induces an automorphism in $P$ such that $P/\Phi(P)$ is a faithful and irreducible $Q$-module, and $x$ centralizes $\Phi(P)$. Furthermore, $|P/\Phi(P)| = p^{2m}$ and $|P'| \leq p^m$;

4. $G = PQ$, where $P = \langle a_0, a_1, \ldots, a_{q-1} \rangle$ is an elementary abelian $p$-group of order $p^q$, $Q = \langle x \rangle$ is cyclic of order $q'$, $q^f$ is the highest power of $q$ dividing
$p - 1$ and $r > f \geq 1$. Define $a_j^x = a_{j+1}$ for $0 \leq j < q - 1$ and $a^x_{q-1} = a_0$, where $i$ is a primitive $q^f$-th root of unity modulo $p$.

Coincidentally, by comparing with main result in [11], minimal non-GPST-groups are precisely minimal non-PST-groups. In addition, Ballester-Bolinches and Esteban-Romero [3] introduced an interesting definition. Let $p$ be a prime. A group $G$ is said to be a $\mathcal{Y}_p$-group if, for all $p$-subgroups $H$ and $S$ of $G$ such that $H \leq S$, $H$ is $S$-permutable in $N_G(S)$. They also gave that: a group $G$ is a soluble PST-group if and only if $G$ is a $\mathcal{Y}_p$-group for all primes $p$ [3, Theorem 4]. Hence the classification of minimal non-$\mathcal{Y}_p$-group in [4, Theorem 2] may be regarded as a local approach to the classification of minimal non-PST-groups. Our result is given naturally.

**Corollary 1.2.** Let $G$ be a group. Then the following conditions are equivalent:

(i) $G$ is a minimal non-PST-group.

(ii) $G$ is a minimal non-GPST-group.

(iii) $G$ is a minimal non-$\mathcal{Y}_p$-group for every prime divisor $p$ of the order of $G$.

**2. Preliminary results**

We collect some lemmas which will be frequently used in the sequel.

**Lemma 2.1 ([7]).** Let $\{P_1, P_2, \ldots, P_r\}$ be a Sylow basis of a soluble group $G$. Then the following statements are equivalent:

(a) Every subgroup of $P_i$ permutes every subgroup of $P_j$ for $i \neq j$.

(b) The nilpotent residual $G^N$ of $G$ is an abelian Hall subgroup of $G$, and every element of $G$ induces a power automorphism in $G^N$.

**Lemma 2.2.** Let $G$ be a minimal non-GPST-group. Then there exists a normal non-cyclic Sylow $p$-subgroup $P$ of $G$ and a non-normal cyclic Sylow $q$-subgroup $Q$ of $G$ with $p \neq q$ such that $|G| = p^aq^b$ for positive integers $a$ and $b$.

**Proof.** Since every proper subgroup of $G$ is a GPST-group, $G$ is supersoluble or minimal non-supersoluble by Theorem A. By a result of Doerk [5], $G$ is soluble and $G$ has a nontrivial normal Sylow $p$-subgroup $P = O_p(G) \neq 1$, for some prime $p$. Let $G = P \rtimes H$ and $W = \langle P, H_1, \ldots, H_t \rangle$ of nilpotent Hall subgroups of $G$ be a complete Wielandt set, where $H$ is a $p'$-group of $G$. If $t \geq 2$, then $PH_1, PH_2, \ldots, PH_t$ are GPST-groups. Thus $H_1, H_2, \ldots, H_t$ permute every maximal subgroup of $P$ whether or not $P$ is cyclic. Since the normality of $P$ and the fact that $H$ is a GPST-group again, $G$ is a GPST-group, a contradiction. Hence $t = 1$. Similar arguments as above, if $|\pi(H)| \geq 2$, then $G$ is a GPST-group, a contradiction. So $H = Q \in \text{Syl}_q(G)$ with $q \neq p$ a prime. If $Q$ is non-cyclic, then $\langle x, P \rangle \neq G$ for every element $x$ of $Q$. The minimality of $G$ implies that $\langle x, P \rangle$ is a GPST-group. By applying Theorem A, every maximal subgroup of $P$ is normal in $G$. By induction again, every subgroup of
Suppose \( P \) permutes every subgroup of \( \langle x \rangle \). By Lemma 2.1, \( P \) is abelian and \( x \) induces a power automorphism on \( P \). So \( G \) is a GPST-group, a contradiction. This induces that \( H = \langle x \rangle \), where \( |x| = q^b > 1 \). Clearly, \( P \) is non-cyclic by the definition of GPST-group. The proof of Lemma 2.2 is complete. \( \square \)

