THE HOMO SEPARATION ANALYSIS METHOD FOR SOLVING THE PARTIAL DIFFERENTIAL EQUATION

M. Zuriqat
Department of Mathematics
Al al-Bayt University
P.O. Box 130095, Al Mafraq
Jordan
d.alzuriqat@yahoo.com

Abstract. In this work,the homo separation analysis method (HSAM) is applied to obtain the exact solution for linear and nonlinear partial differential equation. The proposed algorithm presents a procedure of constructing the set of base functions and gives the one-order deformation equation in a simple form. This analytical method is a combination of the homotopy analysis method (HAM) with the separation of variables method. The exact solution is constructed by choosing an initial guess in addition to only one term of the series obtained by HAM. This work verifies the validity and the potential of the HSAM for the study of nonlinear partial differential equation.

Keywords: homotopy analysis method, separation of variables, partial differential equations, analytical method, Black-scholes equation.

1. Introduction

The study of nonlinear partial differential equation is of crucial importance in all areas of physics and engineering, as well as in other disciplines. It is very difficult to solve nonlinear problems and in general it is often more difficult to get an exact solution to a given nonlinear problem. The importance of obtaining the exact solutions of nonlinear PDE in mathematics is stell a significant problem that needs new methods. Several numerical and analytical methods have been developed and successfully employed to solve linear and nonlinear PDE. Such methods include variational iteration method [4, 12], Adomian decomposition method [5, 10, 21], differential transform method [2, 6, 17], the homopety perturbation method [15], the exp method [13], Legendre polynomial method [8, 9] and the homotopy analysis method [18, 19, 22]. Some of these methods use specific transformations and others give the solution as a series which converges to the exact solution. Recently, a lot of attention has been focused on the studies to getting exact solution for linear and nonlinear PDEs. Zhang and others [3, 14] give the exact solution for some specific nonlinear PDE. However, Yang [7] used the modified homotopy perturbation method to obtain the exact solution of the Fokker-Plank equation. Furthermore, Karbalaie et al.[1] used homotopy perturbation method with sepration of variables to find exact solution of PDE. The HAM yields rapidly convergent series solutions by using few iterations for both linear and nonlinear differential equations. The HAM was successfully applied to solve many nonlinear problems such as Riccati differential equation of fractional order [11], fractional KdV-Burgers-Kuramoto equation [16] and systems of fractional algebraic-differential equations [20]. In this paper, we developed a symbolic algorithm to find the exact solution of nonlinear PDE by using the construct of mth-order deformation equation of HAM and the technique of sepration of variables. We present an elegant fast approach by designing and utilizing a proper initial guess which satisify the initial condition of PDE as follows

$$u_0(x,t) = u(x,0)c_1(t) + \frac{\partial}{\partial x}u(x,0)c_2(t),$$

where $u(x,0)$ is the initial condition of the PDE. The Initial guess $u_0(x,t)$ has the form of sepration of variables, as an initial condition for HAM. By using this method, the other of the PDE to be solved is reduced into an ODE or system of ODEs. The organization of this paper is as follows: in section 2, we present the basic idea of HAM and the construct of homo sepration analysis method (HSAM). In Section 3, four examples are solved to illustrated the applicability of the considered method. Finally, relevant conclusions are drawn in section 4.

2. Homo sepration analysis method

The homotopy analysis method based on the concept of the homotopy, a fundamental concept in topology and differential geometry. In this section, the algorithm of this method is briefly illustrated. To achieve our gool, we consider the nonlinear partial differential equation

$$u_t = F(x,t,u,u_x,u_{xx},u_{xt}), \qquad t \geq 0, \tag{2.1}$$

subject to the initial condition

$$u(x,0) = f(x). \tag{2.2}$$

The so-called zero-order deformation equation of the equation (2.1) can be constructed as follows

$$\begin{aligned}(1-q)L[\phi(x,t,q) - \; u_0(x,t)] \;\; &= \;\; qh[\frac{\partial}{\partial t}\phi(x,t,q) - F(x,t,\phi(x,t,q), \\ &\quad \frac{\partial}{\partial x}\phi(x,t,q), \frac{\partial^2}{\partial x^2}\phi(x,t,q), \frac{\partial^2}{\partial x \partial t}\phi(x,t,q))],\end{aligned} \tag{2.3}$$

where $q \in [0,1]$ is an embedding parameter, L is an auxiliary linear operator, $h \neq 0$ is an auxiliary parameter, $\phi(x,t,q)$ is unknown function and $u_0(x,t)$ is an initial guess of $u(x,t)$ which satisfy the initial condition. Obviously, when $q = 0$

$$\phi(x,t,0) = u_0(x,t), \tag{2.4}$$

and when $q = 1$, we have

$$\phi(x,t,1) = u(x,t). \tag{2.5}$$

Expanding $\phi(x,t,q)$ in Taylor series with respect to q, we get

$$\phi(x,t,q) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t)q^m, \tag{2.6}$$

where

$$u_m(x,t) = \frac{1}{m!}\frac{\partial^m \phi(x,t,q)}{\partial q^m}|_{q=0}. \tag{2.7}$$

