SOME NEW k-FRACTIONAL INTEGRAL INEQUALITIES CONTAINING MULTIPLE PARAMETERS VIA GENERALIZED (s, m)-PREINVEXITY

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Abstract. We establish some new k-fractional integral inequalities for differentiable functions based on generalized (s, m)-preinvexity. We also prove Hadamard-type inequalities involving products of two generalized (s, m)-preinvex functions. These inequalities include some previously known results as special cases.

Keywords: Hadamard-type inequalities; generalized (s, m)-preinvex functions; k-fractional integrals.

1. Introduction

The following double inequality is notable in the literature as the Hermite-Hadamard inequality.

Theorem 1.1. *Suppose that $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is a convex function defined on the interval I of real numbers and $a, b \in I$ along with $a < b$. The following double*

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inequality holds:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

A large number of generalizations and refinements on the inequality (1.1) have been presented, for example, see [7, 8, 10, 13, 16, 17, 19, 20, 21, 27, 28] and the references therein.

In 2013, Sarikaya et al. established the following Hadamard-type inequalities by utilizing Riemann-Liouville fractional integrals.

Theorem 1.2 ([30]). *Let $f : [a, b] \to \mathbb{R}$ be a positive function along with $0 \leq a < b$ and let $f \in L^1[a, b]$. Suppose that f is a convex function on $[a, b]$, then the following inequalities for fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\mu+1)}{2(b-a)^{\mu}}[J^{\mu}_{a^+}f(b) + J^{\mu}_{b^-}f(a)] \leq \frac{f(a)+f(b)}{2}, \tag{1.2}$$

where the symbols $J^{\mu}_{a^+}f$ and $J^{\mu}_{b^-}f$ denote respectively the left-sided and right-sided Riemann-Liouville fractional integrals of order $\mu > 0$ defined by

$$J^{\mu}_{a^+}f(x) = \frac{1}{\Gamma(\mu)}\int_a^x (x-t)^{\mu-1}f(t)\mathrm{d}t, \quad a < x$$

and

$$J^{\mu}_{b^-}f(x) = \frac{1}{\Gamma(\mu)}\int_x^b (t-x)^{\mu-1}f(t)\mathrm{d}t, \quad x < b.$$

Here, $\Gamma(\mu)$ is the gamma function and its definition is $\Gamma(\mu) = \int_0^{\infty} e^{-t}t^{\mu-1}\mathrm{d}t$. It is to be noted that $J^{0}_{a^+}f(x) = J^{0}_{b^-}f(x) = f(x)$.

In the case of $\mu = 1$, the fractional integral reduces to the classical integral.

In 2016, Sarikaya and Yildirim presented another form with respect to Riemann-Liouville fractional Hadamard-type inequalities as follows.

Theorem 1.3 ([31]). *Let $f : [a, b] \to \mathbb{R}$ be a positive function with $0 \leq a < b$ and $f \in L^1[a, b]$. If f is a convex function on $[a, b]$, then the following inequalities for fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{2^{\mu-1}\Gamma(\mu+1)}{(b-a)^{\mu}}\left[J^{\mu}_{(\frac{a+b}{2})^+}f(b) + J^{\mu}_{(\frac{a+b}{2})^-}f(a)\right] \leq \frac{f(a)+f(b)}{2} \tag{1.3}$$

with $\mu > 0$.

Due to the extensive application of Riemann-Liouville fractional integrals, there have been many studies involving this integral operator, for example, see [14, 15, 22, 26, 33] and the references therein.

In 2012, Mubeen and Habibullah presented the following k-fractional integrals.

Definition 1.1 ([24]). *Let $f \in L^1[a,b]$, then Riemann-Liouville k-fractional integrals ${}_kJ^{\mu}_{a^+}f(x)$ and ${}_kJ^{\mu}_{b^-}f(x)$ of order $\mu > 0$ are given as*

$$ {}_kJ^{\mu}_{a^+}f(x) = \frac{1}{k\Gamma_k(\mu)}\int_a^x (x-t)^{\frac{\mu}{k}-1}f(t)\mathrm{d}t, \quad (0 \le a < x < b) $$

and

$$ {}_kJ^{\mu}_{b^-}f(x) = \frac{1}{k\Gamma_k(\mu)}\int_x^b (t-x)^{\frac{\mu}{k}-1}f(t)\mathrm{d}t, \quad (0 \le a < x < b), $$

respectively, where $k > 0$ and $\Gamma_k(\mu)$ is the k-gamma function defined by $\Gamma_k(\mu) = \int_0^\infty t^{\mu-1}e^{-\frac{t^k}{k}}\mathrm{d}t$. Furthermore, $\Gamma_k(\mu+k) = \mu\Gamma_k(\mu)$ and ${}_kJ^0_{a^+}f(x) = {}_kJ^0_{b^-}f(x) = f(x)$.

In the case of $k = 1$, the k-fractional integrals reduces to Riemann-Liouville fractional integrals. For some recent results related to the k-fractional integral inequalities see [1, 2, 5, 29, 32].

In 2016, Farid et al. popularized Theorem 1.3 to the form of k-fractional integrals.

Theorem 1.4 ([11]). *Let $f : [a,b] \to \mathbb{R}$ be a positive function with $0 \le a < b$ and $f \in L^1[a,b]$. If f is a convex function on $[a,b]$, then the following inequalities for k-fractional integrals hold:*

$$ f\Big(\frac{a+b}{2}\Big) \le \frac{2^{\frac{\mu}{k}-1}\Gamma_k(\mu+k)}{(b-a)^{\frac{\mu}{k}}}\Big[{}_kJ^{\mu}_{(\frac{a+b}{2})^+}f(b) + {}_kJ^{\mu}_{(\frac{a+b}{2})^-}f(a)\Big] \le \frac{f(a)+f(b)}{2} \tag{1.4} $$

with $\mu, k > 0$.

The main aim of this article is to establish some new k-fractional integral inequalities related to generalized (s,m)-preinvex functions. The obtained k-fractional integral inequalities can be viewed as the extension of the results of [6, 11, 18, 23] and [25].

To end this section, let us recall some special functions and basic definitions as follows.

(1) The beta function:

$$ \beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^1 t^{x-1}(1-t)^{y-1}\mathrm{d}t, \quad x, y > 0, $$

(2) The hypergeometric function:

$$ {}_2F_1(a,b;c;z) = \frac{1}{\beta(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\mathrm{d}t, \quad c > b > 0, |z| < 1. $$

Definition 1.2 ([12])**.** *A function $f : [0,\infty) \to \mathbb{R}$ is named s-convex in the second sense with $s \in (0,1]$, if*

$$f(\alpha x + \beta y) \le \alpha^s f(x) + \beta^s f(y)$$

holds for all $x, y \in [0,\infty)$ and $\alpha, \beta \ge 0$ along with $\alpha + \beta = 1$.

