PRODUCT-TYPE OPERATORS FROM AREA NEVANLINNA SPACES TO BLOCH-ORLICZ SPACES

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Abstract. Let $\mathbb{D}$ be the unit disk in the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ the class of all analytic functions on $\mathbb{D}$. Let φ be an analytic self-map of $\mathbb{D}$ and $u \in H(\mathbb{D})$. By constructing some more effective test functions in area Nevanlinna space, in this paper we characterize the boundedness and compactness of product-type operators $D^n M_u C_\varphi$, $D^n C_\varphi M_u$, $C_\varphi D^n M_u$, $M_u D^n C_\varphi$, $M_u C_\varphi D^n$ and $C_\varphi M_u D^n$ from area Nevanlinna spaces to Bloch-Orlicz spaces.
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1. Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ the class of all analytic functions on $\mathbb{D}$. Let φ be an analytic self-map of $\mathbb{D}$ and $u \in H(\mathbb{D})$. The weighted composition operator $W_{\varphi,u}$ on $H(\mathbb{D})$ is defined by

$$W_{\varphi,u} f(z) = u(z) f(\varphi(z)),\ z \in \mathbb{D}.$$

If $u \equiv 1$, it is reduced to the composition operator, usually denoted by C_φ. While if $\varphi(z) = z$, it is reduced to the multiplication operator, usually denoted by M_u. Since $W_{\varphi,u} = M_u C_\varphi$, it can be regarded as a product-type operator. It is a standard problem how to provide function theoretic characterizations when φ and u induce a bounded or compact weighted composition operator (see, e.g., [2, 4, 5, 18, 21, 23] and the references therein).

Let D be the differentiation operator on $H(\mathbb{D})$, that is

$$Df(z) = f'(z),\ z \in \mathbb{D}.$$

A systematic study of other product-type operators started with Stević et al. in [14, 16]. Before that there were a few papers in the topic, e.g., [6]. The next two product-type operators DC_φ and $C_\varphi D$, attracted some attention first (see, e.g., [17, 19, 27, 29] and the references therein). The publication of [16] attracted some attention in product-type operators involving integral-type ones (see, e.g., [10, 28, 30, 31, 34] and the references therein). Since that time there has been a great interest in various product-type operators. For example, the following six operators

$$M_uC_\varphi D,\ C_\varphi M_uD,\ M_uDC_\varphi,\ C_\varphi DM_u,\ DC_\varphi M_u,\ DM_uC_\varphi \tag{1}$$

were studied by Sharma in [25]. The product-type operators $W_{\varphi,u}D$ and $DW_{\varphi,u}$, which were considered by Jiang in [7, 8], are included in (1) as the first and sixth operators, respectively. For some other product-type operators, we refer the reader to [12, 13, 15, 20, 22, 24, 39] and the references therein.

The nth differentiation operator D^n on $H(\mathbb{D})$ is defined by

$$D^nf(z) = f^{(n)}(z),\ z \in \mathbb{D}.$$

Zhu in [38] introduced the so-called generalized weighted composition operator:

$$D^n_{\varphi,u}f(z) = u(z)f^{(n)}(\varphi(z)),\ z \in \mathbb{D}.$$

Since $D^n_{\varphi,u} = M_uC_\varphi D^n$, it is also a product-type operator.

The product-type operator $M_uC_\varphi D^n$ from area Nevanlinna space to Bloch-type and Zygmund spaces was studied by Yang et al. in [35, 36]. The weighted composition operator from weighted Bergman-Nevanlinna space to Zygmund and Bloch-type spaces was also studied in [11, 26]. It must be mentioned that in these studies, there is no need to construct more complex test functions in area Nevanlinna space or weighted Bergman-Nevanlinna space. But we find that the used test functions become invalid in the study of the following product-type operators

$$D^nM_uC_\varphi,\ D^nC_\varphi M_u,\ C_\varphi D^nM_u,\ M_uD^nC_\varphi,\ M_uC_\varphi D^n,\ C_\varphi M_uD^n. \tag{2}$$

By constructing some more suitable test functions in area Nevanlinna space, in this paper we characterize the boundedness and compactness of the operators in (2) from area Nevanlinna space to Bloch-Orlicz space. This paper can be regarded as a continuation of the investigation of concrete operators between these spaces.

Let $dA(z) = \frac{1}{\pi}dxdy$ be the normalized Lebesgue measure on $\mathbb{D}$ and $dA_\alpha(z) = (\alpha+1)(1-|z|^2)^\alpha dA(z)$ the weighted Lebesgue measure on $\mathbb{D}$. For $\alpha > -1$ and $p \geq 1$, the area Nevanlinna space $\mathcal{N}^p_\alpha$ consists of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{\mathcal{N}^p_\alpha} = \int_{\mathbb{D}} \big[\log(1+|f(z)|)\big]^p dA_\alpha(z) < \infty.$$

From [3], we see that the area Nevanlinna space $\mathcal{N}^p_\alpha$ is a Fréchet space with the translation invariant metric given by $d(f,g) = \|f-g\|_{\mathcal{N}^p_\alpha}$. If $p = 1$, it becomes the weighted Bergman-Nevanlinna space, usually denoted by $\mathcal{A}^\alpha_{\log}$ (see, [11, 26]).

