HOPF BIFURCATION ANALYSIS AND AMPLITUDE CONTROL OF A NEW 4D HYPER-CHAOTIC SYSTEM

Ping Cai
School of Mathematics and Statistics
Minnan Normal University
Zhangzhou 363000
China
caiping0596@163.com

Abstract. Hopf bifurcation and amplitude control in a new 4D hyper-chaotic system are investigated in this paper. Theoretical analysis shows that the system will exhibit Hopf bifurcation at equilibrium when the Hopf bifurcation conditions are satisfied. Relationship between the amplitude and control gains is given. Hence the amplitude of the limit cycle can be controlled by choosing suitable control gains, ensuring the stability of the bifurcating period solution. Finally, some applications of the amplitude control are carried out to illustrate the effectiveness of the main theoretical results. The accuracy of different kinds of control function are also compared.

Keywords: 4D hyper-chaotic system, Hopf bifurcation, limit cycle, amplitude control.

1. Introduction

Dynamics of nonlinear system is very rich in terms of bifurcation and chaos, and they have great potential applications in many areas of science, biology and engineering. Bifurcation analysis and control have been studied as early as in 60s of the last century, which play an important role in modern nonlinear dynamics [1, 2, 3, 4, 5]. In general, bifurcation control deals with designing a control law to modify the bifurcation characteristics. More specifically, in dynamic bifurcation control, Hopf bifurcation control has an essential role. During the last few years, great efforts have been devoted to investigating chaotic systems, such as Lorenz system [6], Chua's system [7], Chen system [8], Lü system [9], Liu system [10], T system [11] and other new chaotic system [12]. The problem of amplitude control of the bifurcated solution is becoming more and more widely concerned by researchers [13, 14, 15]. On the one hand, decreasing the amplitude can inhibit the harmful vibration behavior of the system. On the other hand, increasing the amplitude can make the vibration used by people. 4D hyper-chaotic system has more complicated dynamical behavior, which has recently become a hot topic [16, 17, 18, 19]. Base on Lorenz system, a new four-dimensional quadratic autonomous hyper-chaotic attractors is present in Ref.[20],

$$
\begin{cases}
\dot{x_1} = a(x_2 - x_1), \\
\dot{x_2} = bx_1 - x_2 + ex_4 - x_1x_3, \\
\dot{x_3} = -cx_3 + x_1x_2 + x_1^2, \\
\dot{x_4} = -dx_2,
\end{cases} \tag{1}
$$

where x_1, x_2, x_3, x_4 are the state variables, a, b, c, d, e are positive real parameters. Several properties of system (1) were investigated, including analysis of Hopf bifurcation and estimation of ultimate bound. In this paper, we would like to investigate Hopf bifurcation and amplitude control of the system.

The rest of this paper is organized as follows. In Section 2, the local stability and Hopf bifurcation at the equilibrium are analyzed, then the bifurcation behavior in the model is given. In Section 3, a control strategy based on the introduction control parameters with quadratic nonlinearities is applied to the model. The relationship between the amplitude of the limit cycle and the control gains is given by using the normal form theory and the center manifold theorem. In Section 4, some applications of the amplitude control are given, and the effectiveness of the control strategy is verified through numerical simulation. The accuracy for different cases of control functions are compared. Conclusions are given in Section 5 finally.

2. Local stability and Hopf bifurcation analysis

Obviously, system (1) has only one equilibrium at $S_0(0,0,0,0)$. The Jacobian matrix of system (1) at the equilibrium S_0 is given by

$$
J = \begin{pmatrix} -a & a & 0 & 0 \\ b & -1 & 0 & e \\ 0 & 0 & -c & 0 \\ 0 & -d & 0 & 0 \end{pmatrix}. \tag{2}
$$

And the characteristic equation is

$$
\lambda^4 + k_1\lambda^3 + k_2\lambda^2 + k_3\lambda + k_4 = 0, \tag{3}
$$

where

$$
\begin{aligned}
k_1 &= 1 + a + c, \\
k_2 &= a - ab + c + ac + de, \\
k_3 &= ac - abc + ade + cde, \\
k_4 &= acde.
\end{aligned}
$$