Definition 1.3 ([9])**.** *A set $K \subseteq \mathbb{R}^n$ is named m-invex with respect to the mapping $\eta : K \times K \times (0,1] \to \mathbb{R}^n$ for some fixed $m \in (0,1]$, if $mx + \lambda\eta(y,x,m) \in K$ holds for all $x, y \in K$ and $\lambda \in [0,1]$.*

Definition 1.4 ([9])**.** *Let $K \subseteq \mathbb{R}^n$ be an open m-invex subset with respect to $\eta : K \times K \times (0,1] \to \mathbb{R}^n$. For some fixed $s, m \in (0,1]$, f is said to be generalized (s,m)-preinvex, if*

$$f\big(mx + t\eta(y,x,m)\big) \le m(1-t)^s f(x) + t^s f(y)$$

is valid for all $x, y \in K$ and $t \in [0,1]$.

Definition 1.5 ([3])**.** *Let $K \subseteq \mathbb{R}^n$ be an invex set with respect to the mapping $\eta : K \times K \to \mathbb{R}^n$. For every $x, y \in K$, the η-path P_{xv} joining the points x and $v = x + \eta(y,x)$ is defined by*

$$P_{xv} = \big\{z | z = x + t\eta(y,x), t \in [0,1]\big\}.$$

Definition 1.6. *Let $K \subseteq \mathbb{R}^n$ be an m-invex set with respect to $\eta : K \times K \times (0,1] \to \mathbb{R}^n$. For every $u, v \in K$ and $m \in (0,1]$, the η_m-path P_{vw} joining the points mv and $w = mv + \eta(u,v,m)$ is defined by*

$$P_{vw} = \big\{z | z = mv + \lambda\eta(u,v,m), \lambda \in [0,1]\big\}.$$

Remark 1.1. If $\eta(u,v,m)$ with $m = 1$ reduces to $\eta(u,v)$, then Definition 1.6 reduces to Definition 1.5.

2. k-Fractional inequalities involving differentiable functions

Throughout this section, let $\mathbb{R}$ be the set of all real numbers, $\mathbb{N}^*$ be the set of all positive integers and let $K \subseteq \mathbb{R}$ be an open m-invex subset with respect to $\eta : K \times K \times (0,1] \to \mathbb{R} \setminus \{0\}$ for some fixed $m \in (0,1]$, $a, b \in K$ with $a < b$. Assume that $f : K \to \mathbb{R}$ is a differentiable function such that f' is integrable on the η_m-path $P_{vw} : w = mv + \eta(u,v,m)$ for arbitrary $u, v \in [a,b]$. Before stating the results we define the following notations:

$$\begin{aligned} &\mathcal{H}_{\eta_m}(\mu,k;n,x) \\ &:= \frac{n+1}{2}\Bigg[\frac{\eta^{\frac{\mu}{k}}(x,a,m) f\big(ma + \eta(x,a,m)\big) + \eta^{\frac{\mu}{k}}(b,x,m) f(mx)}{\eta(b,a,m)} \end{aligned} \tag{2.1}$$

$$+\frac{\eta^{\frac{\mu}{k}}(x,a,m)f(ma)+\eta^{\frac{\mu}{k}}(b,x,m)f\big(mx+\eta(b,x,m)\big)}{\eta(b,a,m)}\Bigg]$$
$$-\frac{(n+1)^{\frac{\mu}{k}+1}\Gamma_k(\mu+k)}{2\eta(b,a,m)}$$
$$\times\Bigg[{}_kJ^{\mu}_{(ma)^+}f\bigg(ma+\frac{1}{n+1}\eta\big(x,a,m\big)\bigg)$$
$$+{}_kJ^{\mu}_{(ma+\eta(x,a,m))^-}f\bigg(ma+\frac{n}{n+1}\eta\big(x,a,m\big)\bigg)$$
$$+{}_kJ^{\mu}_{(mx)^+}f\bigg(mx+\frac{1}{n+1}\eta\big(b,x,m\big)\bigg)$$
$$+{}_kJ^{\mu}_{(mx+\eta(b,x,m))^-}f\bigg(mx+\frac{n}{n+1}\eta\big(b,x,m\big)\bigg)\Bigg].$$

Especially if $\eta(u,v,m)=u-mv$ with $m=1$ for $u,v\in[a,b]$, equation (2.1) reduces to

$$\mathcal{H}(\mu,k;n,x)$$
$$:=\frac{n+1}{2}\Bigg[\frac{(x-a)^{\frac{\mu}{k}}+(b-x)^{\frac{\mu}{k}}}{b-a}f(x)+\frac{(x-a)^{\frac{\mu}{k}}f(a)+(b-x)^{\frac{\mu}{k}}f(b)}{b-a}\Bigg]$$
$$-\frac{(n+1)^{\frac{\mu}{k}+1}\Gamma_k(\mu+k)}{2(b-a)}\Bigg[{}_kJ^{\mu}_{a^+}f\bigg(\frac{n}{n+1}a+\frac{1}{n+1}x\bigg)+{}_kJ^{\mu}_{x^-}f\bigg(\frac{1}{n+1}a+\frac{n}{n+1}x\bigg)$$
$$+{}_kJ^{\mu}_{x^+}f\bigg(\frac{n}{n+1}x+\frac{1}{n+1}b\bigg)+{}_kJ^{\mu}_{b^-}f\bigg(\frac{1}{n+1}x+\frac{n}{n+1}b\bigg)\Bigg].$$

We need the succeeding lemma.

Lemma 2.1. *The following k-fractional integral identity along with $x\in(a,b)$, $n\in\mathbb{N}^*$, $\mu>0$ and $k>0$ holds:*

$$\mathcal{H}_{\eta_m}(\mu,k;n,x)$$
$$=\frac{\eta^{\frac{\mu}{k}+1}(x,a,m)}{2\eta(b,a,m)}\Bigg\{\int_0^1 t^{\frac{\mu}{k}}f'\bigg(ma+\frac{n+t}{n+1}\eta\big(x,a,m\big)\bigg)\mathrm{d}t$$
$$-\int_0^1 t^{\frac{\mu}{k}}f'\bigg(ma+\frac{1-t}{n+1}\eta\big(x,a,m\big)\bigg)\mathrm{d}t\Bigg\}\tag{2.2}$$
$$-\frac{\eta^{\frac{\mu}{k}+1}(b,x,m)}{2\eta(b,a,m)}\Bigg\{\int_0^1 t^{\frac{\mu}{k}}f'\bigg(mx+\frac{1-t}{n+1}\eta\big(b,x,m\big)\bigg)\mathrm{d}t$$
$$-\int_0^1 t^{\frac{\mu}{k}}f'\bigg(mx+\frac{n+t}{n+1}\eta\big(b,x,m\big)\bigg)\mathrm{d}t\Bigg\}.$$