Let Ψ be a strictly increasing convex function on $[0,+\infty)$ such that $\Psi(0)=0$. The Bloch-Orlicz space $\mathcal{B}^\Psi$ was introduced in [21] by Ramos Fernández, is the class of all $f\in H(\mathbb{D})$ such that

$$\sup_{z\in\mathbb{D}}(1-|z|^2)\Psi(\lambda|f'(z)|)<\infty$$

for some $\lambda>0$ depending on f. Ramos Fernández in [21] proved that $\mathcal{B}^\Psi$ is isometrically equal to μ_Ψ-Bloch space, where

$$\mu_\Psi(z)=\frac{1}{\Psi^{-1}(\frac{1}{1-|z|^2})},\ \ z\in\mathbb{D}.$$

Hence, $\mathcal{B}^\Psi$ is a Banach space with the norm given by $\|f\|_{\mathcal{B}^\Psi}=|f(0)|+b_{\mu_\Psi}(f)$, where

$$b_{\mu_\Psi}(f)=\sup_{z\in\mathbb{D}}\mu_\Psi(z)|f'(z)|.$$

This space generalizes some other spaces. For example, if $\Psi(t)=t^p$ with $p>0$, then the space $\mathcal{B}^\Psi$ coincides with the weighted Bloch space $\mathcal{B}_\alpha$, where $\alpha=1/p$. Also, if $\Psi(t)=t\log(1+t)$, then $\mathcal{B}^\Psi$ coincides with the Log-Bloch space (see [1]).

Let X be a topological vector space whose topology is given by the translation invariant metric d_X. A linear operator $L:X\to\mathcal{B}^\Psi$ is metrically bounded if there exists a positive constant K such that

$$\|Lf\|_{\mathcal{B}^\Psi}\le Kd_X(f,0)$$

for all $f\in X$. The operator $L:X\to\mathcal{B}^\Psi$ is metrically compact if it maps bounded sets into relatively compact sets.

In this paper, an operator is bounded (respectively, compact), if it is metrically bounded (respectively, metrically compact). Constants are denoted by C, they are positive and may differ from one occurrence to the next. The notation $a\lesssim b$ means that there exists a positive constant C such that $a\le Cb$.

2. Auxiliary results

In [33], Stević used the Faà di Bruno's formula of the following version

$$(f\circ\varphi)^{(n)}(z)=\sum_{k=0}^{n}f^{(k)}(\varphi(z))B_{n,k}(\varphi'(z),\dots,\varphi^{(n-k+1)}(z)),\tag{3}$$

where $B_{n,k}(x_1,\dots,x_{n-k+1})$ is the Bell polynomial. For $n\in\mathbb{N}$, the sum can go from $k=1$ since $B_{n,0}(\varphi'(z),\dots,\varphi^{(n+1)}(z))=0$, however we will keep the summation since for $n=0$ the only existing term $B_{0,0}=1$.

Now we present some useful information of the Bell polynomials from [9]. The Bell polynomials are associated with set partitions. To the partition $\{1\}$ we associate the monomial x_1; this is the only partition of the set $\{1\}$, and we define

$B_{1,1}(x_1) = x_1$. The set $\{1,2\}$ has the two partitions $\{1,2\}$ and $\{1\}$, $\{2\}$, the former with one block and the latter with two, and we associate to them the monomials x_2 and x_1^2, respectively. Then $B_{2,1}(x_1,x_2) = x_2$ and $B_{2,2}(x_1,x_2) = x_1^2$. There are five partitions of the set $\{1,2,3\}$. Three of these have two blocks, namely $\{1,2\}$, $\{3\}$ and $\{1,3\}$, $\{2\}$ and $\{1\}$, $\{2,3\}$; we associate the monomial x_1x_2 to each of these, and so $B_{3,2}(x_1,x_2) = 3x_1x_2$. The other Bell polynomials of order three are $B_{3,3}(x_1) = x_1^3$, corresponding to $\{1\}$, $\{2\}$, $\{3\}$; and $B_{3,1}(x_1,x_2,x_3) = x_3$, corresponding to $\{1,2,3\}$. In general,

$$B_{n,k}(x_1,x_2,\dots,x_{n-k+1}) = \frac{1}{k!}\sum_{\substack{j_1+\cdots+j_k=n\\ j_i\geq 1}} \binom{n}{j_1,j_2,\dots,j_k} x_{j_1}\cdots x_{j_k}.$$

The sum is effectively over set partitions of $\{1,2,\dots,n\}$ with block sizes j_1, …, j_k, with the factor $1/k!$ correcting for the multiple counting inside the sum.

From (3) and the Leibnitz formula the next result follows.

Lemma 2.1. *Let f, $u \in H(\mathbb{D})$ and φ be an analytic self-map of $\mathbb{D}$. Then*

$$\begin{aligned}&\big(u(z)f(\varphi(z))\big)^{(m)}\\ &= \sum_{k=0}^{m} f^{(k)}(\varphi(z))\sum_{j=k}^{m} C_m^j u^{(m-j)}(z)B_{j,k}\big(\varphi'(z),\dots,\varphi^{(j-k+1)}(z)\big).\end{aligned} \tag{4}$$

To find some useful test functions, we first introduce the following functions. For a fixed $w \in \mathbb{D}$ and $i \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}$, we define the function

$$k_{w,i}(z) = \frac{(1-|w|^2)^{\frac{\alpha+2}{p}+i}}{(1-\overline{w}z)^{\frac{2(\alpha+2)}{p}+i}}, \quad z \in \mathbb{D}.$$

Then from Lemma 4.2.2 in [37], it follows that

$$\sup_{w\in\mathbb{D}}\int_{\mathbb{D}} |k_{w,i}(z)|^p dA_\alpha(z) \lesssim 1. \tag{5}$$

By using the functions $k_{w,i}$, the following result provides some useful test functions.

