# CERTAIN GENERATING FUNCTIONS OF GENERALIZED HYPERGEOMETRIC 2D POLYNOMIALS FROM TRUESDELL'S METHOD

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**Abstract.** In this paper, the generating functions for generalized Hypergeometric 2D polynomials $U_n(\beta, \gamma, x, y)$ are obtained by using the Truesdell's method giving a suitable interpretation to the index n. Further, a pair of linearly independent differential recurrence relations are used in order to derive generating functions for $U_n(\beta, \gamma, x, y)$. The principal interest in our results lies in the fact that, how the Truesdell's method is utilized in an effective and suitable way to generalized Hypergeometric 2D polynomials in order to derive two generating functions independently from ascending and descending recurrence relations respectively.
**Keywords:** special functions, generalized hypergeometric 2D polynomials $U_n(\beta, \gamma, x, y)$ generating functions.

## 1. Introduction

Generating functions play a very important role in the investigation of various properties of the sequences, which they generate. They are used with good effect for the determination of the asymptotic behaviour of the generated sequence $\{f_n\}$ as $n \longrightarrow \infty$[2].In recent years, the development of advanced computers has made it necessary to study the hypergeometric polynomials with series represen-

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tations from the numerical point of view. Because of the important role which hypergeometric polynomials play important role in problems of applied mathematics, the theory of generating functions has been developed various directions and found wide applications in different branches of science and technology.

The aim of present paper is to derive the generating functions for the generalized hypergeometric 2D polynomials by using the Truesdell's method , giving suitable interpretation to the index n. It is worth recalling that this method yields two generating functions for the generalized hypergeometric 2D polynomials, independently from ascending and descending recurrence relations, where as the simultaneous use of these recurrence relations in other group theoretic methods. The results obtained for generalized hypergeometric 2D polynomials are new in the theory of special functions.

The generalized hypergeometric polynomials $U_n\left(\beta,\gamma,x,y\right)$ satisfy the following descending and ascending recurrence relations, respectively:

$$DU_n\left(\beta,\gamma,x,y\right)=nU_{n-1}\left(\beta,\gamma,x,y\right). \tag{1.1}$$

$$\begin{aligned}DU_n(\beta,\gamma,x,y)=\frac{1}{y(x-y)}&\{(\gamma+n)U_{n+1}(\beta,\gamma,x,y)\\&+[(n+\beta)x-(\gamma+2n)y]U_n(\beta,\gamma,x,y)\}.\end{aligned} \tag{1.2}$$

These two independent differential recurrence relations determine the second order linear ordinary differential equation

$$\begin{aligned}&y\left(x-y\right)D^2U_n\left(x,y\right)-\left[\left(n+\beta-1\right)x-\left(\gamma+2n-2\right)y\right]DU_n\left(x,y\right)\\&-n\left(\gamma+n-1\right)U_n\left(x,y\right)=0,\end{aligned} \tag{1.3}$$

where $D=\frac{d}{dy}$. The proof of these results are obvious.

## 2. Generating funcion derived from the ascending recurrence relation

We shell use the Truesdell's $F$-equation

$$\frac{\partial}{\partial t}F\left(z,\alpha\right)=F\left(t,\alpha+1\right). \tag{2.1}$$

To find the generating function for the ser of polynomials $U_n\left(\beta,\gamma,x,y\right)$ as follows:

The polynomials $U_n\left(\beta,\gamma,x,y\right)$ satisfies the asending recurrence relation

$$\begin{aligned}\frac{d}{dy}U_n(\beta,\gamma,x,y)=\frac{1}{y(x-y)}&\{(\gamma+n)U_{n+1}(\beta,\gamma,x,y)\\&+[(n+\beta)x-(\gamma+2n)y]U_n(\beta,\gamma,x,y)\}.\end{aligned} \tag{2.2}$$

Let $f(z,\alpha) = U_n(\beta,\gamma,x,y)$, so that we have

$$\frac{\partial}{\partial z}f(z,\alpha) = \frac{[(\alpha+\beta)-(\gamma+2\alpha)z]}{z(x-z)}U_\alpha(\beta,\gamma,x,y)\}$$

(2.3)
$$+\frac{(\gamma+\alpha)}{z(x-z)}U_{\alpha+1)}(\beta;\gamma;x,y).$$

This equation is called the f-type equation and can be written as

$$\frac{\partial}{\partial z}f(z,\alpha) = A(z,\alpha)f(z,\alpha) + B(z,\alpha)f(z,\alpha+1),$$

with $A(z,\alpha) = \frac{[(\alpha+\beta)-(\gamma+2\alpha)z]}{z(x-z)}$ and $B(z,\alpha) = \frac{(\gamma+\alpha)}{z(x-z)}$.