Proof. By integration by parts and changing the variable, we can state

$$\begin{aligned}
&\int_0^1 t^{\frac{\mu}{k}} f'\left(ma + \frac{n+t}{n+1}\eta(x,a,m)\right)\mathrm{d}t \\
&= \frac{(n+1)t^{\frac{\mu}{k}} f\big(ma+\frac{n+t}{n+1}\eta(x,a,m)\big)}{\eta(x,a,m)}\Bigg|_0^1 - \int_0^1 \frac{\mu(n+1)t^{\frac{\mu}{k}-1} f\big(ma+\frac{n+t}{n+1}\eta(x,a,m)\big)}{k\eta(x,a,m)}\mathrm{d}t \\
&= \frac{(n+1)f\big(ma+\eta(x,a,m)\big)}{\eta(x,a,m)} \\
&\quad - \frac{\mu(n+1)^{\frac{\mu}{k}+1}}{k\eta^{\frac{\mu}{k}+1}(x,a,m)} \int_{ma+\frac{n}{n+1}\eta(x,a,m)}^{ma+\eta(x,a,m)} \left[u - \left(ma + \frac{n}{n+1}\eta(x,a,m)\right)\right]^{\frac{\mu}{k}-1} f(u)\mathrm{d}u \\
&= \frac{(n+1)f\big(ma+\eta(x,a,m)\big)}{\eta(x,a,m)} \\
&\quad - \frac{(n+1)^{\frac{\mu}{k}+1}\Gamma_k(\mu+k)}{\eta^{\frac{\mu}{k}+1}(x,a,m)} {}_kJ^{\mu}_{(ma+\eta(x,a,m))^-} f\left(ma + \frac{n}{n+1}\eta(x,a,m)\right).
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
&\int_0^1 t^{\frac{\mu}{k}} f'\left(ma + \frac{1-t}{n+1}\eta(x,a,m)\right)\mathrm{d}t \\
&= -\frac{(n+1)f(ma)}{\eta(x,a,m)} + \frac{(n+1)^{\frac{\mu}{k}+1}\Gamma_k(\mu+k)}{\eta^{\frac{\mu}{k}+1}(x,a,m)} {}_kJ^{\mu}_{(ma)^+} f\left(ma + \frac{1}{n+1}\eta(x,a,m)\right),
\end{aligned}$$

$$\begin{aligned}
&\int_0^1 t^{\frac{\mu}{k}} f'\left(mx + \frac{1-t}{n+1}\eta(b,x,m)\right)\mathrm{d}t \\
&= -\frac{(n+1)f(mx)}{\eta(b,x,m)} + \frac{(n+1)^{\frac{\mu}{k}+1}\Gamma_k(\mu+k)}{\eta^{\frac{\mu}{k}+1}(b,x,m)} {}_kJ^{\mu}_{(mx)^+} f\left(mx + \frac{1}{n+1}\eta(b,x,m)\right)
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1 t^{\frac{\mu}{k}} f'\left(mx + \frac{n+t}{n+1}\eta(b,x,m)\right)\mathrm{d}t \\
&= \frac{(n+1)f\big(mx+\eta(b,x,m)\big)}{\eta(b,x,m)} \\
&\quad - \frac{(n+1)^{\frac{\mu}{k}+1}\Gamma_k(\mu+k)}{\eta^{\frac{\mu}{k}+1}(b,x,m)} {}_kJ^{\mu}_{(mx+\eta(b,x,m))^-} f\left(mx + \frac{n}{n+1}\eta(b,x,m)\right).
\end{aligned}$$

After suitable rearrangements we obtain the desired result. This ends the proof.

Corollary 2.1. *In Lemma 2.1, if* $\eta(u,v,m)=u-mv$ *with* $m=1$ *for* $u,v\in[a,b]$*, we have*

$$\begin{aligned}
&\mathcal{H}(\mu,k;n,x)\\
&=\frac{(x-a)^{\frac{\mu}{k}+1}}{2(b-a)}\bigg\{\int_0^1 t^{\frac{\mu}{k}}f'\Big(\frac{n+t}{n+1}x+\frac{1-t}{n+1}a\Big)\mathrm{d}t\\
&-\int_0^1 t^{\frac{\mu}{k}}f'\Big(\frac{1-t}{n+1}x+\frac{n+t}{n+1}a\Big)\mathrm{d}t\bigg\}\\
&-\frac{(b-x)^{\frac{\mu}{k}+1}}{2(b-a)}\bigg\{\int_0^1 t^{\frac{\mu}{k}}f'\Big(\frac{n+t}{n+1}x+\frac{1-t}{n+1}b\Big)\mathrm{d}t\\
&-\int_0^1 t^{\frac{\mu}{k}}f'\Big(\frac{1-t}{n+1}x+\frac{n+t}{n+1}b\Big)\mathrm{d}t\bigg\}.
\end{aligned} \tag{2.3}$$

Remark 2.1. (i) In Lemma 2.1, if $\eta(u,v,m)$ with $m=1$ reduces to $\eta(u,v)$ for $u,v\in[a,b]$, putting $k=1$ along with $n=1$, we have Lemma 2.8 in [25].

(ii) In Corollary 2.1,

(a) putting $k=1$, we have Lemma 2.5 in [4],

(b) putting $k=1=n$, we have Lemma 1 in [23]. Further, putting $\mu=1$, we have Lemma 1 in [18].

Utilizing Lemma 2.1, the following theorem can be obtained.

Theorem 2.1. *Suppose that* $|f'|^q$ *for* $q\ge 1$ *is generalized* (s,m)*-preinvex, then for* $x\in(a,b)$*,* $n\in\mathbb{N}^*$*,* $\mu>0$ *and* $k>0$*, the following* k*-fractional integral inequality holds:*

$$\begin{aligned}
&\big|\mathcal{H}_{\eta_m}(\mu,k;n,x)\big|\le\bigg(\frac{1}{\frac{\mu}{k}+1}\bigg)^{1-\frac{1}{q}}\bigg\{\frac{|\eta^{\frac{\mu}{k}+1}(x,a,m)|}{2|\eta(b,a,m)|}\bigg[\Big(m\Phi_1(\mu,k,n,s)\big|f'(a)\big|^q\\
&+\Phi_2(\mu,k,n,s)\big|f'(x)\big|^q\Big)^{\frac{1}{q}}\\
&+\Big(m\Phi_2(\mu,k,n,s)\big|f'(a)\big|^q+\Phi_1(\mu,k,n,s)\big|f'(x)\big|^q\Big)^{\frac{1}{q}}\bigg]\\
&+\frac{|\eta^{\frac{\mu}{k}+1}(b,x,m)|}{2|\eta(b,a,m)|}\bigg[\Big(m\Phi_2(\mu,k,n,s)\big|f'(x)\big|^q+\Phi_1(\mu,k,n,s)\big|f'(b)\big|^q\Big)^{\frac{1}{q}}\\
&+\Big(m\Phi_1(\mu,k,n,s)\big|f'(x)\big|^q+\Phi_2(\mu,k,n,s)\big|f'(b)\big|^q\Big)^{\frac{1}{q}}\bigg]\bigg\},
\end{aligned} \tag{2.4}$$

where

$$\Phi_1(\mu,k,n,s)=\int_0^1 t^{\frac{\mu}{k}}\left(\frac{1-t}{n+1}\right)^s\mathrm{d}t=\frac{\beta(\frac{\mu}{k}+1,s+1)}{(n+1)^s}$$

and

$$\Phi_2(\mu,k,n,s)=\int_0^1 t^{\frac{\mu}{k}}\left(\frac{n+t}{n+1}\right)^s \mathrm{d}t=\begin{cases}\dfrac{{}_2F_1[-s,1;\frac{\mu}{k}+2;\frac{1}{2}]}{\frac{\mu}{k}+1}, & n=1,\\ \dfrac{n^s{}_2F_1[-s,\frac{\mu}{k}+1;\frac{\mu}{k}+2;-\frac{1}{n}]}{(\frac{\mu}{k}+1)(n+1)^s}, & n>1.\end{cases}$$