Lemma 2.2. *Let $w \in \mathbb{D}$ and $m \in \mathbb{N}$. Then for each fixed $k \in \{0,1,\dots,m\}$, there exist two groups of constants $a_{0,k}$, $a_{1,k}$,…, $a_{m,k}$ and $b_{0,k}$, $b_{1,k}$,…, $b_{m,k}$, such that the function*

$$h_{w,k}(z) = \sum_{i=0}^{m} a_{i,k}k_{w,i}(z)\exp\sum_{i=0}^{m} b_{i,k}k_{w,i}(z) \tag{6}$$

satisfies

$$h_{w,k}^{(k)}(w) = \frac{\overline{w}^k}{(1-|w|^2)^{\frac{\alpha+2}{p}+k}}\exp\frac{1}{(1-|w|^2)^{\frac{\alpha+2}{p}}} \quad \text{and} \quad h_{w,k}^{(j)}(w) = 0 \tag{7}$$

for each $j \in \{0, 1, \ldots, m\} \setminus \{k\}$. *Moreover,*

$$\sup_{w\in\mathbb{D}} \|h_{w,k}\|_{\mathcal{N}^p_\alpha} \lesssim 1.$$

Proof. First we prove that if the function $h_{w,k}$ has the expression (6), then $h_{w,k} \in \mathcal{N}^p_\alpha$ and $\sup\limits_{w\in\mathbb{D}} \|h_{w,k}\|_{\mathcal{N}^p_\alpha} \lesssim 1$. Indeed, from the facts that

$$\log(1 + xy) \le \log(1 + x) + \log(1 + y) \;\text{ and }\; \log(1 + x) \le x,\; x, y > 0,$$

we have

$$\begin{aligned}\log(1 + |h_{w,k}(z)|) &= \log\Big(1 + \big|\sum_{i=0}^{m} a_{i,k}k_{w,i}(z)\big|\big|\exp\sum_{i=0}^{m} b_{i,k}k_{w,i}(z)\big|\Big)\\ &\le \big|\sum_{i=0}^{m} a_{i,k}k_{w,i}(z)\big| + \log\Big(1 + \exp\big|\sum_{i=0}^{m} b_{i,k}k_{w,i}(z)\big|\Big)\\ &\le 1 + \sum_{i=0}^{m}(|a_{i,k}| + |b_{i,k}|)|k_{w,i}(z)|.\end{aligned}$$

Then from this, (5) and a elementary inequality, we get

$$\begin{aligned}\|h_{w,k}\|_{\mathcal{N}^p_\alpha} &\le \int_{\mathbb{D}} \Big(1 + \sum_{i=0}^{m}(|a_{i,k}| + |b_{i,k}|)|k_{w,i}(z)|\Big)^p dA_\alpha(z)\\ &\le (m + 2)^p\Big(1 + \sum_{i=0}^{m}(|a_{i,k}| + |b_{i,k}|)^p \int_{\mathbb{D}} |k_{w,i}(z)|^p dA_\alpha(z)\Big)\\ &\le C.\end{aligned}$$

From this, the desired result follows.

Write

$$f_{w,k}(z) := \sum_{i=0}^{m} a_{i,k}k_{w,i}(z) \;\text{ and }\; g_{w,k}(z) := \sum_{i=0}^{m} b_{i,k}k_{w,i}(z).$$

We first consider the case of $k = 0$. By some calculation, we obtain that the function $h_{w,0}$ satisfies (7) if and only if the functions $f_{w,0}$ and $g_{w,0}$ satisfy the following systems

$$\begin{cases} f_{w,0}(w) = \frac{1}{(1-|w|^2)^a}\\ f'_{w,0}(w) = 0\\ f''_{w,0}(w) = 0\\ \quad\cdots\cdots\\ f^{(m)}_{w,0}(w) = 0\end{cases} \tag{8}$$

and

$$(9)\qquad \begin{cases} g_{w,0}(w) = \frac{1}{(1-|w|^2)^a} \\ g'_{w,0}(w) = 0 \\ g''_{w,0}(w) = 0 \\ \quad \cdots\cdots \\ g^{(m)}_{w,0}(w) = 0, \end{cases}$$

respectively, where $a := (\alpha + 2)/p$. According to the expressions of $f_{w,0}$ and $g_{w,0}$, by calculating $f^{(j)}_{w,0}(w)$ and $g^{(j)}_{w,0}(w)$ we see that $f_{w,0}$ and $g_{w,0}$ satisfy the systems (8) and (9), respectively, if and only if unknowns $a_{0,0}$, $a_{1,0}$, $\dots$, $a_{m,0}$, $b_{0,0}$, $b_{1,0}$, $\dots$, $b_{m,0}$, satisfy the following linear system of equations

$$(10)\qquad \begin{cases} \sum\limits_{i=0}^{m} a_{i,0} = 1 \\ \sum\limits_{i=0}^{m} (2a+i)a_{i,0} = 0 \\ \sum\limits_{i=0}^{m} (2a+i)(2a+i+1)a_{i,0} = 0 \\ \qquad\qquad \cdots\cdots \\ \sum\limits_{i=0}^{m} \prod\limits_{j=1}^{m} (2a+i+j-1)a_{i,0} = 0 \end{cases}$$

