PAIRWISE CONNECTEDNESS IN FUZZY BITOPOLOGICAL SPACES IN QUASI-COINCIDENCE SENSE

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Abstract. In this paper, we have defined a new notion of fuzzy connectedness in fuzzy bitopological spaces in sense of quasi-coincidence sense. We have found the relations among our and other such notions. We have observed that our notion is stronger than some other such notions. We have shown that the pairwise fuzzy connectedness is preserved under the FP-continuous mapping. Moreover, we have obtained productivity and some other properties of this new concept.

Keywords: quasi-coincidence, fuzzy bitopological spaces, fuzzy pairwise connectedness.

1. Introduction

Zadeh [17] introduced the concept of a fuzzy set in his classical paper. Using this notion Chang [3] introduced the concept of fuzzy topological space. Since then, many authors have studied successfully to generalize several concepts of general topology to the fuzzy setting. The notion of bitopological spaces was initially introduced by Kelly [8] in 1963. Kandil and El-Shafee [7] introduced and studied the notion of fuzzy bitopological spaces as a natural generalization of fuzzy topological spaces. The concepts of fuzzy connectedness have been introduced earlier by Lowen and Srivastava [9], Wuyts [16], Dewan Muslim Ali [1], Tapi and Deole [13], Fatteh and Bassan [4] and G. Jager [6]. In [11], Park and Lee generalized the concept of fuzzy extremally disconnected spaces due to Ghosh [5] into a fuzzy bitopoogical setting and discuss some of its properties. In 2002,

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Chandrasekar and Balasubramanian [2] defined weaker forms of connectedness and stronger forms of disconnectedness in fuzzy bitopological spaces.

The purpose of this paper is to introduce a new notion of fuzzy connectedness in fuzzy bitopological space in the light of quasi-coincidence sense and compare with [2]. We investigate that pairwise fuzzy connectedness is preserved under FP-continuous mapping. Moreover, we obtain some other properties of this new concept.

2. Priliminaries

For the purpose of the main results, we need to introduce some definitions and notations. Through this paper, X will be a nonempty set, $I = [0, 1]$ and FP stands for fuzzy pairwise. The notations (X, t) and (X, s, t) will be denoted fuzzy topological space and fuzzy bitopological space respectively and μ_A will be denoted the characteristics function on the subset A of X.

Definition 2.1 ([17]). *A fuzzy set μ in a set X is a function from X into the closed unit interval $I = [0, 1]$. For every $x \in X$, $\mu(x) \in I$ is called the grade of membership of x. A member of I^X may also be called fuzzy subset of X.*

Definition 2.2 ([17]). *Let f be a mapping from a set X into a set Y and u a fuzzy set in X. Then the image of u, written as $f(u)$, is a fuzzy set in Y whose membership function is given by*

$$f(u)(y) = \begin{cases} \sup\{u(x)\}, & \text{if } f^{-1}[\{y\}] \neq \Phi,\ x \in X; \\ 0, & \text{otherwise.} \end{cases}$$

Definition 2.3 ([17]). *Let f be a mapping from a set X into a set Y and v be a fuzzy set in Y. Then the inverse of v, denoted by $f^{-1}(v)$ a fuzzy set in X, is defined by $f^{-1}(v)(x) = v(f(x))$, for all $x \in X$.*

Definition 2.4 ([15]). *A fuzzy set μ in X is called a fuzzy singleton if and only if $\mu(x) = r$, $(0 < r \leq 1)$ for a certain $x \in X$ and $\mu(y) = 0$ for all points y of X except x. The fuzzy singleton is denoted by x_r and x is its support. We call x_r is a fuzzy point if $0 < r < 1$. The class of all fuzzy singletons in X is denoted by $S(X)$.*