Computing the following determinants:

$$\Delta_1 = k_1,$$
$$\Delta_2 = \begin{vmatrix} k_1 & 1 \\ k_3 & k_2 \end{vmatrix} = k_1k_2 - k_3,$$
$$\Delta_3 = \begin{vmatrix} k_1 & 1 & 0 \\ k_3 & k_2 & k_1 \\ 0 & k_4 & k_3 \end{vmatrix} = k_3(k_1k_2 - k_3) - k_1^2k_4,$$
$$\Delta_4 = \begin{vmatrix} k_1 & 1 & 0 & 0 \\ k_3 & k_2 & k_1 & 1 \\ 0 & k_4 & k_3 & k_2 \\ 0 & 0 & 0 & k_4 \end{vmatrix} = k_4\Delta_3.$$

If $k_1 > 0, k_3 > 0, k_4 > 0$ and $k_3(k_1k_2 - k_3) - k_1^2k_4 > 0$, then $\Delta_i > 0 (i = 1, 2, 3, 4)$. Based on Routh-Hurwitz criteria, all roots of the characteristic equation have negative real parts. Thus, S_0 is locally asymptotically stable. If $k_3(k_1k_2 - k_3) - k_1^2k_4 \le 0$, and $k_i > 0 (i = 1, 2, 3, 4)$, S_0 is unstable and non-hyperbolic. Taking b as the Hopf bifurcation parameter, by the equation

$$k_3(k_1k_2 - k_3) - k_1^2k_4 = 0, \tag{4}$$

we get the critical value

$$b_0 = \frac{a + a^2 + de}{a + a^2}. \tag{5}$$

When $b = b_0$, the Jacobian matrix J has a pair of imaginary eigenvalues as follows:

$$\lambda_{1,2} = \pm i\omega_0 = \pm i\sqrt{\frac{ade}{1 + a}}. \tag{6}$$

The other two eigenvalues are

$$\lambda_3 = -1 - a < 0 \tag{7}$$

and

$$\lambda_4 = -c < 0. \tag{8}$$

Under these conditions, the following transversality condition is also satisfied:

$$\alpha'(0) = Re(\lambda'(0)|_{\lambda=i\omega_0}) = \frac{(1 + a)(3adek_1k_5 + 2c\omega_0k_6k_7)}{ade(9k_8 + 4k_9)} \neq 0, \tag{9}$$

where

$$\begin{aligned}
k_5 &= cde + a(c + de + cde) + a^2(c + 2de),\\
k_6 &= c + 2ac + a^2c - ade,\\
k_7 &= de + (1 + \omega_0)(a^2 + a),\\
k_8 &= ade(1 + a)(1 + a + c)^2,\\
k_9 &= ((1 + a)^2c - ade)^2.
\end{aligned}$$

Therefore, system (1) undergoes Hopf bifurcation at $S_0(0, 0, 0, 0)$ based on Hopf bifurcation theory [21]. The bifurcation results a family of limit cycles emerging from the equilibrium S_0 at the sufficiently small neighborhood of b_0. Next, a control strategy is applied to the model to control the amplitude of the limit cycle.

3. Relationship between the amplitude of limit cycle and control gains

In this section, a control strategy is applied to the model. The control functions are introduced for the quadratic nonlinearities of system (1), as shown below:

$$\begin{cases}
\dot{x_1} = a(x_2 - x_1),\\
\dot{x_2} = bx_1 - x_2 + ex_4 - f(m) * x_1x_3,\\
\dot{x_3} = -cx_3 + g(n) * (x_1x_2 + x_1^2),\\
\dot{x_4} = -dx_2,
\end{cases} \tag{10}$$

where $f(m)$ and $g(n)$ are control functions. In general, the threshold of bifurcation is determined by the linear parts of the system, and the stability of bifurcating solution is determined by the non-linear parts of the system. So, the control approach do not shift the bifurcation critical value. And the original equilibrium $S_0(0, 0, 0, 0)$ is also preserved. By the linear transform $(x_1, x_2, x_3, x_4)^T = P(X_1, X_2, X_3, X_4)^T$, where

$$P = \begin{pmatrix}
\frac{(a^2+a)\omega_0}{d(a^2+a+de)} & \frac{-ae}{a^2+a+de} & \frac{-a^2-a}{d} & 0\\
\frac{\omega_0}{d} & 0 & \frac{1+a}{d} & 0\\
0 & 0 & 0 & 1\\
0 & 1 & 1 & 0
\end{pmatrix}, \tag{11}$$