$$\frac{\partial}{\partial z}g(z,\alpha) = C(z,\alpha)g(z,\alpha+1)$$

by supposing

$$g(z,\alpha) = f(z,\alpha)\exp\left\{-\int_{z_0}^{z}A(\nu,\alpha)d\nu\right\}$$
$$= f(z,\alpha)\, z_0^{\alpha+\beta} z^{-\alpha-\beta}(x-z_0)^{\alpha-\beta+\gamma}(x-z)^{-\alpha+\beta-\gamma}$$

and by choosing $z_0 = 1$, then we get

(2.4)
$$g(z,\alpha) = (x-1)^{\alpha-\beta+\gamma} z^{-\alpha-\beta}(x-z)^{\beta-\gamma-\alpha} f(z,\alpha).$$

Now, it can easily verified that this satisfies g-type equation

(2.5)
$$\frac{\partial}{\partial z}g(z,\alpha) = (\alpha+\gamma)(x-1)^{-1}g(z,\alpha+1).$$

Let $C(z,\alpha)$ denote the factorable coefficient of $g(z,\alpha+1)$ in (2.5), then

$$C(z,\alpha) = (\alpha+\gamma)(x-1)^{-1}$$

with $C(z,\alpha) = A(\alpha)Z(z)$. Where $A(\alpha) = (\alpha+\gamma)(x-1)^{-1}$ and $Z(z) = 1$. We effect the transformation of $f(z,\alpha)$ into $F(t,\alpha)$ by letting

(2.6)
$$t = -\int_{z_1}^{z} Z(\nu)d\nu = z - z_1$$

and $F_0 F(t,\alpha) = g(z,\alpha)\exp\left\{\int_{\alpha_0}^{\alpha}\log A(\nu)\Delta\nu\right\}$ on choosing $\alpha_0 = -\gamma$, we get $\int_{-\gamma}^{\alpha}\left\{(V+\gamma)(x-1)^{-1}\right\}\Delta V = \int_{-\gamma}^{\alpha}\log\left((x-1)^{-1}\right) + \log(V+\gamma)\Delta V$ Since

(2.7)
$$\int_0^x \log z\Delta z = \log\Gamma(x) - \log\sqrt{2\pi}$$

we have $\int_{-\gamma}^{\alpha}\left\{(V+\gamma)(x-1)^{-1}\right\}\Delta V=\log\left[\frac{\Gamma(\alpha+\gamma)\{(x-1)\}^{-(\alpha+\gamma)}}{\sqrt{2\pi}}\right]$. This implies

$$F_0F\,(t,\alpha)=g\,(z,\alpha)\exp\left\{\log\left[\frac{\Gamma(\alpha+\gamma)\,\{(x-1)\}^{-(\alpha+\gamma)}}{\sqrt{2\pi}}\right]\right\}.$$

Now by choosing $\alpha_0=-\gamma, z_1=v$ and $F_0=\frac{1}{\sqrt{2\pi}}$, we get

$$F(t,\alpha)=\Gamma(\alpha+\gamma)(x-1)^{-(\alpha+\gamma)}g(t+v,\alpha). \tag{2.8}$$

To show that $F(t,\alpha)$ does indeed satisfy the F-equation we determine $\frac{\partial}{\partial t}F(t,\alpha)$ as follows:

$$\begin{aligned}F(t,\alpha)&=\Gamma(\alpha+\gamma)(x-1)^{-(\alpha+\gamma)}\frac{\partial g(t+v,\alpha)}{\partial(t+v)}\frac{\partial g(t+v)}{\partial t}\\&=\Gamma(\alpha+\gamma+1)(x-1)^{-(\alpha+\gamma+1)}g(v+t,\alpha+1).\end{aligned} \tag{2.9}$$

Therefore,

$$\frac{\partial}{\partial t}F(t,\alpha)=F(t,\alpha+1).$$

For later use express $F(t,\alpha)$ in the following form:

$$\begin{aligned}F(t,\alpha)&=\Gamma(\alpha+\gamma)(x-1)^{-(\alpha+\gamma)}g(t+v,\alpha)\\&=\Gamma(\alpha+\gamma)(x-1)^{-(\alpha+\gamma)}(x-1)^{\alpha+\beta-\gamma}(v+t)^{-\alpha-\beta}\\&(x-v-t)^{-(\alpha-\beta+\gamma)}f(v+t,\alpha)\\&=\Gamma(\alpha+\gamma)(x-1)^{-\beta}(v+t)^{-(\alpha+\beta)}\\&(x-v-t)^{-(\alpha-\beta+\gamma)}U_\alpha(\beta;\gamma;x,v+t).\end{aligned} \tag{2.10}$$