and

$$(11)\qquad \begin{cases} \sum\limits_{i=0}^{m} b_{i,0} = 1 \\ \sum\limits_{i=0}^{m} (2a+i)b_{i,0} = 0 \\ \sum\limits_{i=0}^{m} (2a+i)(2a+i+1)b_{i,0} = 0 \\ \qquad\qquad \cdots\cdots \\ \sum\limits_{i=0}^{m} \prod\limits_{j=1}^{m} (2a+i+j-1)b_{i,0} = 0, \end{cases}$$

respectively. Hence we only need to prove that (10) and (11) have nonzero solutions, respectively. Indeed, if we can show that the determinants of the systems (10) and (11) are nonzero, then (10) and (11) have nonzero solutions, respectively. By Lemma 3 in [33], the determinants of the systems (10) and (11) equal $\prod_{j=1}^{m} j!$, respectively, which is different from zero. This finishes the proof of the lemma for the case $k = 0$.

We next consider the case of $k \neq 0$. From (4) we have

$$
\begin{aligned}
h_{w,k}^{(k)}(z) &= \sum_{j=0}^{k}\sum_{i=j}^{k} C_k^i f_{w,k}^{(k-i)}(z) B_{i,j}(g'_{w,k}(z),\dots,g_{w,k}^{(i-j+1)}(z)) \exp g_{w,k}(z) \\
&= f_{w,k}^{(k)}(z)\exp g_{w,k}(z) \\
&+ \sum_{j=1}^{k}\sum_{i=j}^{k} C_k^i f_{w,k}^{(k-i)}(z) B_{i,j}(g'_{w,k}(z),\dots,g_{w,k}^{(i-j+1)}(z)) \exp g_{w,k}(z) \\
&\quad + \sum_{i=1}^{k} C_k^i f_{w,k}^{(k-i)}(z) B_{i,j}(g'_{w,k}(z),\dots,g_{w,k}^{(i-j+1)}(z)) \exp g_{w,k}(z).
\end{aligned}
\tag{12}
$$

From (12) we see that if $f_{w,k}^{(l)}(w) = 0$ for all $l < k$, then $h_{w,k}^{(l)}(w) = 0$. On the other hand, for all $s > k$, from (4) we have

$$
\begin{aligned}
h_{w,k}^{(s)}(z) &= \sum_{j=0}^{s}\sum_{i=j}^{s} C_s^i f_{w,k}^{(s-i)}(z) B_{i,j}(g'_{w,k}(z),\dots,g_{w,k}^{(i-j+1)}(z)) \exp g_{w,k}(z) \\
&= C_s^k f_{w,k}^{(k)}(z)\exp g_{w,k}(z) \sum_{j=0}^{s-k} B_{s-k,j}(g'_{w,k}(z),\dots,g_{w,k}^{(s-k-j+1)}(z)) \\
&+ \sum_{j=0}^{s-k}\sum_{i=j,i\neq s-k}^{s} C_s^i f_{w,k}^{(s-i)}(z) B_{i,j}(g'_{w,k}(z),\dots,g_{w,k}^{(i-j+1)}(z)) \exp g_{w,k}(z) \\
&+ \sum_{j=s-k+1}^{s}\sum_{i=j}^{s} C_s^i f_{w,k}^{(s-i)}(z) B_{i,j}(g'_{w,k}(z),\dots,g_{w,k}^{(i-j+1)}(z)) \exp g_{w,k}(z).
\end{aligned}
\tag{13}
$$

From (13), for each $s > k$ we see that if $g'_{w,k}(w) = 0, \dots, g_{w,k}^{(s-k+1)}(w) = 0$, $f_{w,k}(w) = 0$, $f'_{w,k}(w) = 0, \dots, f_{w,k}^{(k-1)}(w) = 0$, $f_{w,k}^{(k+1)} = 0, \dots, f_{w,k}^{(s)}(w) = 0$, then $h_{w,k}^{(s)}(w) = 0$.

Now letting $s = k+1, \dots, m$, and noticing condition (7), we see that if we can prove that there exist two groups of constants $a_{0,k}, a_{1,k}, \dots, a_{m,k}$ and $b_{0,k}$, $b_{1,k}, \dots, b_{m,k}$, such that the following systems hold

$$
\begin{cases}
\sum_{i=0}^{m} a_{i,k} = 0, \quad \sum_{i=0}^{m} (2a+i)a_{i,k} = 0 \\
\sum_{i=0}^{m} (2a+i)(2a+i+1)a_{i,k} = 0 \\
\qquad \cdots\cdots \\
\sum_{i=0}^{m}\prod_{j=1}^{k} (2a+i+j-1)a_{i,k} = 1 \\
\qquad \cdots\cdots \\
\sum_{i=0}^{m}\prod_{j=1}^{m} (2a+i+j-1)a_{i,k} = 0
\end{cases}
\tag{14}
$$

and

$$
\begin{cases}
\sum\limits_{i=0}^{m} b_{i,k} = 1\\
\sum\limits_{i=0}^{m} (2a+i)b_{i,k} = 0\\
\sum\limits_{i=0}^{m} (2a+i)(2a+i+1)b_{i,k} = 0\\
\qquad\qquad \cdots\cdots\\
\sum\limits_{i=0}^{m} \prod\limits_{j=1}^{m-k+1} (2a+i+j-1)b_{i,k} = 0,
\end{cases}
\tag{15}
$$