then system (10) has the following normal form:

$$\begin{cases}
\dot{X}_1 = -\omega_0 X_2 + F_1(X_1, X_2, X_3, X_4),\\
\dot{X}_2 = \omega_0 X_1 + F_2(X_1, X_2, X_3, X_4),\\
\dot{X}_3 = \lambda_3 X_3 + F_3(X_1, X_2, X_3, X_4),\\
\dot{X}_4 = \lambda_4 X_4 + F_4(X_1, X_2, X_3, X_4),
\end{cases} \tag{12}$$

where F_1, F_2, F_3, F_4 are high order nonlinear functions about X_1, X_2, X_3, X_4 , which are shown as follows:

$$F_1(X_1, X_2, X_3, X_4) = \frac{(1+a)(a^2+2a+de+1)f(m)(-a^2\omega_0 X_1X_4 + a\omega_0^2 X_2X_4 + (a^4+a^3+a^2de)X_3X_4)}{\omega_0(a^2+de+a)(a^3+3a^2+3a+ade+1)},$$

$$F_2(X_1, X_2, X_3, X_4) = \frac{af(m)((1+a)^2\omega_0 X_1X_4 - (1+a)deX_2X_4 - (a^4+3a^3+3a^2+a+(1+a)^2de)X_3X_4)}{a^5+4a^4+2a^3(3+de)+4a^2(1+de)+a(1+3de+d^2e^2)+de},$$

$$F_3(X_1, X_2, X_3, X_4) = \frac{(a^2+a)f(m)(-(1+a)\omega_0 X_1X_4 + deX_2X_4 + (1+a)(a+a^2de)X_3X_4)}{a^5+4a^4+2a^3(3+de)+4a^2(1+de)+a(1+3de+d^2e^2)+de},$$

$$F_4(X_1, X_2, X_3, X_4) = \frac{ag(n)((1+a)(2a+2a^2+de)\omega_0^2X_1^2 - (3a^2de+3ade+d^2e^2)\omega_0X_1X_2 + ad^2e^2X_2^2)}{d(a+a^2+de)} + a(a+1)g(n)\left(\frac{(1-de-2a-3a^2)\omega_0}{d}X_1X_3 + ae(2a-1)X_2X_3 + \frac{(a^2-1)(a^2+a+de)}{d}X_3^2\right).$$

Based on the center manifold theory and normal form reduction, the curvature coefficient is expressed by the following according to Ref. [21]:

$$\sigma_1 = Re\left\{\frac{g_{20}g_{11}}{2\omega_0}i + G^1_{110}w^1_{11} + G^2_{110}w^2_{11} + \frac{G_{21}+G^1_{101}w^1_{20}+G^2_{101}w^2_{20}}{2}\right\}. \tag{13}$$

The characteristic quantities can be calculated from system (12) as follows:

$$g_{11}=0, g_{20}=0, G^1_{110}=0, G^1_{101}=0, G_{21}=0, w^1_{11}=0, w^1_{20}=0,$$

$$G^2_{110} = \frac{-(a(a+1)^2+i(a^2+a)\omega_0)f(m)}{2(a^3+3a^2+3a+ade+1)},$$

$$G^2_{101} = \frac{-a^5-3a^3-3a^2-a^2de+ade-a+i\omega_0(a^3+4a^2+5a+ade+de+2)}{2(a^2+a+de)(a^3+3a^2+3a+ade+1)},$$

$$w^2_{11} = \frac{a^2eg(n)}{cd(a^2+a+de)},$$

$$w^2_{20} = \frac{((-2a^3-2a^2)i+(3a^2+3a+de)\omega_0)aeg(n)}{2d(a^2+a+de)^2(2\omega_0-ic)}.$$