Proof. Using Lemma 2.1, the power-mean inequality and the generalized (s,m)-preinvexity of $|f'|^q$, we get

$$\begin{aligned}\left|\mathcal{H}_{\eta_m}(\mu,k;n,x)\right| &\le \frac{\left|\eta^{\frac{\mu}{k}+1}(x,a,m)\right|}{2|\eta(b,a,m)|}\left(\int_0^1 t^{\frac{\mu}{k}}\mathrm{d}t\right)^{1-\frac{1}{q}}\left[(I_1)^{\frac{1}{q}}+(I_2)^{\frac{1}{q}}\right]\\ &+\frac{\left|\eta^{\frac{\mu}{k}+1}(b,x,m)\right|}{2|\eta(b,a,m)|}\left(\int_0^1 t^{\frac{\mu}{k}}\mathrm{d}t\right)^{1-\frac{1}{q}}\left[(I_3)^{\frac{1}{q}}+(I_4)^{\frac{1}{q}}\right],\end{aligned}$$

where

$$\begin{aligned}I_1&=\int_0^1 t^{\frac{\mu}{k}}\left|f'\left(ma+\frac{n+t}{n+1}\eta(x,a,m)\right)\right|^q\mathrm{d}t\\ &\le\int_0^1 t^{\frac{\mu}{k}}\left[m\left(\frac{1-t}{n+1}\right)^s|f'(a)|^q+\left(\frac{n+t}{n+1}\right)^s|f'(x)|^q\right]\mathrm{d}t,\\ I_2&=\int_0^1 t^{\frac{\mu}{k}}\left|f'\left(ma+\frac{1-t}{n+1}\eta(x,a,m)\right)\right|^q\mathrm{d}t\\ &\le\int_0^1 t^{\frac{\mu}{k}}\left[m\left(\frac{n+t}{n+1}\right)^s|f'(a)|^q+\left(\frac{1-t}{n+1}\right)^s|f'(x)|^q\right]\mathrm{d}t,\\ I_3&=\int_0^1 t^{\frac{\mu}{k}}\left|f'\left(mx+\frac{1-t}{n+1}\eta(b,x,m)\right)\right|^q\mathrm{d}t\\ &\le\int_0^1 t^{\frac{\mu}{k}}\left[m\left(\frac{n+t}{n+1}\right)^s|f'(x)|^q+\left(\frac{1-t}{n+1}\right)^s|f'(b)|^q\right]\mathrm{d}t\end{aligned}$$

and

$$\begin{aligned}I_4&=\int_0^1 t^{\frac{\mu}{k}}\left|f'\left(mx+\frac{n+t}{n+1}\eta(b,x,m)\right)\right|^q\mathrm{d}t\\ &\le\int_0^1 t^{\frac{\mu}{k}}\left[m\left(\frac{1-t}{n+1}\right)^s|f'(x)|^q+\left(\frac{n+t}{n+1}\right)^s|f'(b)|^q\right]\mathrm{d}t.\end{aligned}$$

Hence the proof is completed.

Corollary 2.2. *In Theorem 2.1,*

(*i*) *if we put $q=1$, we have:*

$$\begin{aligned}&\left|\mathcal{H}_{\eta_m}(\mu,k;n,x)\right|\\ &\le\frac{\Phi_1(\mu,k,n,s)+\Phi_2(\mu,k,n,s)}{2|\eta(b,a,m)|}\left[\left|\eta^{\frac{\mu}{k}+1}(x,a,m)\right|\left(m\left|f'(a)\right|+\left|f'(x)\right|\right)\right.\\ &\left.+\left|\eta^{\frac{\mu}{k}+1}(b,x,m)\right|\left(m\left|f'(x)\right|+\left|f'(b)\right|\right)\right].\end{aligned}$$

Especially if $\eta(u,v,m)$ *with* $m=1$ *reduces to* $\eta(u,v)$ *for* $u,v\in[a,b]$*, and choosing* $k=1=n$*, we get Theorem* 3.1 *proved by Noor et al. in [25],*

(*ii*) *if* $\eta(u,v,m)=u-mv$ *with* $m=1$ *for* $u,v\in[a,b]$*, we have:*

$$\begin{aligned}&\left|\mathcal{H}(\mu,k;n,x)\right|\\&\le\left(\frac{1}{\frac{\mu}{k}+1}\right)^{1-\frac{1}{q}}\left\{\frac{(x-a)^{\frac{\mu}{k}+1}}{2(b-a)}\left[\left(\Phi_1(\mu,k,n,s)\left|f'(a)\right|^q+\Phi_2(\mu,k,n,s)\left|f'(x)\right|^q\right)^{\frac{1}{q}}\right.\right.\\&\left.+\left(\Phi_2(\mu,k,n,s)\left|f'(a)\right|^q+\Phi_1(\mu,k,n,s)\left|f'(x)\right|^q\right)^{\frac{1}{q}}\right]\\&+\frac{(b-x)^{\frac{\mu}{k}+1}}{2(b-a)}\left[\left(\Phi_2(\mu,k,n,s)\left|f'(x)\right|^q+\Phi_1(\mu,k,n,s)\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right.\\&\left.\left.+\left(\Phi_1(\mu,k,n,s)\left|f'(x)\right|^q+\Phi_2(\mu,k,n,s)\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right]\right\}.\end{aligned}$$

Especially if we choose $k=1=n$ *along with* $s=1$*, we get Theorem* 3 *established by Mihai and Mitroi in [23]. Further, if we take* $\mu=1$*, we obtain Theorem* 3 *presented by Latif in [18],*