Definition 2.5 ([3]). *A fuzzy topology t on X is a collection of members of I^X which is closed under arbitrary suprema and finite infima and which contains constant fuzzy sets 1 and 0. The pair (X, t) is called a fuzzy topological space (fts, in short) and members of t are called t-open (or simply open) fuzzy sets. A fuzzy set μ is called t-closed (or simply closed) fuzzy set if $1 - \mu \in t$.*

Definition 2.6 ([3]). *A function f from a fuzzy topological space (X, t) into a fuzzy topological space (Y, s) is called fuzzy continuous if and only if for every $u \in s$, $f^{-1}(u) \in t$.*

Definition 2.7 ([7]). *A fuzzy singleton x_r is said to be quasi-coincident with a fuzzy set μ, denoted by $x_r q \mu$ iff $r + \mu(x) > 1$. If x_r is not quasi-coincident with μ, we write $x_r \bar{q} \mu$.*

Definition 2.8 ([8]). *Let X be any non empty set and S and T be any two general topologies on X then the triple (X, S, T) is called bitopological space.*

Definition 2.9 ([7]). *A fuzzy bitopological space (fbts, in short) is a triple (X, s, t) where s and t are arbitrary fuzzy topologies on X.*

Definition 2.10 ([10]). *A function f from a fuzzy bitopological space (X, s, t) into a fuzzy bitopological space (Y, s_1, t_1) is called FP-continuous if and only if $f : (X, s) \to (Y, s_1)$ and $f : (X, t) \to (Y, t_1)$ are both fuzzy continuous.*

Definition 2.11 ([14]). *Let $\{(X_i, s_i, t_i), i \in \bigwedge\}$ be a family of fuzzy bitopological spaces. Then the space $(\prod X_i, \prod s_i, \prod t_i)$ is called product fuzzy bitopological space of the family $\{(X_i, s_i, t_i), i \in \bigwedge\}$, where $\prod s_i$, $\prod t_i$ respectively denote the usual product fuzzy topologies of the families $\{\prod s_i : i \in \bigwedge\}$ and $\{\prod t_i : i \in \bigwedge\}$ of the fuzzy topologies on X.*

A fuzzy topological property P is called productive if the product of a family of fbts, each having property P, has property P.

3. The main results

The aim of this section is to introduce a new notion of fuzzy pairwise connectedness in fuzzy bitopological spaces by using quasi-coincidence sense.

Definition 3.1 ([2]). *An fbts (X, s, t) is called pairwise fuzzy connected if X has no proper fuzzy sets $u \in s$, $v \in t$ such that $u + v = 1$.*

Definition 3.2 ([2]). *An fbts (X, s, t) is called pairwise fuzzy strongly connected if it has no proper fuzzy sets u, $v \in s^c \cup t^c$ such that $u + v \leq 1$.*

Definition 3.3. *An fbts (X, s, t) is called pairwise fuzzy q-connected if it has no proper fuzzy sets $u \in s^c$, $v \in t^c$ such that $u \bar{q} v$.*

Theorem 3.1. *In a fuzzy bitopological space, the following implications hold:*

Pairwise fuzzy strongly connectedness

$$\Downarrow$$

Pairwise fuzzy q-connectedness

$$\Downarrow$$

Pairwise fuzzy connectedness.

Proof. The proof is obvious.

However, the converses are not true in general as shown in the following examples.

Example. Let $X = \{a, b\}$ and let s be indiscrete fuzzy topology on X and t be discrete fuzzy topology on X. Then (X, s, t) is pairwise fuzzy q-connected but not pairwise strongly fuzzy connected.

Example. Let $X = [0, 1]$. Let s be a fuzzy topology on X generated by $\{u\}$ where $u(x) = 0.7$ for all $x \in X$. Again, let t be a fuzzy topology on X generated by $\{v\}$ where $v(x) = 0.6$ for all $x \in X$. Then (X, s, t) is pairwise fuzzy connected but not pairwise fuzzy q-connected.