then this finishes the proof of the lemma for the case of $k \neq 0$. From Lemma 3 in [33] and a calculation, we get that the determinant of the system (14) equals $(-1)^{k-1}\prod_{j=1}^{m} j!$, which is different from zero. By this, there must exist $a_{0,k}$, $a_{1,k}$, $\ldots$, $a_{m,k}$ such that the system (14) holds. Since the number of equations in the system (15) is less than the number of variables, there must exist $b_{0,k}$, $b_{1,k}$, $\ldots$, $b_{m,k}$, such that the system (15) holds. $\square$

To characterize the compactness, we need the following result, which is proved in a standard way (see [23]). So the proof is omitted.

Lemma 2.3. *Let $T \in \{D^nM_uC_\varphi, D^nC_\varphi M_u, C_\varphi D^nM_u, M_uD^nC_\varphi, M_uC_\varphi D^n, C_\varphi M_uD^n\}$. Then the bounded operator $T : \mathcal{N}_\alpha^p \to \mathcal{B}^\Psi$ is compact if and only if for every bounded sequence $\{f_j\}$ in $\mathcal{N}_\alpha^p$ such that $f_j \to 0$ uniformly on every compact subset of $\mathbb{D}$ as $j \to \infty$, it follows that*

$$\lim_{j\to\infty} \|Tf_j\|_{\mathcal{B}^\Psi} = 0.$$

We need also the following estimate for derivative of functions in area Nevanlinna spaces. We refer the reader to [36] for a complete proof.

Lemma 2.4. *For a fixed $k \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}$, there exists a positive constant $C_k = C(\alpha, p, k)$ independent of $f \in \mathcal{N}_\alpha^p$ and $z \in \mathbb{D}$ such that*

$$|f^{(k)}(z)| \leq \frac{1}{(1-|z|^2)^k} \exp \frac{C_k\|f\|_{\mathcal{N}_\alpha^p}}{(1-|z|^2)^{\frac{\alpha+2}{p}}}.$$

3. Boundedness and compactness

We first characterize the boundedness and compactness of $D^nM_uC_\varphi : \mathcal{N}_\alpha^p \to \mathcal{B}^\Psi$.

Theorem 3.1. *Let φ be an analytic self-map of $\mathbb{D}$ and $u \in H(\mathbb{D})$. Then the following statements are equivalent*:

(*i*) *The operator $D^nM_uC_\varphi : \mathcal{N}_\alpha^p \to \mathcal{B}^\Psi$ is bounded.*

(*ii*) *The operator* $D^n M_u C_\varphi : \mathcal{N}_\alpha^p \to \mathcal{B}^\Psi$ *is compact.*

(*iii*) *For all* $c > 0$ *and* $k \in \{0, 1, \ldots, n+1\}$, u *and* φ *satisfy the following conditions:*

$$I_k = \sup_{z\in\mathbb{D}} \mu_\Psi(z) I_k(z) < \infty$$

and

$$\lim_{|\varphi(z)|\to 1} \frac{\mu_\Psi(z) I_k(z)}{(1-|\varphi(z)|^2)^k} \exp \frac{c}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}} = 0,$$

where

$$I_k(z) = \Big| \sum_{j=k}^{n+1} C_{n+1}^j u^{(n+1-j)}(z) B_{j,k}(\varphi'(z), \ldots, \varphi^{(j-k+1)}(z)) \Big|.$$

Proof. $(i) \Rightarrow (iii)$. Let $h_k(z) = z^k \in \mathcal{N}_\alpha^p$, $k = 0, 1, \ldots, n+1$. Applying the operator $D^n M_u C_\varphi : \mathcal{N}_\alpha^p \to \mathcal{B}^\Psi$ to the function h_0, we have

$$\begin{aligned} b_{\mu_\Psi}(D^n M_u C_\varphi h_0) &= \sup_{z\in\mathbb{D}} \mu_\Psi(z) \Big| (D^n M_u C_\varphi h_0)'(z) \Big| \\ &= \sup_{z\in\mathbb{D}} \mu_\Psi(z) \Big| \sum_{j=0}^{n+1} C_{n+1}^j u^{(n+1-j)}(z) B_{j,0}(\varphi'(z), \ldots, \varphi^{(j+1)}(z)) \Big| \\ &= \sup_{z\in\mathbb{D}} \mu_\Psi(z) I_0(z) = I_0. \end{aligned} \tag{16}$$

Since the operator $D^n M_u C_\varphi : \mathcal{N}_\alpha^p \to \mathcal{B}^\Psi$ is bounded, we have

$$b_{\mu_\Psi}(D^n M_u C_\varphi h_0) \le \|D^n M_u C_\varphi h_0\| \le C \|D^n M_u C_\varphi\|. \tag{17}$$

From (16) and (17), we obtain that $I_0 < \infty$.