So an explicit expression of the curvature coefficient is written as:

$$\sigma_1 = -\frac{a^2e\sqrt{1+a}f(m)g(n)(k_{10}+k_{11})}{8cdk_{12}k_{13}}, \tag{14}$$

where

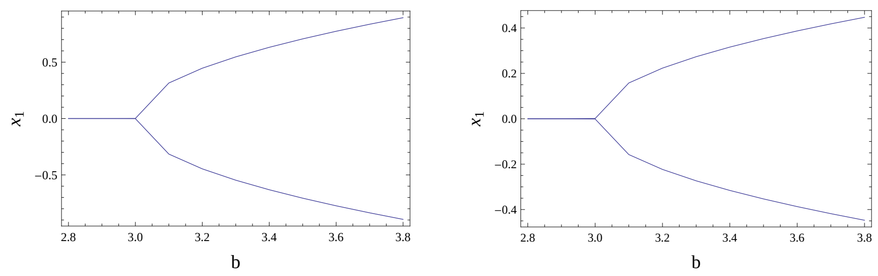
$$
\begin{aligned}
k_{10} &= \sqrt{1+a}(-2c\omega_0^2 + ac^2(6+de) + 2a^3(3c^2+8de)),\\
k_{11} &= 2a^2\sqrt{1+a}(6c^2+8de+c\omega_0^2),\\
k_{12} &= (1+a)c^2 + 4ade,\\
k_{13} &= a^5 + 4a^4 + 2a^3(3+de) + 4a^2(1+de) + a(1+3de+d^2e^2) + de.
\end{aligned}
$$

Therefore, the approximate amplitude in close vicinity to the Hopf bifurcation point is

$$
\begin{aligned}
r &= \sqrt{-\frac{\alpha'(0)}{\sigma_1}(b-b_0)}\\
&= \sqrt{\frac{8\sqrt{1+a}ck_{12}k_{13}(3adek_1k_5+2cw_0k_6k_7)}{a^3e^2f(m)g(n)(9k_8+4k_9)(k_{10}+k_{11})}(b-b_0)}, \quad |b-b_0| \ll 1. \qquad (15)
\end{aligned}
$$

4. Application of amplitude control

It should be pointed out that Eq.(15) describes the relationship between the amplitude of the state variable x_1 and the control functions. Other linear transformation can be chosen to describe the amplitude of the other state variables. We choose $a = 2, c = 1, d = 3, e = 4$, then $b_0 = 3, \alpha'(0) = 0.484654, \sigma_1 = -0.0546643f(m)g(n)$. Obviously, σ_1 degrade into the curvature coefficient of the original system(1) when $f(m)g(n) = 1$. If $f(m)g(n) > 0$, the bifurcated limit cycle is stable, and then the parameter $\mu_2 = -\frac{\sigma_1}{\alpha'(0)} > 0$, where the Hopf bifurcation is supercritical and the bifurcating periodic solutions exist for $b > b_0$. Since this control strategy does not change the bifurcation critical value of the system, it means that both the original system and the controlled system bifurcated at $b_0 = 3$. The bifurcation figures are shown in Fig.1.



(a) the original system (b) the controlled system

Fig. 1. Bifurcation diagrams of the original system and the controlled system at equilibrium S_0.

Setting $b = 3.10$, time displacement curves of the period solutions under different value of control gains are shown in Fig.2 and Fig.3, respectively. The solid lines denote the period solutions of the original system while the dashed lines represent the period solutions of the controlled system. It can be seen that under different values of the control functions, the amplitude can be large or small. Other values can be similar.

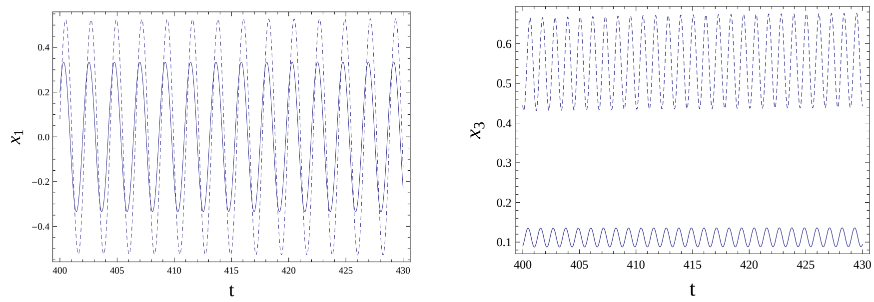


Fig. 2. Time displacement curves of period solutions ($0 < f(m)g(n) < 1, 0 < f(m) < 1$).