Theorem 3.2. *An fbts (X, s, t) is pairwise fuzzy q-connected iff it has no proper fuzzy sets $u \in s$, $v \in t$ such that $u + v \geq 1$.*

Proof. Suppose (X, s, t) is not pairwise fuzzy q-connected. Then there exist fuzzy sets $u \in s^c$, $v \in t^c$ such that $u\bar{q}v$. That is, $u(x) + v(x) \leq 1$ for all $x \in X$. Put $\alpha = 1 - u$, $\beta = 1 - v$. Then $\alpha \in s$, $\beta \in t$ such that

$$\alpha(x) + \beta(x) \geq 1 \text{ for all } x \in X.$$

Hence $\alpha + \beta \geq 1$ which is a contradiction.

Conversely, suppose that (X, s, t) contains fuzzy sets $u \in s$, $v \in t$ such that $u + v \geq 1$. Put $\lambda = 1 - u$, $\mu = 1 - v$. Then $\lambda \in s^c$, $\mu \in t^c$. Now

$$\lambda(x) + \mu(x) = 1 - u(x) + 1 - v(x) = 2 - (u(x) + v(x)) \leq 1,$$

since $u + v \geq 1$. Thus (X, s, t) is not pairwise fuzzy q-connected which is a contradiction and hence the proof is complete.

Theorem 3.3. *An fbts (X, s, t) is pairwise fuzzy q-connected if X contains no proper fuzzy set u such that u is both s-open and t-closed or both t-open and s-closed.*

Proof. Suppose that X contains no proper fuzzy set u such that u is both s-open and t-closed. Then by definition, $1 - u$ is s-closed fuzzy set. Now,

$$(1 - u)(x) + u(x) = 1 \leq 1 \text{ for all } x \in X.$$

That is, $(1 - u)\bar{q}u$. Hence (X, s, t) is not pairwise fuzzy q-connected

Theorem 3.4. *Let (X, s, t) be a fuzzy topological space, $A \subset X$. If A is a pairwise fuzzy q-connected subset of X, then for any fuzzy sets $u \in s$, $v \in t$, $\mu_A \leq u + v$ implies either $\mu_A \leq u$ or $\mu_A \leq v$.*

Proof. Suppose that A is not pairwise fuzzy q-connected subset of X. Then there exist fuzzy sets $\lambda \in s^c, \delta \in t^c$ such that (i) $\lambda/A \neq 0$, (ii) $\delta/A \neq 0$ and *(iii)* $(\lambda/A)\bar{q}(\delta/A)$. Now, (iii) implies that

$$(\lambda/A)(x) + (\delta/A)(x) \leq 1 \quad \ldots\ldots\ldots \qquad (iv)$$

Now, if we put $u = 1 - \lambda$, $v = 1 - \delta$ then $u/A = 1 - \lambda/A$ and $v/A = 1 - \delta/A$. Hence (i), (ii) and (iv) imply that $\mu_A \leq u + v$ but $\mu_A \not\leq u$ and $\mu_A \not\leq v$.

Theorem 3.5. *If F is a subset of an fbts $(X,\, s,\, t)$ such that μ_F is both s-open and t-closed in X, then X is pairwise fuzzy q-connected implies that F is a pairwise q-connected subset of X.*

Proof. Suppose F is not pairwise q-connected subset of X. Then there exist fuzzy sets $u \in s^c$, $v \in t^c$ such that

$$(i)\, u/F \neq 0,\ (ii) v/F \neq 0 \text{ and } (iii)\, (u/F)\bar{q}(v/F).$$

Now (iii) implies that

$$(u/F)(x) + (v/F)(x) \leq 1 \text{ for all } x \in F.$$

Since μ_F is s-open, then $1 - \mu_F$ is s-closed. Now we have

$$(u \cap (1 - \mu_F))(x) + (v \cap \mu_F)(x) \leq 1, \text{ for all } x \in X.$$

Also by (i) and (ii), we get

$$u \cap (1 - \mu_F) \neq 0 \text{ and } v \cap \mu_F \neq 0.$$

So, X is not pairwise fuzzy q-connected which is a contradiction.