(*iii*) *if* $\eta(u,v,m)=u-mv$ *with* $m=1$ *for* $u,v\in[a,b]$*, and putting* $x=\frac{a+b}{2}$*,* $n=1$*, we obtain:*

$$\begin{aligned}&\left|\left(\frac{2}{b-a}\right)^{\frac{\mu}{k}-1}\mathcal{H}\left(\mu,k;1,\frac{a+b}{2}\right)\right|\\&=\left|\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]-\frac{2^{\frac{2\mu}{k}-1}\Gamma_k(\mu+k)}{(b-a)^{\frac{\mu}{k}}}\left[{}_kJ^{\mu}_{a^+}f\left(\frac{3a+b}{4}\right)\right.\right.\\&\left.\left.+{}_kJ^{\mu}_{(\frac{a+b}{2})^-}f\left(\frac{3a+b}{4}\right)+{}_kJ^{\mu}_{(\frac{a+b}{2})^+}f\left(\frac{a+3b}{4}\right)+{}_kJ^{\mu}_{b^-}f\left(\frac{a+3b}{4}\right)\right]\right|\\&\le\frac{b-a}{8}\left(\frac{1}{\frac{\mu}{k}+1}\right)^{1-\frac{1}{q}}\left\{\left[\Phi_1(\mu,k,1,s)\left|f'(a)\right|^q+\Phi_2(\mu,k,1,s)\left|f'\left(\frac{a+b}{2}\right)\right|^q\right]^{\frac{1}{q}}\right.\\&+\left[\Phi_2(\mu,k,1,s)\left|f'(a)\right|^q+\Phi_1(\mu,k,1,s)\left|f'\left(\frac{a+b}{2}\right)\right|^q\right]^{\frac{1}{q}}\\&+\left[\Phi_2(\mu,k,1,s)\left|f'\left(\frac{a+b}{2}\right)\right|^q+\Phi_1(\mu,k,1,s)\left|f'(b)\right|^q\right]^{\frac{1}{q}}\\&\left.+\left[\Phi_1(\mu,k,1,s)\left|f'\left(\frac{a+b}{2}\right)\right|^q+\Phi_2(\mu,k,1,s)\left|f'(b)\right|^q\right]^{\frac{1}{q}}\right\}\\&\le\frac{b-a}{8}\left(\frac{1}{\frac{\mu}{k}+1}\right)^{1-\frac{1}{q}}\left(1+2^{1-\frac{s}{q}}\right)\left(\Phi_1^{\frac{1}{q}}(\mu,k,1,s)+\Phi_2^{\frac{1}{q}}(\mu,k,1,s)\right)\left[\left|f'(a)\right|+\left|f'(b)\right|\right].\end{aligned}$$

The second inequality is obtained by utilizing the s-convexity of $|f'|^q$ and the fact that

$$\sum_{i=1}^{n}(u_i+v_i)^\theta \le \sum_{i=1}^{n}(u_i)^\theta+\sum_{i=1}^{n}(v_i)^\theta,\ u_i,v_i\ge 0,\ 1\le i\le n,\ 0\le\theta\le 1.$$

If $|f'|^q$ for $q>1$ is also generalized (s,m)-preinvex, we obtain the following result.

Theorem 2.2. *Assume that $|f'|^q$ for $q>1$ is generalized (s,m)-preinvex with $\frac{1}{p}+\frac{1}{q}=1$, then for $x\in(a,b)$, $n\in\mathbb{N}^*$, $\mu>0$ and $k>0$, the following k-fractional integral inequality holds:*

$$\begin{aligned}
&\big|\mathcal{H}_{\eta_m}(\mu,k;n,x)\big|\\
&\le\left(\frac{1}{\frac{\mu p}{k}+1}\right)^{\frac{1}{p}}\left\{\frac{\big|\eta^{\frac{\mu}{k}+1}(x,a,m)\big|}{2|\eta(b,a,m)|}\left[\Big(m\Psi_1\big|f'(a)\big|^q+\Psi_2\big|f'(x)\big|^q\Big)^{\frac{1}{q}}\right.\right.\\
&\quad\left.+\Big(m\Psi_2\big|f'(a)\big|^q+\Psi_1\big|f'(x)\big|^q\Big)^{\frac{1}{q}}\right]\\
&\quad+\frac{\big|\eta^{\frac{\mu}{k}+1}(b,x,m)\big|}{2|\eta(b,a,m)|}\left[\Big(m\Psi_2\big|f'(x)\big|^q+\Psi_1\big|f'(b)\big|^q\Big)^{\frac{1}{q}}\right.\\
&\quad\left.\left.+\Big(m\Psi_1\big|f'(x)\big|^q+\Psi_2\big|f'(b)\big|^q\Big)^{\frac{1}{q}}\right]\right\},
\end{aligned}\tag{2.5}$$

where

$$\Psi_1=\int_0^1\left(\frac{1-t}{n+1}\right)^s\mathrm{d}t=\frac{1}{(s+1)(n+1)^s},$$
$$\Psi_2=\int_0^1\left(\frac{n+t}{n+1}\right)^s\mathrm{d}t=\frac{(n+1)^{s+1}-n^{s+1}}{(s+1)(n+1)^s}.$$

Proof. From Lemma 2.1, utilizing the Hölder inequality and the generalized (s,m)-preinvexity of $|f'|^q$, we have

$$\begin{aligned}
\big|\mathcal{H}_{\eta_m}(\mu,k;n,x)\big|\le{}&\frac{\big|\eta^{\frac{\mu}{k}+1}(x,a,m)\big|}{2|\eta(b,a,m)|}\left(\int_0^1 t^{\frac{\mu p}{k}}\mathrm{d}t\right)^{\frac{1}{p}}\left[(J_1)^{\frac{1}{q}}+(J_2)^{\frac{1}{q}}\right]\\
&+\frac{\big|\eta^{\frac{\mu}{k}+1}(b,x,m)\big|}{2|\eta(b,a,m)|}\left(\int_0^1 t^{\frac{\mu p}{k}}\mathrm{d}t\right)^{\frac{1}{p}}\left[(J_3)^{\frac{1}{q}}+(J_4)^{\frac{1}{q}}\right],
\end{aligned}$$

where

$$\begin{aligned}
J_1&=\int_0^1\left|f'\Big(ma+\frac{n+t}{n+1}\eta(x,a,m)\Big)\right|^q\mathrm{d}t\\
&\le\int_0^1 m\Big(\frac{1-t}{n+1}\Big)^s\big|f'(a)\big|^q+\Big(\frac{n+t}{n+1}\Big)^s\big|f'(x)\big|^q\mathrm{d}t,\\
J_2&=\int_0^1\left|f'\Big(ma+\frac{1-t}{n+1}\eta(x,a,m)\Big)\right|^q\mathrm{d}t
\end{aligned}$$

$$\begin{aligned} &\leq \int_0^1 m\left(\frac{n+t}{n+1}\right)^s\left|f'(a)\right|^q+\left(\frac{1-t}{n+1}\right)^s\left|f'(x)\right|^q \mathrm{d}t,\\ J_3 &= \int_0^1\left|f'\left(mx+\frac{1-t}{n+1}\eta(b,x,m)\right)\right|^q \mathrm{d}t\\ &\leq \int_0^1 m\left(\frac{n+t}{n+1}\right)^s\left|f'(x)\right|^q+\left(\frac{1-t}{n+1}\right)^s\left|f'(b)\right|^q \mathrm{d}t \end{aligned}$$

and

$$\begin{aligned} J_4 &= \int_0^1\left|f'\left(mx+\frac{n+t}{n+1}\eta(b,x,m)\right)\right|^q \mathrm{d}t\\ &\leq \int_0^1 m\left(\frac{1-t}{n+1}\right)^s\left|f'(x)\right|^q+\left(\frac{n+t}{n+1}\right)^s\left|f'(b)\right|^q \mathrm{d}t. \end{aligned}$$

This completes the proof.