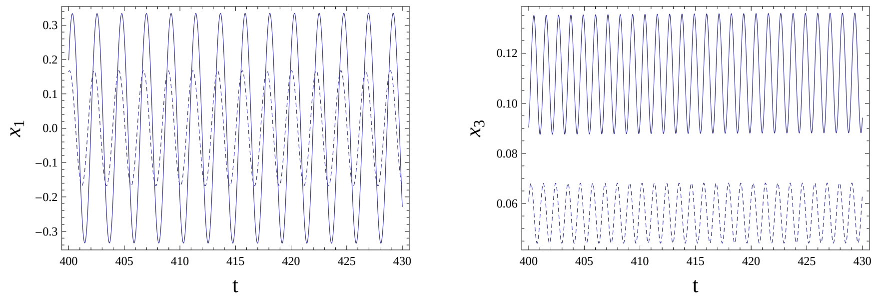
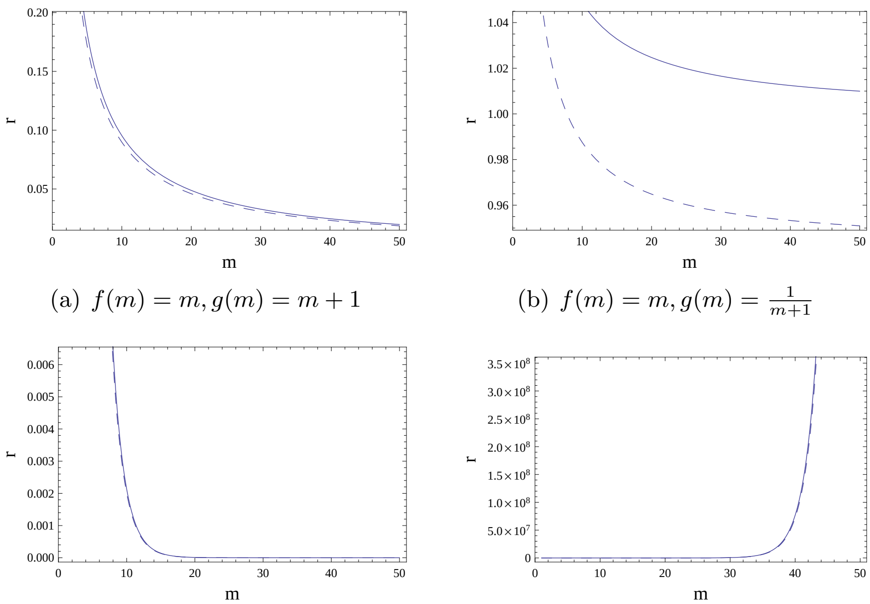


Fig. 3. Time displacement curves of period solutions ($f(m)g(n) > 1, f(m) > 1$).

For simplicity, suppose $m = n$. Let $f(m) = m$, while $g(m) = m + 1, g(m) = \frac{1}{m+1}, g(m) = e^m$ and $g(m) = \frac{1}{e^m}$, the accuracy of approximate solution of amplitude(15) and numerical solution is compared in Fig.4. In this case, the dotted lines represent the approximate solution, and the solid lines represent the numerical solution.

5. Conclusion

This paper is concerned about Hopf bifurcation and amplitude control of a hyper-chaotic system. Applying Routh-Hurwitz criterion, the stability of the equilibrium is investigated. Then the existence of Hopf bifurcation is given based on Hopf bifurcation theory. A control approach is applied to the system, which not only keeps the equilibrium structure of the original system, but also not change the Hopf bifurcation critical value. By the normal form theory and the center manifold theorem, the relationship between the amplitude of the limit



(a) $f(m) = m, g(m) = m + 1$ (b) $f(m) = m, g(m) = \frac{1}{m+1}$

(c) $f(m) = m, g(m) = e^{m}$ (d) $f(m) = m, g(m) = e^{-m}$

Fig. 4. The accuracy of approximated solution of amplitude and numerical solution.

cycle and the control gains is presented. Numerical simulations show, in case of the stability of the bifurcating periodic solutions, the control law can increase or decrease the amplitude of the periodic solution effectively under different control gains. The accuracy of different kinds of control functions is also compared.

Acknowledgment

The author wishes to thank the anonymous reviewers for their valuable comments and suggestions.

Project supported by Natural Science Foundation of Fujian Province of China (No. 2017J05012); and National Natural Science Foundation of China for Youth (Nos. 61403181, 61603174); and the Institute of Meteorological Big Data-Digital Fujian and Fujian Key Laboratory of Date Science and Statistics.