**Lemma 2.3** ([14, Lemma 5]). Suppose \( G = P \langle x \rangle \), \( P \) is a normal \( p \)-subgroup of \( G \) and \( x \) is a \( q \)-element. If all maximal subgroups of Sylow subgroups of \( G \) are normal in \( G \), then \( x \) induces a power automorphism on \( P/\Phi(P) \).

**Lemma 2.4** ([9], 13.4.3). Let \( \alpha \) be a power automorphism of an abelian group \( A \). If \( A \) is a \( p \)-group of finite exponent, then there is a positive integer \( l \) such that \( a^{\alpha} = a^l \) for all \( a \) in \( A \). If \( \alpha \) is nontrivial and has order prime to \( p \), then \( \alpha \) is fixed-point-free.

**Lemma 2.5** ([5]). Let \( G \) be a minimal non-supersoluble group. Then

1. \( G \) has a unique normal Sylow \( p \)-subgroup \( P \);
2. \( P/\Phi(P) \) is a minimal normal subgroup of \( G/\Phi(P) \), and \( P/\Phi(P) \) is non-cyclic;
3. If \( p \neq 2 \), then the exponent of \( P \) is \( p \);
4. If \( P \) is non-abelian and \( p = 2 \), then the exponent of \( P \) is \( 4 \);
5. If \( P \) is abelian, then the exponent of \( P \) is \( p \).

### 3. The proof of Theorem 1.1

**Proof.** If \( G \) is a minimal non-GPST-group, then we may assume \( G = PQ \) by Lemma 2.2, where \( P \) is a non-cyclic normal Sylow \( p \)-subgroup of \( G \) and \( Q = \langle x \rangle \) is a non-normal Sylow \( q \)-subgroup of \( G \) of order \( q^r \). Since all Sylow \( q \)-subgroups are conjugate in \( G \), we only consider the case that \( Q \) acts on \( P \). So we investigate the following two cases.

1. Assume that \( G \) is supersoluble and \( d(P) = k \), where \( d(P) \) is the rank of \( P \).

   Let \( 1 \leq \cdots \leq R \leq P \leq \cdots \leq G \) be an arbitrary chief series of \( G \). By Maschke’s Theorem [9, Theorem 8.1.2], there exists a normal subgroup \( N \) of \( G \) contained in \( P \) such that \( P/\Phi(P) = R/\Phi(P) \times N/\Phi(P) \), where \( |N/\Phi(P)| = p \). Clearly, \( N \nsubseteq R \) and \( 1 \leq N \leq P \leq G \) is a normal series of \( G \). By applying Schreier’s Refinement Theorem [9, Theorem 3.1.2], \( P \) has another maximal subgroup \( K \neq R \) such that \( K \) is normal in \( G \). Therefore, \( P \) has at least two maximal subgroups \( R \) and \( K \) which are normal in \( G \).

   Now we prove \( k = 2 \). If \( k \geq 3 \), then we can let \( P/\Phi(P) = \langle \bar{a}_1 \rangle \times \langle \bar{a}_2 \rangle \times \cdots \times \langle \bar{a}_k \rangle \) where \( a_1, a_2, \ldots, a_{k-1} \in R, a_2, a_3, \ldots, a_k \in K \). Since \( R/\langle x \rangle \) is a GPST-group, every maximal subgroup of \( R \) is normal in \( G \). By Lemma 2.3, \( (y\Phi(R))^x = y^l\Phi(R) \) for every \( y \in R \), where \( l \) is a positive integer. Thus, \( (y\Phi(P))^x = y^l\Phi(P) \) for every \( y \in R \). Similarly, \( (z\Phi(P))^x = z^m\Phi(P) \) for every \( z \in K \), where \( m \) is a positive integer. Furthermore, \( a_1^l\Phi(P) = (a_2\Phi(P))^x = a_2^m\Phi(P) \), and so \( l \equiv m \text{ (mod } p \) \). Hence, \( (a_i\Phi(P))^x = a_i^l\Phi(P) \) for \( i = 1, 2, \ldots, k \). It is easy to
see that every maximal subgroup of $P$ is normal in $G$. By Theorem A, $G$ is a GPST-group. This contradiction implies $k = 2$.