Assume now that we have proved the following inequalities

$$I_l = \sup_{z\in\mathbb{D}} \mu_\Psi(z) I_l(z) \le C \|D^n M_u C_\varphi\| \tag{18}$$

for each $l \in \{0, 1, ..., k-1\}$ and a $k \le n+1$. Applying Lemma 2.1 to the function h_k, and noticing that $h_k^{(s)}(z) \equiv 0$ for $s > k$, we get

$$\begin{aligned} \big(D^n M_u C_\varphi h_k\big)'(z) &= \sum_{j=0}^{k} h_k^{(j)}(\varphi(z)) \sum_{i=j}^{n+1} C_{n+1}^i u^{(n+1-i)}(z) \cdot \\ &\quad \cdot B_{i,j}(\varphi'(z), \ldots, \varphi^{(i-j+1)}(z)) \\ &= \sum_{j=0}^{k} k \cdots (k-j+1)(\varphi(z))^{k-j} \sum_{i=j}^{n+1} C_{n+1}^i u^{(n+1-i)}(z) \cdot \\ &\quad \cdot B_{i,j}(\varphi'(z), \ldots, \varphi^{(i-j+1)}(z)). \end{aligned} \tag{19}$$

From (19), using the boundedness of $D^nM_uC_\varphi : \mathcal{N}_\alpha^p \to \mathcal{B}^\Psi$, the boundedness of $\varphi(z)$ and the triangle inequality, noticing that the coefficient at

$$\sum_{j=k}^{n+1} C_{n+1}^j u^{(n+1-j)}(z)B_{j,k}(\varphi'(z),\dots,\varphi^{(j-k+1)}(z))$$

is independent of z, finally using hypothesis (18) we easily obtain

$$I_k = \sup_{z\in\mathbb{D}} \mu_\Psi(z)I_k(z) \le C\|D^nM_uC_\varphi\|. \tag{20}$$

By induction we get that (20) holds for each $k \in \{0,1,\dots,n+1\}$.

Let $w \in \mathbb{D}$ and $c > 0$. Then for a fixed $k \in \{0,1,\dots,n+1\}$, by Lemma 2.2 there exists a function $h_{\varphi(w),k} \in \mathcal{N}_\alpha^p$ such that

$$h_{\varphi(w),k}^{(k)}(\varphi(w)) = \frac{\overline{\varphi(w)}^k}{(1-|\varphi(w)|^2)^{\frac{\alpha+2}{p}+k}} \exp\frac{c}{(1-|\varphi(w)|^2)^{\frac{\alpha+2}{p}}} \quad \text{and} \tag{21}$$
$$h_{\varphi(w),k}^{(j)}(\varphi(w)) = 0$$

for each $j \in \{0,1,\dots,n+1\}\setminus\{k\}$. Hence, from the boundedness of $D^nM_uC_\varphi : \mathcal{N}_\alpha^p \to \mathcal{B}^\Psi$, we have

$$\begin{aligned}\mu_\Psi(w)\big|(D^nM_uC_\varphi h_{\varphi(w)})'(w)\big| &= \frac{\mu_\Psi(w)|\varphi(w)|^kI_k(w)}{(1-|\varphi(w)|^2)^{\frac{\alpha+2}{p}+k}} \exp\frac{c}{(1-|\varphi(w)|^2)^{\frac{\alpha+2}{p}}}\\ &\le \|D^nM_uC_\varphi h_{\varphi(w)}\|_{\mathcal{B}^\Psi} \le C\|D^nM_uC_\varphi\|. \end{aligned}\tag{22}$$

Then from (22) we get

$$\frac{\mu_\Psi(w)|\varphi(w)|^kI_k(w)}{(1-|\varphi(w)|^2)^k} \exp\frac{c}{(1-|\varphi(w)|^2)^{\frac{\alpha+2}{p}}} \le C(1-|\varphi(w)|^2)^{\frac{\alpha+2}{p}}. \tag{23}$$

Taking the limit in (23) as $|\varphi(w)| \to 1$ gives the following

$$\lim_{|\varphi(w)|\to 1} \frac{\mu_\Psi(w)I_k(w)}{(1-|\varphi(w)|^2)^k} \exp\frac{c}{(1-|\varphi(w)|^2)^{\frac{\alpha+2}{p}}} = 0.$$

This shows that the statement (i) implies (iii).

$(iii) \Rightarrow (ii)$. In order to prove that the operator $D^nM_uC_\varphi : \mathcal{N}_\alpha^p \to \mathcal{B}^\Psi$ is compact, by Lemma 2.3 we just need to prove that, if $\{f_i\}$ is a sequence in $\mathcal{N}_\alpha^p$ such that $\sup_{i\in\mathbb{N}} \|f_i\|_{\mathcal{N}_\alpha^p} \le M$ and $f_i \to 0$ uniformly on any compact subset of $\mathbb{D}$ as $i \to \infty$, then

$$\lim_{i\to\infty} \|D^nM_uC_\varphi f_i\|_{\mathcal{B}^\Psi} = 0.$$

Notice that the second condition in (iii) holds for all $c > 0$. Hence, for arbitrary $\varepsilon > 0$, there is an $\eta \in (0,1)$, such that for any $z \in K = \{z \in \mathbb{D} : |\varphi(z)| > \eta\}$

$$\frac{\mu_\Psi(z)I_k(z)}{(1-|\varphi(z)|^2)^k} \exp\frac{C_kM}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}} < \varepsilon, \tag{24}$$

where C_k is the constant in Lemma 2.4. For such chosen ε and η, by using (24) and Lemma 2.4, we have