If the initial guess $u_0(x,t)$, the auxiliary linear operator L and the nonzero auxiliary parameter h are properly chosen so that the power series (2.6) converges at $q = 1$, one has

$$u(x,t) = \phi(x,t,1) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t). \tag{2.8}$$

Define the vector

$$\overrightarrow{u}_m(x,t) = \{u_0(x,t), u_1(x,t), \ldots, u_m(x,t)\}.$$

Differentiating the zero-order deformation equation (2.3) m times with respective to q, then setting $q = 0$ and dividing them by $m!$, finally using (2.7), we have the so-called high-order deformation equations

$$L[u_m(x,t) - \chi_m \; u_{m-1}(x,t)] = h \; \Re_m(\overrightarrow{u}_{m-1}(x,t)), \tag{2.9}$$

subject to the initial conditions

$$u_m(x,0) = 0,$$

where

$$\begin{aligned} \Re_m(\overrightarrow{u}_{m-1}(x,t)) \quad &= \quad \frac{\partial}{\partial t}u_{m-1}(x,t) - \frac{1}{(m-1)!}\frac{\partial^{m-1}}{\partial q^{m-1}}F(x,t,\phi(x,t,q), \\ & \qquad \frac{\partial}{\partial x}\phi(x,t,q), \frac{\partial^2}{\partial x^2}\phi(x,t,q), \frac{\partial^2}{\partial x \partial t}\phi(x,t,q))|_{q=0}, \end{aligned} \tag{2.10}$$

and

$$\chi_m = \begin{cases} 0, & m \le 1 \\ 1, & m > 1 \end{cases}.$$

Select the auxiliary linear operator $L = \frac{\partial}{\partial t}$, and $h = -1$, then the mth-order deformation equation (2.9) can be written in the form

$$\frac{\partial}{\partial t}(u_m(x,t) - \chi_m \; u_{m-1}(x,t)) + \Re_m(\overrightarrow{u}_{m-1}(x,t)) = 0. \tag{2.11}$$

For special case take $m = 1$ in equation (2.11) and using the relations (2.4),(2.10), then we have

$$\frac{\partial}{\partial t}u_1(x,t) = \frac{\partial}{\partial t}u_0(x,t) - F(x,t,u_0(x,t),$$

$$\frac{\partial}{\partial x}u_0(x,t), \frac{\partial^2}{\partial x^2}u_0(x,t), \frac{\partial^2}{\partial x \partial t}u_0(x,t)) \equiv 0, \tag{2.12}$$

By utilizing the results of equation (2.8), we approximate the analytical solution $u(x,t)$, by the truncated series:

$$u(x,t) = u_0 + u_1 + ... + u_{n-1} = \sum_{i=0}^{n-1} u_i, \tag{2.13}$$

For simplicity, we assume that $u_n(x,t) = 0$, when $n > 1$, which means that the exact solution in equation (2.13) is

$$u(x,t) = u_0(x,t).$$

To illustrate our basic idea, we consider the initial approximation of equation (2.1) as follows

$$\begin{aligned} u(x,t) &= u_0(x,t) = u(x,0)c_1(t) + \frac{\partial}{\partial x}u(x,0)c_2(t), \\ . &= f(x)c_1(t) + f'(x)c_2(t). \end{aligned} \tag{2.14}$$

Our gool in this method is finding $c_1(t)$ and $c_2(t)$. Since equation (2.14) satisfies the initial condition, we get

$$c_1(0) = 1, \ c_2(0) = 0. \tag{2.15}$$

By substituting equation (2.14) into equation (2.12), we obtain

$$\begin{aligned} \frac{\partial}{\partial t}u_1(x,t) &= f(x)c_1'(t) + f'(x)c_2'(t) - F(x,t,f(x)c_1(t) + f'(x)c_2(t), \\ & f'(x)c_1(t) + f''(x)c_2(t), f''(x)c_1(t) + f'''(x)c_2'(t), \end{aligned} \tag{2.16}$$

$$f'(x)c_1'(t) + f''(x)c_2'(t)) \equiv 0.$$

The partial differential equation (2.16) transform into an ordinary differential equation or a system of ordinary differential equations. The exact solution of the partial differential equation is found when the target unknowns $c_1(t)$ and $c_2(t)$ are computed, by utlizing (2.16) and the initial conditions (2.15).

3. Numerical results

In this work, we carefully propose the HSAM, a reliable modification of the HAM, that gives the exact solution of the linear and non linear partial differential equation. To demonstrate the effectiveness of the method, we consider here the following four examples.