Corollary 2.3. *In Theorem 2.2,*

(i) if $\eta(u,v,m)=u-mv$ with $m=1$ for $u,v\in[a,b]$, we have:

$$\begin{aligned} \left|\mathcal{H}(\mu,k;n,x)\right| \leq \left(\frac{1}{\frac{\mu p}{k}+1}\right)^{\frac{1}{p}}&\left\{\frac{(x-a)^{\frac{\mu}{k}+1}}{2(b-a)}\left[\left(\Psi_1\left|f'(a)\right|^q+\Psi_2\left|f'(x)\right|^q\right)^{\frac{1}{q}}\right.\right.\\ &\left.+\left(\Psi_2\left|f'(a)\right|^q+\Psi_1\left|f'(x)\right|^q\right)^{\frac{1}{q}}\right]\\ &+\frac{(b-x)^{\frac{\mu}{k}+1}}{2(b-a)}\left[\left(\Psi_2\left|f'(x)\right|^q+\Psi_1\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right.\\ &\left.\left.+\left(\Psi_1\left|f'(x)\right|^q+\Psi_2\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right]\right\}. \end{aligned}$$

Especially if we choose $k=1=n$ along with $s=1$, we get Theorem 2 established by Mihai and Mitroi in [23]. Further, if we take $\mu=1$, we obtain Theorem 2 presented by Latif in [18],

(ii) if $\eta(u,v,m)=u-mv$ with $m=1$ for $u,v\in[a,b]$, putting $x=\frac{a+b}{2}$, $n=1$, and utilizing similar arguments as in (iii) of Corollary 2.3, we have:

$$\begin{aligned} &\left|\left(\frac{2}{b-a}\right)^{\frac{\mu}{k}-1}\mathcal{H}\left(\mu,k;1,\frac{a+b}{2}\right)\right|\\ &=\left|\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]-\frac{2^{\frac{2\mu}{k}-1}\Gamma_k(\mu+k)}{(b-a)^{\frac{\mu}{k}}}\left[{}_kJ^{\mu}_{a^+}f\left(\frac{3a+b}{4}\right)\right.\right.\\ &\quad\left.\left.+{}_kJ^{\mu}_{(\frac{a+b}{2})^-}f\left(\frac{3a+b}{4}\right)+{}_kJ^{\mu}_{(\frac{a+b}{2})^+}f\left(\frac{a+3b}{4}\right)+{}_kJ^{\mu}_{b^-}f\left(\frac{a+3b}{4}\right)\right]\right| \end{aligned}$$

$$\leq \frac{b-a}{8}\left(\frac{1}{\frac{\mu p}{k}+1}\right)^{\frac{1}{p}}\left\{\left[\frac{1}{2^s(s+1)}|f'(a)|^q+\frac{2^{s+1}-1}{2^s(s+1)}\left|f'\Big(\frac{a+b}{2}\Big)\right|^q\right]^{\frac{1}{q}}\right.$$
$$+\left[\frac{2^{s+1}-1}{2^s(s+1)}|f'(a)|^q+\frac{1}{2^s(s+1)}\left|f'\Big(\frac{a+b}{2}\Big)\right|^q\right]^{\frac{1}{q}}$$
$$+\left[\frac{2^{s+1}-1}{2^s(s+1)}\left|f'\Big(\frac{a+b}{2}\Big)\right|^q+\frac{1}{2^s(s+1)}|f'(b)|^q\right]^{\frac{1}{q}}$$
$$\left.+\left[\frac{1}{2^s(s+1)}\left|f'\Big(\frac{a+b}{2}\Big)\right|^q+\frac{2^{s+1}-1}{2^s(s+1)}|f'(b)|^q\right]^{\frac{1}{q}}\right\}$$
$$\leq \frac{b-a}{8}\left(\frac{1}{\frac{\mu p}{k}+1}\right)^{\frac{1}{p}}\left(1+2^{1-\frac{s}{q}}\right)\left\{\left[\frac{1}{2^s(s+1)}\right]^{\frac{1}{q}}+\left[\frac{2^{s+1}-1}{2^s(s+1)}\right]^{\frac{1}{q}}\right\}\Big[|f'(a)|+|f'(b)|\Big].$$

Remark 2.2. In Theorem 2.2, if $\eta(u,v,m)$ with $m=1$ reduces to $\eta(u,v)$ for $u,v\in[a,b]$, and choosing $k=1=n$, we get Theorem 3.3 proved by Noor et al. in [25].

3. k-Fractional inequalities for products of two functions

We next establish k-fractional integral inequality involving products of two generalized (s,m)-preinvex functions.

Theorem 3.1. *Let $K\subseteq\mathbb{R}$ be an open m-invex subset with respect to $\eta: K\times K\times(0,1]\to\mathbb{R}\setminus\{0\}$ for some fixed $m\in(0,1]$, $a,b\in K$ with $0\leq a<b$. If $f,g: K\to(0,+\infty)$ are generalized (s_1,m)-preinvex and generalized (s_2,m)-preinvex, respectively, then the following inequality holds:*

$$\frac{\Gamma_k(\mu+k)}{2\eta^{\frac{\mu}{k}}(b,a,m)}\Big[{}_kJ^{\mu}_{(ma)^+}(fg)\big(ma+\eta(b,a,m)\big)+{}_kJ^{\mu}_{(ma+\eta(b,a,m))^-}(fg)(ma)\Big]$$
$$\leq\left[\frac{\mu}{2(\mu+ks_1+ks_2)}+\frac{\mu\beta(s_1+s_2+1,\frac{\mu}{k})}{2k}\right]\Big[m^2f(a)g(a)+f(b)g(b)\Big] \tag{3.1}$$
$$+\left[\frac{\mu\beta(s_2+1,\frac{\mu}{k}+s_1)}{2k}+\frac{\mu\beta(s_1+1,\frac{\mu}{k}+s_2)}{2k}\right]\Big[mf(a)g(b)+mf(b)g(a)\Big].$$

Proof. Since f is generalized (s_1,m)-preinvex and g is generalized (s_2,m)-preinvex, we get

$$\frac{\Gamma_k(\mu+k)}{2\eta^{\frac{\mu}{k}}(b,a,m)}\,{}_kJ^{\mu}_{(ma)^+}(fg)\big(ma+\eta(b,a,m)\big)$$
$$=\frac{\mu}{2k\eta^{\frac{\mu}{k}}(b,a,m)}\int_{ma}^{ma+\eta(b,a,m)}\big(ma+\eta(b,a,m)-u\big)^{\frac{\mu}{k}-1}f(u)g(u)\mathrm{d}u$$
$$=\frac{\mu}{2k}\int_0^1(1-t)^{\frac{\mu}{k}-1}f\big(ma+t\eta(b,a,m)\big)g\big(ma+t\eta(b,a,m)\big)\mathrm{d}t$$