$$\begin{aligned}
&\sup_{z\in\mathbb{D}}\mu_\Psi(z)\big|(D^nM_uC_\varphi f_i)'(z)\big|\\
&=\sup_{z\in\mathbb{D}}\mu_\Psi(z)\Big|\sum_{k=0}^{n+1}f_i^{(k)}(\varphi(z))\sum_{j=k}^{n+1}C_{n+1}^{j}u^{(n+1-j)}(z)\cdot\\
&\cdot B_{j,k}\big(\varphi'(z),\dots,\varphi^{(j-k+1)}(z)\big)\Big|\\
&\le\sup_{z\in\mathbb{D}}\mu_\Psi(z)\sum_{k=0}^{n+1}\big|f_i^{(k)}(\varphi(z))\big|I_k(z)\\
&\le\Big(\sup_{z\in K}+\sup_{z\in\mathbb{D}\setminus K}\Big)\mu_\Psi(z)\sum_{k=0}^{n+1}\big|f_i^{(k)}(\varphi(z))\big|I_k(z)\\
&\le(n+2)\varepsilon+\sum_{k=0}^{n+1}I_k\sup_{|z|\le\eta}\big|f_i^{(k)}(z)\big|.
\end{aligned}\tag{25}$$

Since $f_i\to 0$ uniformly on compact subsets of $\mathbb{D}$ as $i\to\infty$ implies that for each $k\in\mathbb{N}$, $f_i^{(k)}\to 0$ uniformly on compact subsets of $\mathbb{D}$ as $i\to\infty$, from (25) we get

$$\lim_{i\to\infty}\sup_{z\in\mathbb{D}}\mu_\Psi(z)\big|(D^nM_uC_\varphi f_i)'(z)\big|=0.$$

It is clear that

$$\lim_{i\to\infty}\big|(D^nM_uC_\varphi f_i)(0)\big|=0.\tag{26}$$

Consequently, from (25) and (26) we obtain

$$\lim_{i\to\infty}\|D^nM_uC_\varphi f_i\|_{\mathcal{B}^\Psi}=0.\tag{27}$$

Hence, from Lemma 2.3 we see that the operator $D^nM_uC_\varphi:\mathcal{N}_\alpha^p\to\mathcal{B}^\Psi$ is compact.

$(ii)\Rightarrow(i)$. This implication is obvious. This finishes the proof of the theorem. □

Since $D^nC_\varphi M_u=D^nM_{u\circ\varphi}C_\varphi$, by Faà di Bruno's formula and Theorem 3.1 we obtain the characterizations of the boundedness and compactness for the operator $D^nC_\varphi M_u:\mathcal{N}_\alpha^p\to\mathcal{B}^\Psi$ in the following result.

Corollary 3.1. *Let φ be an analytic self-map of $\mathbb{D}$ and $u\in H(\mathbb{D})$. Then the following statements are equivalent*:

(i) The operator $D^nC_\varphi M_u:\mathcal{N}_\alpha^p\to\mathcal{B}^\Psi$ is bounded.

(ii) The operator $D^nC_\varphi M_u:\mathcal{N}_\alpha^p\to\mathcal{B}^\Psi$ is compact.

(*iii*) *For all* $c > 0$ *and* $k \in \{0, 1, \dots, n+1\}$, u *and* φ *satisfy the following conditions:*

$$\sup_{z\in\mathbb{D}} \mu_\Psi(z) J_k(z) < \infty$$

and

$$\lim_{|\varphi(z)|\to 1} \frac{\mu_\Psi(z) J_k(z)}{(1-|\varphi(z)|^2)^k} \exp \frac{c}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}} = 0,$$

where

$$J_k(z) = \Big| \sum_{j=k}^{n+1} \sum_{i=0}^{n+1-j} C_{n+1}^j u^{(i)}(\varphi(z)) B_{n+1-j,i}\big(\varphi'(z), \dots, \varphi^{(n+2-j-i)}(z)\big) \cdot$$
$$\cdot B_{j,k}\big(\varphi'(z), \dots, \varphi^{(j-k+1)}(z)\big)\Big|.$$

Since

$$(C_\varphi D^n M_u f)'(z) = \sum_{k=0}^{n+1} C_{n+1}^k u^{(n+1-k)}(\varphi(z))\varphi'(z) f^{(k)}(\varphi(z)),$$

we have the following result, whose proof is similar to that of Theorem 3.1. So it is omitted.

Theorem 3.2. *Let* φ *be an analytic self-map of* $\mathbb{D}$ *and* $u \in H(\mathbb{D})$. *Then the following statements are equivalent*:

(*i*) *The operator* $C_\varphi D^n M_u : \mathcal{N}_\alpha^p \to \mathcal{B}^\Psi$ *is bounded.*

(*ii*) *The operator* $C_\varphi D^n M_u : \mathcal{N}_\alpha^p \to \mathcal{B}^\Psi$ *is compact.*

(*iii*) *For all* $c > 0$ *and* $k \in \{0, 1, \dots, n+1\}$, u *and* φ *satisfy the following conditions:*

$$\sup_{z\in\mathbb{D}} \mu_\Psi(z) |u^{(n+1-k)}(\varphi(z))||\varphi'(z)| < \infty$$

and

$$\lim_{|\varphi(z)|\to 1} \frac{\mu_\Psi(z)|u^{(n+1-k)}(\varphi(z))||\varphi'(z)|}{(1-|\varphi(z)|^2)^k} \exp \frac{c}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}} = 0.$$

Next we characterize the boundedness and compactness of $M_u D^n C_\varphi : \mathcal{N}_\alpha^p \to \mathcal{B}^\Psi$.