Example 3.1. Consider the following nonhomogenous partial differential equation

$$u_t = -x^2 e^t u_{xx} + (x+2)u_x + tx, \tag{3.1}$$

with the initial condition

$$u(x,0) = x + 2.$$

Using the relation (2.14), then we get the initial approximation

$$u(x,t) = u_0(x,t) = (x+2)c_1(t) + c_2(t),$$

and by using the relation (2.16), then we have

$$\frac{\partial}{\partial t}u_1(x,t) = x(c_1'(t) - c_1(t) - t) + (c_2'(t) + 2c_1'(t) - 2c_1(t)) \equiv 0.$$

We obtain the system of ordinary differential equations

$$c_1'(t) - c_1(t) - t = 0,\ c_1(0) = 1, \tag{3.2}$$

$$c_2'(t) + 2c_1'(t) - 2c_1(t) = 0,\ c_2(0) = 0. \tag{3.3}$$

Solving the equations (3.2) and (3.3), by using the ODEs properties, we obtain

$$c_1(t) = 2e^t - t - 1, \quad c_2(t) = -t^2,$$

and the exact solution is

$$u(x,t) = (x+2)(2e^t - t - 1) - t^2.$$

Example 3.2. Consider the following Black-scholes equation

$$u_t = u_{xx} + (k-1)u_x - ku, \tag{3.4}$$

with the initial condition

$$u(x,0) = e^x - 1. \tag{3.5}$$

The initial approximation has the form

$$u(x,t) = u_0(x,t) = (e^x - 1)c_1(t) + e^x c_2(t),$$

and by using the relation (2.16), then we have

$$\frac{\partial}{\partial t}u_1(x,t) = e^x(c_1'(t) + c_2'(t)) - (c_1'(t) + kc_1(t)) \equiv 0.$$

We obtain the system of ordinary differential equations

$$c_1'(t) + kc_1(t) = 0, \ c_1(0) = 1, \tag{3.6}$$

$$c_1'(t) + c_2'(t) = 0, \ c_2(0) = 0. \tag{3.7}$$

Solving the equations (3.6) and (3.7), by using the ODEs properties, we obtain

$$c_1(t) = e^{-kt}, \quad c_2(t) = 1 - e^{-kt},$$

and the exact solution is

$$u(x,t) = (e^x - 1)e^{-kt} + e^x(1 - e^{-kt}).$$

Example 3.3. Consider the following nonlinear partial differential equation

$$u_t = u_{xx} + u_x(u + u_{xx}), \tag{3.8}$$

subject to the initial condition

$$u(x,0) = \sin x. \tag{3.9}$$

The initial approximation has the form

$$u(x,t) = u_0(x,t) = \sin x \ c_1(t) + \cos x \ c_2(t),$$

then

$$\frac{\partial}{\partial t}u_1(x,t) = \sin x(c_1'(t) + c_1(t)) + \cos x(c_2'(t) + c_2(t)) \equiv 0.$$

We obtain the system of ordinary differential equations

$$c_1'(t) + c_1(t) = 0, \ c_1(0) = 1, \tag{3.10}$$

$$c_2'(t) + c_2(t) = 0, \ c_2(0) = 0. \tag{3.11}$$

Solving the equations (3.10) and (3.11), by using the ODEs properties, we obtain

$$c_1(t) = e^{-t}, \quad c_2(t) = 0,$$

and the exact solution is

$$u(x,t) = e^{-t}\sin x.$$

Example 3.4. Consider the following nonlinear partial differential equation

$$u_t = u^2 - 4u\ u_x + 2u_{xt} - \frac{1}{8}u\ , \tag{3.12}$$

subject to the initial condition

$$u(x,0) = e^{\frac{1}{4}x}. \tag{3.13}$$

Choose the initial approximation

$$\begin{aligned} u(x,t) &= u_0(x,t) = e^{\frac{1}{4}x}\ c_1(t) + \frac{1}{4}e^{\frac{1}{4}x}\ c_2(t) \\ &= \frac{1}{4}e^{\frac{1}{4}x}(4c_1(t) + c_2(t)), \end{aligned}$$

then

$$\frac{\partial}{\partial t}u_1(x,t) = \frac{1}{32}e^{\frac{1}{4}x}\ [4(4c_1'(t) + c_2'(t)) + (4c_1(t) + c_2(t))] \equiv 0.$$

We obtain the ordinary differential equation

$$4y'(t) + y(t) = 0,\ y(0) = 4, \tag{3.14}$$

where

$$y(t) = 4c_1(t) + c_2(t)$$

Solving the equations (3.14), by using the ODEs properties, we obtain

$$4c_1(t) + c_2(t) = 4e^{-\frac{1}{4}t},$$

and the exact solution is

$$u(x,t) = e^{\frac{1}{4}(x-t)}.$$

4. Conclusions

The fundamental goal of this work is to propose a simple method for the solution of PDEs. A combined form of the HAM with sepration of variables is effectively used to handle linear and nonlinear partial differential equations. The main advantage of the method is its fast and gives exact solution for our problem. In this research work, it was demonstrated through different examples how this new method can be used for solving linear and nonlinear PDE. When compared with the existing published methods, it is easy to notice that the new method has many advantages. It is straightforward, easy to understand and requiring much less computations to perform a limited number of steps of the simple procedure that can be applied to find the exact solution of a wide range of types of PDEs Finally, the recent appearance of nonlinear partial differential equations as models in some fields such as models in science and engineering makes it is necessary to investigate the method of solutions for such equations. and we hope that this work is a step in this direction.

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