$$\begin{aligned}
&\le \frac{\mu}{2k}\int_0^1 (1-t)^{\frac{\mu}{k}-1}\Big[m(1-t)^{s_1}f(a)+t^{s_1}f(b)\Big]\Big[m(1-t)^{s_2}g(a)+t^{s_2}g(b)\Big]\mathrm{d}t\\
&=\frac{\mu}{2k}\int_0^1 \Big[m^2(1-t)^{\frac{\mu}{k}+s_1+s_2-1}f(a)g(a)+mt^{s_2}(1-t)^{\frac{\mu}{k}+s_1-1}f(a)g(b)\\
&\quad+mt^{s_1}(1-t)^{\frac{\mu}{k}+s_2-1}f(b)g(a)+t^{s_1+s_2}(1-t)^{\frac{\mu}{k}-1}f(b)g(b)\Big]\mathrm{d}t\\
&=\frac{\mu}{2(\mu+ks_1+ks_2)}m^2f(a)g(a)+\frac{\mu\beta(s_2+1,\frac{\mu}{k}+s_1)}{2k}mf(a)g(b)\\
&\quad+\frac{\mu\beta(s_1+1,\frac{\mu}{k}+s_2)}{2k}mf(b)g(a)+\frac{\mu\beta(s_1+s_2+1,\frac{\mu}{k})}{2k}f(b)g(b).
\end{aligned}$$

Similarly we get

$$\begin{aligned}
&\frac{\Gamma_k(\mu+k)}{2\eta^{\frac{\mu}{k}}(b,a,m)}\,{}_kJ^{\mu}_{(ma+\eta(b,a,m))^-}(fg)(ma)\\
&\le \frac{\mu\beta(s_1+s_2+1,\frac{\mu}{k})}{2k}m^2f(a)g(a)+\frac{\mu\beta(s_1+1,\frac{\mu}{k}+s_2)}{2k}mf(a)g(b)\\
&\quad+\frac{\mu\beta(s_2+1,\frac{\mu}{k}+s_1)}{2k}mf(b)g(a)+\frac{\mu}{2(\mu+ks_1+ks_2)}f(b)g(b).
\end{aligned}$$

By adding both sides of the above inequalities we can obtain the desired result. This completes the proof.

Corollary 3.1. *In Theorem 3.1, if the mapping $\eta(b,a,m)$ with $m=1$ reduces to $\eta(b,a)$ and $s_1=s_2=s$, we obtain*

$$\begin{aligned}
&\frac{\Gamma_k(\mu+k)}{2\eta^{\frac{\mu}{k}}(b,a)}\Big[{}_kJ^{\mu}_{a^+}(fg)\big(a+\eta(b,a)\big)+{}_kJ^{\mu}_{(a+\eta(b,a))^-}(fg)(a)\Big]\\
&\le\left[\frac{\mu}{2(\mu+2ks)}+\frac{\mu\beta(2s+1,\frac{\mu}{k})}{2k}\right]\Big[f(a)g(a)+f(b)g(b)\Big]\\
&\quad+\left[\frac{\mu\beta(s+1,\frac{\mu}{k}+s)}{k}\right]\Big[f(a)g(b)+f(b)g(a)\Big].
\end{aligned}$$

Especially if $\eta(b,a)=b-a$ and $k=1=s$, we get

$$\begin{aligned}
&\frac{\Gamma(\mu+1)}{2(b-a)^{\mu}}\Big[J^{\mu}_{a^+}f(b)g(b)+J^{\mu}_{b^-}f(a)g(a)\Big]\\
&\le\frac{\mu^2+\mu+2}{2(\mu+1)(\mu+2)}\Big[f(a)g(a)+f(b)g(b)\Big]+\frac{\mu}{(\mu+1)(\mu+2)}\Big[f(a)g(b)+f(b)g(a)\Big],
\end{aligned}$$

which is Theorem 2.1 established by Chen in [6].

Corollary 3.2. *In Theorem* 3.1*, if the mapping $\eta(b,a,m)=b-ma$ with $m=1$, $s_1=1=s_2$ and $g(x)=1$, we obtain*

$$\frac{\Gamma_k(\mu+k)}{2(b-a)^{\frac{\mu}{k}}}\Big[{}_kJ^{\mu}_{a^+}f(b)+{}_kJ^{\mu}_{b^-}f(a)\Big]\le\frac{f(a)+f(b)}{2}.$$

Especially if we take $k = 1$, we get

$$\frac{\Gamma(\mu+1)}{2(b-a)^{\mu}}\Big[J^{\mu}_{a^+}f(b)+J^{\mu}_{b^-}f(a)\Big]\le\frac{f(a)+f(b)}{2},$$

which is the right hand side of the inequality (1.2).

Another k-fractional integral inequality involving products of two generalized (s,m)-preinvex functions is obtained as follows.

Theorem 3.2. *With the same assumptions in Theorem 3.1, we have*

$$\begin{aligned}&\frac{2^{\frac{\mu}{k}-1}\Gamma_k(\mu+k)}{\eta^{\frac{\mu}{k}}(b,a,m)}\Big[{}_kJ^{\mu}_{(ma+\frac{1}{2}\eta(b,a,m))^+}(fg)\big(ma+\eta(b,a,m)\big)\\ &\quad+{}_kJ^{\mu}_{(ma+\frac{1}{2}\eta(b,a,m))^-}(fg)(ma)\Big]\\ &\le\Upsilon_1\big[m^2f(a)g(a)+f(b)g(b)\big]+\Upsilon_2\big[mf(a)g(b)+mf(b)g(a)\big],\end{aligned}\tag{3.2}$$

where

$$\Upsilon_1=\frac{\mu}{2^{s_1+s_2+1}(\mu+ks_1+ks_2)}+\frac{{}_2F_1[-s_1-s_2,\frac{\mu}{k};\frac{\mu}{k}+1;\frac{1}{2}]}{2}$$

and

$$\Upsilon_2=\frac{{}_2F_1[-s_2,\frac{\mu}{k}+s_1;\frac{\mu}{k}+s_1+1;\frac{1}{2}]\mu}{2^{s_1+1}(\mu+ks_1)}+\frac{{}_2F_1[-s_1,\frac{\mu}{k}+s_2;\frac{\mu}{k}+s_2+1;\frac{1}{2}]\mu}{2^{s_2+1}(\mu+ks_2)}.$$