Theorem 3.3. *Let* φ *be an analytic self-map of* $\mathbb{D}$ *and* $u \in H(\mathbb{D})$. *Then the following statements are equivalent*:

(*i*) *The operator* $M_u D^n C_\varphi : \mathcal{N}_\alpha^p \to \mathcal{B}^\Psi$ *is bounded.*

(*ii*) *The operator* $M_u D^n C_\varphi : \mathcal{N}_\alpha^p \to \mathcal{B}^\Psi$ *is compact.*

(*iii*) *For all* $c > 0$ *and* $k \in \{0, 1, \ldots, n\}$, u *and* φ *satisfy the following conditions:*

$$\sup_{z\in\mathbb{D}} \mu_\Psi(z)|u(z)||\varphi'(z)|^{n+1} < \infty,$$
$$\sup_{z\in\mathbb{D}} \mu_\Psi(z)L_k(z) < \infty,$$
$$\lim_{|\varphi(z)|\to 1} \frac{\mu_\Psi(z)L_k(z)}{(1-|\varphi(z)|^2)^k} \exp\frac{c}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}} = 0,$$

where

$$L_k(z) = \big|u'(z)B_{n,k}(\varphi'(z), \ldots, \varphi^{(n-k+1)}(z)) + u(z)B_{n+1,k}(\varphi'(z), \ldots, \varphi^{(n-k+2)}(z))\big|,$$

and

$$\lim_{|\varphi(z)|\to 1} \frac{\mu_\Psi(z)|u(z)||\varphi'(z)|^{n+1}}{(1-|\varphi(z)|^2)^{n+1}} \exp\frac{c}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}} = 0.$$

Proof. By some calculation, we have

$$(M_u D^n C_\varphi f)'(z) = \sum_{k=0}^{n} f^{(k)}(\varphi(z))\Big(u'(z)B_{n,k}\big(\varphi'(z), \ldots, \varphi^{(n-k+1)}(z)\big) + u(z)B_{n+1,k}\big(\varphi'(z), \ldots, \varphi^{(n-k+2)}(z)\big)\Big) + u(z)(\varphi'(z))^{n+1} f^{(n+1)}(\varphi(z)).$$

By this formula, the proof can be given similar to that of Theorem 3.1. So we omit it. □

Since $(M_u C_\varphi D^n f)'(z) = u'(z)f^{(n)}(\varphi(z)) + u(z)\varphi'(z)f^{(n+1)}(\varphi(z))$, we have

Theorem 3.4. *Let* φ *be an analytic self-map of* $\mathbb{D}$ *and* $u \in H(\mathbb{D})$. *Then the following statements are equivalent*:

(*i*) *The operator* $M_u C_\varphi D^n : \mathcal{N}_\alpha^p \to \mathcal{B}^\Psi$ *is bounded.*

(*ii*) *The operator* $M_u C_\varphi D^n : \mathcal{N}_\alpha^p \to \mathcal{B}^\Psi$ *is compact.*

(*iii*) *For all* $c > 0$, *the functions* u *and* φ *satisfy the following conditions:*

$$\sup_{z\in\mathbb{D}} \mu_\Psi(z)|u'(z)| < \infty,$$
$$\sup_{z\in\mathbb{D}} \mu_\Psi(z)|u(z)||\varphi'(z)| < \infty,$$
$$\lim_{|\varphi(z)|\to 1} \frac{\mu_\Psi(z)|u'(z)|}{(1-|\varphi(z)|^2)^n} \exp\frac{c}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}} = 0,$$

and

$$\lim_{|\varphi(z)|\to 1}\frac{\mu_\Psi(z)|u(z)||\varphi'(z)|}{(1-|\varphi(z)|^2)^{n+1}}\exp\frac{c}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}=0.$$

Noticing that $C_\varphi M_u D^n = M_{u\circ\varphi}C_\varphi D^n$, by Theorem 3.4 we have

Corollary 3.2. *Let φ be an analytic self-map of $\mathbb{D}$ and $u\in H(\mathbb{D})$. Then the following statements are equivalent*:

(*i*) *The operator $C_\varphi M_u D^n:\mathcal{N}_\alpha^p\to\mathcal{B}^\Psi$ is bounded.*

(*ii*) *The operator $C_\varphi M_u D^n:\mathcal{N}_\alpha^p\to\mathcal{B}^\Psi$ is compact.*

(*iii*) *For all $c>0$, the functions u and φ satisfy the following conditions:*

$$\sup_{z\in\mathbb{D}}\mu_\Psi(z)|u'(\varphi(z))||\varphi'(z)|<\infty,$$
$$\sup_{z\in\mathbb{D}}\mu_\Psi(z)|u(\varphi(z))||\varphi'(z)|<\infty,$$
$$\lim_{|\varphi(z)|\to 1}\frac{\mu_\Psi(z)|u'(\varphi(z))||\varphi'(z)|}{(1-|\varphi(z)|^2)^{n}}\exp\frac{c}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}=0,$$

and

$$\lim_{|\varphi(z)|\to 1}\frac{\mu_\Psi(z)|u(\varphi(z))||\varphi'(z)|}{(1-|\varphi(z)|^2)^{n+1}}\exp\frac{c}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}=0.$$

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