References

[1] C.X. Liang, J.S. Tang, *Equilibrium points and bifurcation control of a chaotic system*, Chin. Phys. B, 17 (2008), 0135-0139.

[2] S.H. Liu, J.S. Tang, J.Q. Qin and X.B. Yin, *Bifurcation analysis and control of periodic solutions changing into invariant tori in Langford system*, Chin. Phys. B, 17 (2008), 1691-1697.

[3] G.R. Jiang, B.G. Xu and Q.G. Yang, *Bifurcation control and chaos in a linear impulsive system,* Chin. Phys. B, 18 (2009), 5235-5241.

[4] J. Zhou, *Bifurcation analysis of the Oregonator model*, Appl. Math. Lett., 52 (2016), 192-198.

[5] D.W. Ding, X.Y. Zhang, J.D. Cao, N. Wang and D. Liang, *Bifurcation control of complex networks model via pd controller*, Neurocomputing, 175 (2016), 1-9.

[6] H.W. Li and M. Wang, *Hopf bifurcation analysi in a Lorenz-type system*, Nonlinear Dyn., 71 (2013), 235-240.

[7] J.H. Yang and L.Q. Zhao, *Bifurcation analysis and chaos control of the modified Chua's circuit system*, Chaos Soliton Fract., 77 (2015), 332-339.

[8] Z.S. Cheng, *Anti-control of Hopf bifurcation for Chen's system through washout filters*, Neurocomputing, 73 (2010), 3139-3146.

[9] Z.S. Lü and L.X. Duan, *Control of codimension-2 bautin bifurcation in chaotic lu system,* Commun. Theor. Phys., 52 (2009), 631-636.

[10] Z.S. Lü and L.X. Duan, *Anti-control of Hopf bifurcation in the chaotic Liu system with symbolic computation*, Chin. Phys. Lett., 26 (2009), 050504.

[11] R.Y. Zhang, *Bifurcation analysis for T system with delayed feedback and its application to control of chaos,* Nonlinear Dyn., 72 (2013), 629-641.

[12] P. Cai and Z.Z. Yuan, *Hopf bifurcation and chaos control in a new chaotic system via hybrid control strategy*, Chinese J. Phys., 55 (2017), 64-70.

[13] J.S. Tang, F. Han, H. Xiao and X. Wu, *Amplitude control of a limit cycle in a coupled van der Pol system*, Nonlinear Anal.-Theor., 71 (2009), 2491-2496.

[14] C. Yan, S.H. Liu, J.S. Tang and Y.M. Meng, *Amplitude control of limit cycles in Langford system,* Chaos Soliton Fract., 42 (2009), 335-340.

[15] P. Cai, J. S. Tang, *Control of amplitude of limit cycles in a class of strongly nonlinear oscilation systems*, J. Vib. Shock (in Chinese), 32 (2013), 110-112.

[16] C.L. Li, K.L. Su, J. Zhang and D.Q. Wei, *Robust control for fractional-order four-wing hyperchaotic system using LMI*, Optik, 12 (2013), 5807-5810.

[17] W. Xue, G.Y. Qi, J.J. Mu, H.Y. Jia and Y.L. Guo, *Hopf bifurcation analysis and circuit implementation for a novel four-wing hyper-chaotic system,* Chin. Phys. B, 22 (2013), 080504.

[18] P. Cai, J.S. Tang and Z.B. Li, *Controlling Hopf bifurcation of a new modified hyperchaotic system,* Math. Prob. Eng., 2015, 614135.

[19] L.L. Zhou, Z.Q. Chen, Z.L. Wang and J.Z. Wang, *On the analysis of local bifurcation and topological horseshoe of a new 4D hyper-chaotic system,* Chaos Soliton Fract., 91 (2016), 148-156.

[20] A. Zarei and S. Tavakoli, *Hopf bifurcation analysis and ultimate bound estimation of a new 4-D quadratic autonomous hyper-chaotic system,* Appl. Math. Comput., 291 (2016), 323-339.

[21] B.D. Hassard, N.D. Kazarinoff and Y. Wan, *Theory and Applications of Hopf Bifurcation,* Theory and Applications of Hopf Bifurcation, London Cambridge Univ., 1981.

Accepted: 1.09.2017