Now we let $P/\Phi(P) = R/\Phi(P) \times K/\Phi(P) = \langle \tilde{a}_1 \rangle \times \langle \tilde{a}_2 \rangle$, where $a_1 \in R, a_2 \in K$, $\tilde{a}_1^x = \tilde{a}_1^{k_1}$ and $\tilde{a}_2^x = \tilde{a}_2^{k_2}$. If $k_1 = k_2$, then every maximal subgroup of $P$ is normal in $G$, and so $G$ is a GPST-group, a contradiction. Hence, $k_1 \neq k_2$. Furthermore, we have that $P$ has only two maximal subgroups which are normal in $G$. Clearly, at least one action of which $x$ acts on $R$ and $P$ is nontrivial.

Without loss of generality, we may assume that $x$ induces an automorphism $\alpha$ on $R$. Since every subgroup of $P(x)$ is a GPST-group and by induction, it follows from Theorem A that every subgroup of $P$ permutes every subgroup of $\langle x \rangle$. By Lemma 2.1, $R$ is abelian and $\alpha$ is a power automorphism on $R$. By Lemma 2.4, $\alpha$ is fixed-point-free. Hence, we have either $K \cap R = 1$ if $K \langle x \rangle = K \times \langle x \rangle$ or $K \langle x \rangle \neq K \times \langle x \rangle$. If $K \cap R = 1$ and $K \langle x \rangle = K \times \langle x \rangle$, then $P = \langle a, b \rangle$ is an elementary abelian group of order $p^2$. We can easily have that $G$ is of type (I) with $f = 1$ and $k = q - 1$.

If $K \langle x \rangle \neq K \times \langle x \rangle$, similar arguments as above, $K$ is abelian, and $x$ induces a power automorphism in $K$. Thus, $\Phi(P) = R \cap K \leq Z(P)$. If $|P : Z(P)| \leq p$, then $P$ is abelian.

We prove that $P$ is elementary abelian. Let $\Omega_1(P)$ be the group generated by all elements of order $p$ in $P$ and assume that $\Omega_1(P) \neq P$. Then $\langle \Omega_1(P), x \rangle \neq G$ and it is a GPST-group. Therefore $x$ induces a power automorphism in $\Omega_1(P)$, i.e., there is a positive integer $t$, relatively prime to $p$, such that $a^t = a^t$ for all $a \in \Omega_1(P)$. Let $\beta$ be the automorphism of $P$ induced by $x$ and let $\gamma$ be the automorphism of $P$ in which $a \mapsto a^t$. Then $\beta \gamma^{-1}$ is an automorphism of $P$ fixing each element of order $p$ and $\beta \gamma^{-1}$ has order equal to a power of $p$, say $p^d$. Obviously $\beta \gamma = \gamma \beta$, so $\beta^p = \gamma^p \in \langle \gamma \rangle$. But $\beta$ has order prime to $p$, so $\beta \in \langle \gamma \rangle$ and $\beta$ is a power automorphism of $P$, a contradiction.

Assume that $P = \langle a \rangle \times \langle b \rangle$ is elementary abelian. Let $q^l$ be the order of the automorphism of $P$ induced by $x$, $a^x = a^i$ and $b^x = b^j$, where $i$ and $s$ are two distinct primitive $q^l$-th roots of unity modulo $p$. Then $0 < f \leq r$ and $p \equiv 1(\text{mod } q^l)$. Since $P(x^q) \neq G$, $x^q$ induces a power automorphism in $P$ and $i^q = s^q$. So $i$ and $s$ both have order $q^l$. Then $s = ij$ for some integer $j \equiv 1(\text{mod } q^l)$. Now $i^q = s^q = s^q$, so $j \equiv 1(\text{mod } q^l)$, and we can assume that $j = 1 + kq^l$. Hence $G$ is again of type (I).