Proof. Since f is generalized (s_1,m)-preinvex and g is generalized (s_2,m)-preinvex, we have

$$\begin{aligned}&\frac{2^{\frac{\mu}{k}-1}\Gamma_k(\mu+k)}{\eta^{\frac{\mu}{k}}(b,a,m)}{}_kJ^{\mu}_{(ma+\frac{1}{2}\eta(b,a,m))^+}(fg)\big(ma+\eta(b,a,m)\big)\\ &=\frac{\mu2^{\frac{\mu}{k}-1}}{k\eta^{\frac{\mu}{k}}(b,a,m)}\int_{ma+\frac{1}{2}\eta(b,a,m)}^{ma+\eta(b,a,m)}\big(ma+\eta(b,a,m)-u\big)^{\frac{\mu}{k}-1}f(u)g(u)\mathrm{d}u\\ &=\frac{\mu}{2k}\int_0^1(1-t)^{\frac{\mu}{k}-1}f\bigg(ma+\frac{1+t}{2}\eta(b,a,m)\bigg)g\bigg(ma+\frac{1+t}{2}\eta(b,a,m)\bigg)\mathrm{d}t\\ &\le\frac{\mu}{2k}\int_0^1(1-t)^{\frac{\mu}{k}-1}\bigg[m\bigg(\frac{1-t}{2}\bigg)^{s_1}f(a)+\bigg(\frac{1+t}{2}\bigg)^{s_1}f(b)\bigg]\\ &\times\bigg[m\bigg(\frac{1-t}{2}\bigg)^{s_2}g(a)+\bigg(\frac{1+t}{2}\bigg)^{s_2}g(b)\bigg]\mathrm{d}t\\ &=\frac{\mu}{2k}\int_0^1\bigg[\frac{m^2(1-t)^{\frac{\mu}{k}+s_1+s_2-1}}{2^{s_1+s_2}}f(a)g(a)+\frac{m(1-t)^{\frac{\mu}{k}+s_1-1}(1+t)^{s_2}}{2^{s_1+s_2}}f(a)g(b)\\ &+\frac{m(1-t)^{\frac{\mu}{k}+s_2-1}(1+t)^{s_1}}{2^{s_1+s_2}}f(b)g(a)+\frac{(1-t)^{\frac{\mu}{k}-1}(1+t)^{s_1+s_2}}{2^{s_1+s_2}}f(b)g(b)\bigg]\mathrm{d}t\end{aligned}$$

$$= \frac{\mu}{2^{s_1+s_2+1}(\mu+ks_1+ks_2)}m^2 f(a)g(a)+\frac{{}_2F_1[-s_2,\frac{\mu}{k}+s_1;\frac{\mu}{k}+s_1+1;\frac{1}{2}]\mu}{2^{s_1+1}(\mu+ks_1)}mf(a)g(b)$$
$$+\frac{{}_2F_1[-s_1,\frac{\mu}{k}+s_2;\frac{\mu}{k}+s_2+1;\frac{1}{2}]\mu}{2^{s_2+1}(\mu+ks_2)}mf(b)g(a)+\frac{{}_2F_1[-s_1-s_2,\frac{\mu}{k};\frac{\mu}{k}+1;\frac{1}{2}]}{2}f(b)g(b).$$

Similarly we get

$$\frac{2^{\frac{\mu}{k}-1}\Gamma_k(\mu+k)}{\eta^{\frac{\mu}{k}}(b,a,m)}{}_kJ^{\mu}_{(ma+\frac{1}{2}\eta(b,a,m))^-}(fg)(ma)$$
$$\le \frac{{}_2F_1[-s_1-s_2,\frac{\mu}{k};\frac{\mu}{k}+1;\frac{1}{2}]}{2}m^2 f(a)g(a)+\frac{{}_2F_1[-s_1,\frac{\mu}{k}+s_2;\frac{\mu}{k}+s_2+1;\frac{1}{2}]\mu}{2^{s_2+1}(\mu+ks_2)}mf(a)g(b)$$
$$+\frac{{}_2F_1[-s_2,\frac{\mu}{k}+s_1;\frac{\mu}{k}+s_1+1;\frac{1}{2}]\mu}{2^{s_1+1}(\mu+ks_1)}mf(b)g(a)+\frac{\mu}{2^{s_1+s_2+1}(\mu+ks_1+ks_2)}f(b)g(b).$$

By adding both sides of the above inequalities we can obtain the desired result. This ends the proof.

Corollary 3.3. *In Theorem* 3.2*, if the mapping* $\eta(b,a,m)$ *with* $m=1$ *reduces to* $\eta(b,a)$ *and* $s_1=s_2=s$*, we obtain*

$$\frac{2^{\frac{\mu}{k}-1}\Gamma_k(\mu+k)}{\eta^{\frac{\mu}{k}}(b,a)}\Big[{}_kJ^{\mu}_{(a+\frac{1}{2}\eta(b,a))^+}(fg)\big(a+\eta(b,a)\big)+{}_kJ^{\mu}_{(a+\frac{1}{2}\eta(b,a))^-}(fg)(a)\Big]$$
$$\le\left[\frac{\mu}{2^{2s+1}(\mu+2ks)}+\frac{{}_2F_1[-2s,\frac{\mu}{k};\frac{\mu}{k}+1;\frac{1}{2}]}{2}\right]\Big[f(a)g(a)+f(b)g(b)\Big]$$
$$+\left[\frac{{}_2F_1[-s,\frac{\mu}{k}+s;\frac{\mu}{k}+s+1;\frac{1}{2}]\mu}{2^s(\mu+ks)}\right]\Big[f(a)g(b)+f(b)g(a)\Big].$$

Especially if $\eta(b,a)=b-a$ *and* $s=1$ *, we get*

$$\frac{2^{\frac{\mu}{k}-1}\Gamma_k(\mu+k)}{(b-a)^{\frac{\mu}{k}}}\Big[{}_kJ^{\mu}_{(\frac{a+b}{2})^+}(fg)(b)+{}_kJ^{\mu}_{(\frac{a+b}{2})^-}(fg)(a)\Big]$$
$$\le\left[\frac{1}{2}-\frac{\mu}{2(\mu+k)}+\frac{\mu}{4(\mu+2k)}\right]\Big[f(a)g(a)+f(b)g(b)\Big]$$
$$+\left[\frac{\mu}{2(\mu+k)}-\frac{\mu}{4(\mu+2k)}\right]\Big[f(a)g(b)+f(b)g(a)\Big].$$

Corollary 3.4. *In Theorem* 3.2*, if the mapping* $\eta(b,a,m)=b-ma$ *with* $m=1$*,* $s_1=s_2=s$ *and* $g(x)=1$*, we obtain*

$$\frac{2^{\frac{\mu}{k}-1}\Gamma_k(\mu+k)}{(b-a)^{\frac{\mu}{k}}}\Big[{}_kJ^{\mu}_{(\frac{a+b}{2})^+}f(b)+{}_kJ^{\mu}_{(\frac{a+b}{2})^-}f(a)\Big]$$
$$\le\left[\frac{\mu}{2^{2s+1}(\mu+2ks)}+\frac{{}_2F_1[-2s,\frac{\mu}{k};\frac{\mu}{k}+1;\frac{1}{2}]}{2}+\frac{{}_2F_1[-s,\frac{\mu}{k}+s;\frac{\mu}{k}+s+1;\frac{1}{2}]\mu}{2^s(\mu+ks)}\right]$$
$$\times\Big[f(a)+f(b)\Big].$$

Especially for $s = 1$, *we get*

$$\frac{2^{\frac{\mu}{k}-1}\Gamma_k(\mu+k)}{(b-a)^{\frac{\mu}{k}}}\left[{}_kJ^{\mu}_{(\frac{a+b}{2})^+}f(b)+{}_kJ^{\mu}_{(\frac{a+b}{2})^-}f(a)\right]\leq\frac{f(a)+f(b)}{2},$$

which is the right hand side of the inequality (1.4).

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