Theorem 3.6. *If A and B are two subsets of a fuzzy bitopological space (X, s, t) and $\mu_A \leq \mu_B \leq \bar{\mu}_A$ and A is a pairwise fuzzy q-connected subset of X, then B is a pairwise fuzzy q-connected subset of X.*

Proof. Suppose B is not pairwise fuzzy q-connected subset of X. Then there exist fuzzy sets $u \in s^c$, $v \in t^c$ such that

$$(i)\, u/B \neq 0,\ (ii) v/B \neq 0 \text{ and } (iii)\, (u/B)\bar{q}(v/B).$$

We first show that $u/A \neq 0$. Suppose $u/A = 0$. Then it is clear that

$$u(x) + \mu_A(x) \leq 1 \text{ for all } x \in A.$$

This implies that

$$u(x) + \bar{\mu}_A(x) \leq 1 \text{ for all } x \in A.$$

So,

$$u(x) + \mu_B(x) \leq 1 \text{ for all } x \in A$$

since $\mu_B \leq \bar{\mu}_A$. Hence $u/B = 0$ which is a contradiction as $u/B \neq 0$. Therefore, $u/A \neq 0$. Similarly, we can show that $v/A \neq 0$.

Now from (iii) we get

$$u/B(x) + v/B(x) \leq 1 \text{ for all } x \in B.$$

So, we have $u/A(x)+v/A(x) \leq 1$ for all $x \in A$. as $\mu_A \leq \mu_B$. Hence $(u/A)\bar{q}(v/A)$. Therefore A is not pairwise fuzzy q-connected subset of X which is a contradiction.

In following theorem, we observe here that our concept is preserved under fuzzy continuous mapping.

Theorem 3.7. *Let (X, s, t) and (Y, s_1, t_1) be two fuzzy bitopological spaces and $f : X \to Y$ be FP-continuous. Then X is pairwise fuzzy q-connected implies Y is pairwise fuzzy q-connected.*

Proof. Suppose Y is not pairwise fuzzy q-connected. Then there exist fuzzy sets $u \in s_1^c$, $v \in t_1^c$ such that $u\bar{q}v$. That is, $u(y) + v(y) \leq 1$ for all $y \in Y$. Since f is continuous, then $f^{-1}(u)$ and $f^{-1}(v)$ are non-zero s-open and t-closed respectively. Now

$$f^{-1}(u)(x) + f^{-1}(v)(x) = u(f(x)) + v(f(x)) \leq 1,$$

for all $x \in X$ since $u\bar{q}v$. Hence $f^{-1}(u)\bar{q}f^{-1}(v)$. Therefore X is not pairwise fuzzy q-connected which is a contradiction.

In the following theorem, we show that the productivity property holds in pairwise fuzzy q-connected spaces.

Theorem 3.8. *Product space is pairwise fuzzy q-connected if coordinate spaces are pairwise fuzzy q-connected.*

Proof. Let (X, s_1, t_1) and (Y, s_2, t_2) be two fuzzy bitopological spaces. Suppose the product space $(X \times Y, s_1 \times s_2, t_1 \times t_2)$ is not pairwise fuzzy q-connected. Then there exist fuzzy sets $u \times v \in t_1 \times t_2$, $\alpha \times \beta \in t_1 \times t_2$ such that $u \times v \neq 1$, $\alpha \times \beta \neq 1$ and for every $x \in X$, $y \in Y$

$$(u \times v)(x) + (\alpha \times \beta)(x, y) \geq 1.$$

That is, $\min\{u(x), v(y)\} + \min\{\alpha(x), \beta(y)\} \geq 1$. Hence $u(x) + \alpha(x) \geq 1$ and $v(y) + \beta(y) \geq 1$. Therefore, the coordinate spaces are not pairwise fuzzy q-connected which is a contradiction.

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