Thus

$$\begin{aligned}F(t+z,\alpha)&=\Gamma(\alpha+\gamma)(x-1)^{-\beta}(v+t+z)^{-(\alpha+\beta)}\\&(x-v-t-z)^{-(\alpha-\beta+\gamma)}U_\alpha(\beta;\gamma;x,v+t+z).\end{aligned} \tag{2.11}$$

$$\begin{aligned}F(t,\alpha+n)&=\Gamma(\alpha+n+\gamma)(x-1)^{-\beta}(v+t)^{-(\alpha+n+\beta)}\\&(x-v-t)^{-(\alpha-\beta+\gamma-n)}U_{\alpha+n}(\beta;\gamma;x,v+t).\end{aligned} \tag{2.12}$$

Now let us apply Truesdell's generating function theorem, if the function $F(t,\alpha)$. Satisfies the F-equation and $f(t+z,\alpha)$ possesses a Taylor's series in power of z, then this series may be expressed as

$$F(t+z,\alpha)=\sum_{n=0}^{\infty}\frac{z^n}{n!}F(t,\alpha+n). \tag{2.13}$$

It follows that

$$(v+t+z)^{-(\alpha+\beta)}(x-v-t-z)^{-(\alpha-\beta+\gamma)}U_\alpha(\beta;\gamma;x,v+t+z)$$
$$\sum_{n=0}^{\infty}\frac{\Gamma(\alpha+n+\gamma)}{\Gamma(\alpha+\gamma)}(v+t)^{-(\alpha+n+\beta)}(x-v-t)^{-(\alpha-\beta+\gamma-n)}U_{\alpha+n}(\beta;\gamma;x,v+t).$$

Now replacing $v+t$ by y and z by $yt(x-y)$, we get the generating relation

$$(1-yt)^{-(\alpha-\beta+\gamma)}\left\{1+t(x-y)\right\}^{-(\alpha+\beta)}U_\alpha\left[\beta;\gamma;x,y+yt(x-y)\right]$$
$$\sum_{n=0}^{\infty}\frac{(\gamma+\alpha)_n}{n!}U_{\alpha+n}(\beta;\gamma;x,y)t^n. \tag{2.14}$$

Which is a generating relation for $U_n(\beta;\gamma;x,y)$.

## 3. Generating function derived from the descending recurrence relation

Similarly, by using the Truesdell's G-equation

$$\frac{\partial}{\partial t}G(t,\alpha)=G(t,\alpha-1). \tag{3.1}$$

We have derived a generating relation for the set of polynomials $U_{\alpha-n}\left(\beta;\gamma;x,y\right)$ as follows:

The decsending recurrence relation for $U_n\left(\beta;\gamma;x,y\right)$ is

$$\frac{d}{dy}U_n\left(\beta;\gamma;x,y\right)=nU_{n-1}\left(\beta;\gamma;x,y\right). \tag{3.2}$$

Let $f(z,\alpha)=U_n\left(\beta;\gamma;x,y\right)$, so that we have

$$\frac{\partial}{\partial z}f(z,\alpha)=\alpha f(z,\alpha-1). \tag{3.3}$$

This equation is called the F-type equation and can be written as

$$\frac{\partial}{\partial z}f(z,\alpha)=A(z,\alpha)f(z,\alpha)+B(z,\alpha)f(z,\alpha-1)$$

with $A(z,\alpha)=0$ and $B(z,\alpha)=\alpha$.

Now let us transform $f(z,\alpha)$ into $g(z,\alpha)$ so that

$$\frac{\partial}{\partial z}g(z,\alpha)=C(z,\alpha)g(z,\alpha-1).$$

Let us suppose that

$$g(z,\alpha)=f(z,\alpha)\exp\left\{-\int_{z_0}^{z}A(\nu,\alpha)d\nu\right\}=\exp(c)f(z,\alpha),$$

where $c$ being an integration constant.

Now in particular, if we write $z_0 = 0$, then we get

$$g(z,\alpha) = \exp(c) f(z,\alpha).$$

Now

$$\begin{aligned} g(z,\alpha) &= \exp(c) f(z,\alpha) \\ &= \alpha \exp(c) f(z,\alpha-1) \\ &= \alpha g(z,\alpha-1). \end{aligned} \tag{3.4}$$

Let $C(z,\alpha)$ denote the factorable coefficient of $g(z,\alpha-1)$,then $C(z,\alpha) = \alpha$ with $Z(z) = 1$ and $A(\alpha) = \alpha$. Further let us effect the transformation of $g(z,\alpha)$ into $g(t,\alpha)$ by letting

$$t = -\int_{z_1}^{z} Z(\nu) d\nu = z - z_1$$

and

$$\begin{aligned} G_0 G(t,\alpha) &= g(z,\alpha) \exp\left\{ -\int_{\alpha_0}^{\alpha+1} \log A(\nu) \Delta\nu \right\} \\ &= g(z,\alpha) \exp\left\{ -\int_{\alpha_0}^{\alpha+1} \log \nu \Delta\nu \right\}. \end{aligned}$$