If $|P : Z(P)| = p^2$, then $\Phi(P) = R \cap K = Z(P)$, and so $P$ is minimal non-abelian and $|P'| = p$. Let $P_1 = \langle a, P' \rangle$ and $P_2 = \langle b, P' \rangle$. Then $P_1 Q$ and $P_2 Q$ are GPST-groups. By hypothesis, $x$ induces power automorphisms in $P_1$ and $P_2$, say $g \mapsto g^{n_1}$ and $g \mapsto g^{n_2}$ respectively. By Lemma 2.4, these two power automorphisms are fixed-point-free. However, they must agree on $P'$, so $n_1 \equiv n_2(\text{mod } p)$ and we can assume $n_1 = n_2$. Since $[a, b]^{n_1} = [a, b]^x = [a^{n_1}, b^{n_1}]$, $n_1^2 \equiv n_1(\text{mod } p)$ and $n_1 \equiv 1(\text{mod } p)$, a contradiction.

(2) Assume that $G$ is minimal non-supersoluble.

Let $M$ be a maximal subgroup of $G$ such that $Q \leq M$. Then $M = P_3 Q$, where $P_3$ is a Sylow $p$-subgroup of $M$. By $[P_3, Q] \leq P \cap P_3 Q = P_3$, we have
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\[ N_G(P_3) \geq P_3Q = M. \] Since \( N_P(P_3) > P_3 \), \( P_3 \) is normal in \( G \). By Lemma 2.5 and the maximality of \( M \), \( P_3 = \Phi(P) \) is the Sylow \( p \)-subgroup of \( M \).

**Case 1.** If \( G \) is also a minimal non-nilpotent group, then by applying [4, Theorem 3], \( G \) is of either type (II) or type (III).

**Case 2.** If \( G \) is not a minimal non-nilpotent group and \( P \) is abelian, then by applying [2, Theorem 9, 10], we assume that \( G = PQ \), where \( P = \langle a_0, a_1, \ldots, a_{q-1} \rangle \) is an elementary abelian \( p \)-group of order \( p^q \), \( Q = \langle x \rangle \) is cyclic of order \( q^r \), \( q^f \) is the highest power of \( q \) dividing \( p - 1 \) and \( r > f \geq 1 \). Define \( a_j^x = a_{j+1} \) for \( 0 \leq j < q - 1 \) and \( a_{q-1}^x = a_0^i \), where \( i \) is a primitive \( q^f \)-th root of unity modulo \( p \).

For a maximal subgroup \( P^Q \) of \( G \) and any element \( a_k \) of \( P \), by computation, \( a_k^x = a_k^i \). So \( G \) is of type (IV).

**Case 3.** Assume that \( G \) is not a minimal non-nilpotent group and \( P \) is non-abelian. By applying [2, Theorem 9, 10], we may assume that \( G = P^Q \) such that \( P = \langle a_0, a_1 \rangle \) is an extraspecial group of order \( p^3 \) with exponent \( p \), \( Q = \langle x \rangle \) is a cyclic group of order \( 2^f \) with \( 2^f \) the highest power of \( 2 \) dividing \( p - 1 \) and \( r > f \geq 1 \) and \( a_0^y = a_1 \) and \( a_1^y = a_0^i \), where \( y \in \langle a_0, a_1 \rangle \) and \( i \) is a primitive \( 2^f \)-th root of unity modulo \( p \).

Since every subgroup of \( P^Q \) is a GPST-group, every subgroup of \( P \) permutes every subgroup of \( \langle x^2 \rangle \) by induction. If \( P^Q \neq P \times \langle x^2 \rangle \), then \( P \) is abelian by Lemma 2.1, a contradiction. Hence \( P^Q = P \times \langle x^2 \rangle \) and \( a_0^y = a_1^y = a_0^i \), which implies that \( x = 1 \) and \( i \equiv 1 \mod p \), a contradiction. Therefore, \( G \) is not of the type as above.

Conversely, it is easy to check that all groups satisfied types (I)—(IV) are minimal non-GPST-groups.

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**References**


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