In particular if we choose $\alpha_0 = 0$ then we have

$$\begin{aligned} G_0 G(t,\alpha) &= g(z,\alpha) \exp\left\{ -\int_{0}^{\alpha+1} \log A(\nu) \Delta\nu \right\} \\ &= g(z,\alpha) \exp\left\{ -\log \frac{\Gamma(\alpha+1)}{\sqrt{2\pi}} \right\} \\ &= g(z,\alpha) \frac{\sqrt{2\pi}}{\Gamma(\alpha+1)}. \end{aligned}$$

Suppose $z_1 = \nu, \alpha_0 = 0$ and $G_0 = \sqrt{2\pi}$ then

$$G(t,\alpha) = \frac{\exp c}{\Gamma(\alpha+1)} g(\nu+t,\alpha).$$

To show that $G(t,\alpha)$ does indeed satisfy the G-equation we determine $\frac{\partial}{\partial t} G(t,\alpha)$ as follows:

$$\begin{aligned} \frac{\partial}{\partial t} G(t,\alpha) &= \frac{\exp c}{\Gamma(\alpha+1)} \frac{\partial g(\nu+t,\alpha)}{\partial(\nu+t)} \frac{\partial(\nu+t)}{\partial t} \\ &= \frac{\exp c}{\Gamma(\alpha+1)} g(\nu+t,\alpha-1). \end{aligned}$$

Therefore,

$$\frac{\partial}{\partial t} G(t-\alpha).$$

For later use let us express $G(t,\alpha)$ in the following form:

$$\begin{aligned} G(t,\alpha) &= \frac{\exp c}{\Gamma(\alpha+1)} g(\nu+t,\alpha) \\ &= \frac{\exp(2c)}{\Gamma(\alpha+1)} f(\nu+t,\alpha) \\ &= \frac{\exp c}{\Gamma(\alpha+1)} U_\alpha(\beta;\gamma;x,\nu+t). \end{aligned}$$

Also

$$G(t+z,\alpha) = \frac{\exp(2c)}{\Gamma(\alpha+1)} U_\alpha(\beta;\gamma;x,\nu+t+z)$$

and

$$G(t,\alpha-n) = \frac{\exp(2c)}{\Gamma(\alpha-n+1)} U_{\alpha-n}(\beta;\gamma;x,\nu+t).$$

By Tailor's series in powers of $z$

$$G(t+z,\alpha) = \sum_{n=0}^{\infty} \frac{z^n}{n!} G(t,\alpha-n). \tag{3.5}$$

Which implies that

$$\begin{aligned} &\frac{\exp(2c)}{\Gamma(\alpha+1)} U_\alpha(\beta;\gamma;x,\nu+t+z) \\ &= \sum_{n=0}^{\infty} \frac{z^n}{n!} \frac{\exp(2c)}{\Gamma(\alpha-n+1)} U_{\alpha-n}(\beta;\gamma;x,\nu+t) \end{aligned}$$

or

$$U_\alpha(\beta;\gamma;x,\nu+t+z) = \sum_{n=0}^{\infty} \frac{(-1)^n(-\alpha)_n}{n!} U_{\alpha-n}(\beta;\gamma;x,\nu+t).$$

Now replacing $\nu+t$ by $y$ and $z$ by $-t$, we get

$$U_\alpha(\beta;\gamma;x,y-t) = \sum_{n=0}^{\infty} \frac{(-\alpha)_n}{n!} U_{\alpha-n}(\beta;\gamma;x,y)t^n.$$

which is the another generating relation for

$$U_n(\beta;\gamma;x,y). \tag{3.6}$$

## 4. Applications

From the relations (2.14) and (3.6),we can derive the following generating functions for Laguerre polynomials of two variables:

$$\begin{aligned} &(1-yt)^{-n-\alpha-1} \exp\left(\frac{-tx}{(1-ty)}\right) L_n^{(\alpha)}\left(x, \frac{x}{1-ty}\right) \\ &= \sum_{l=0}^{n} \frac{(1+n)_l}{l!} L_{n+l}^{(\alpha)}(x) t^l. \end{aligned} \tag{1}$$

$$(2) \qquad (1-t)^n L_n^{(\alpha)}\left(x, \frac{1-t}{y}\right) = \sum_{l=0}^{n} \frac{(\alpha-n)_l}{l!} L_{n-1}^{(\alpha)}(x,y)t^l.$$

## 5. Conclusion

Generating functions involving generalized Hypergeometric 2D polynomials are derived by Truesdell's Method. Certain known generating relations to two variable Laguerre polynomials are discussed as applications. The applications of generalized Hypergeometric 2D polynomials in communications including wireless, mobile, and satellite communications with new ideas and approaches to design communications system with high performance in comparison with employed communication systems is the further scope of this research.

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