1-MOVABLE DOUBLY CONNECTED DOMINATION IN GRAPHS

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Abstract. This paper presents some characterizations involving the concept of 1-movable doubly connected domination and investigates the 1-movable doubly connected dominating sets in the join of graphs. Moreover, the 1-movable doubly connected domination number of the join of graphs is determined.
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1. Introduction

Let $G = (V(G), E(G))$ be a graph and $v \in V(G)$. The *open neighborhood* of v is the set $N_G(v) = N(v) = \{u \in V(G) : uv \in E(G)\}$ and the *closed neighborhood* of v is the set $N_G[v] = N[v] = N(v) \cup \{v\}$. If $S \subseteq V(G)$, then the *open neighborhood* of S is the set $N_G(S) = N(S) = \cup_{v\in S} N_G(v)$ and the *closed neighborhood* of S is the set $N_G[S] = N[S] = S \cup N(S)$. A subset S of $V(G)$ is a *dominating set* of G if for every $v \in V(G)\backslash S$, there exists $u \in S$ such that $uv \in E(G)$, that is, $N_G[S] = V(G)$. The *domination number* of G denoted by $\gamma(G)$, is the smallest cardinality of a dominating set of G. A dominating set of G with cardinality equal to $\gamma(G)$ is called a γ-set of G. If S is a dominating set of G, then a vertex w is a *private neighbor* of $v \in S$ with respect to S if $N(w) \cap S = \{v\}$. If $w \in S$, then w is an *internal private neighbor* of $v \in S$, otherwise, w is an *external private neighbor* of $v \in S$. The set of private neighbors of $v \in S$ is denoted by $pn\,(v; S)$, the set of internal private neighbors of $v \in S$ is denoted by $ipn\,(v; S)$

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and the set of external private neighbors of $v \in S$ is denoted by $epn\,(v; S)$. A dominating set S of $V(G)$ is a *connected dominating set* of G if the subgraph $\langle S \rangle$ induced by S is connected. The *connected domination number* of G, denoted by $\gamma_c(G)$, is the smallest cardinality of a connected dominating set of G. A connected dominating set of G with cardinality equal to $\gamma_c(G)$ is called a γ_c*-set* of G. A non-empty subset S of $V(G)$ is a *doubly connected dominating set* of G if S is a connected dominating set and the subgraph $\langle V(G) \setminus S \rangle$ induced by $V(G) \setminus S$ is connected. The *doubly connected domination number* of G, denoted by $\gamma_{cc}(G)$, is the smallest cardinality of a doubly connected dominating set of G. A doubly connected dominating set of G with cardinality equal to $\gamma_{cc}(G)$ is called a γ_{cc}*-set* of G. Doubly connected domination in graphs was studied in [1], [2], [5] and [8].

A non-empty set $S \subseteq V(G)$ is a *1-movable doubly connected dominating set* of G if (i) $S = V(G)$ and for each $v \in S$, $S \setminus \{v\}$ is a doubly connected dominating set of G or (ii) S is a doubly connected dominating set of G and for each $v \in S$, $S \setminus \{v\}$ is a doubly connected dominating set of G or $(S \setminus \{v\}) \cup \{u\}$ is a doubly connected dominating set of G for some $u \in (V(G) \setminus S) \cap N_G(v)$. The *1-movable doubly connected domination number* of a graph G, denoted by $\gamma^1_{mcc}(G)$, is the smallest cardinality of a 1-movable doubly connected dominating set of G. A 1-movable doubly connected dominating set of G with cardinality equal to $\gamma^1_{mcc}(G)$ is called γ^1_{mcc}*-set* of G.

Consider for example the graph as shown in Figure 1. The sets $S = \{x, y\}$, $(S \setminus \{x\}) \cup \{z_3\} = \{y, z_3\}$ and $(S \setminus \{y\}) \cup \{w_1\} = \{x, w_1\}$ are doubly connected dominating sets of G. Moreover, S is a γ^1_{mcc}-set of G. Hence, $\gamma^1_{mcc}(G) = |S| = 2$.

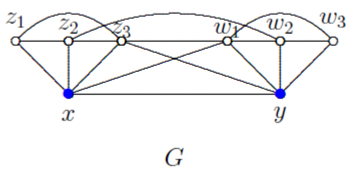

Figure 1: A graph G with a 1-movable doubly connected dominating set

The movability of sets in some variants of domination were introduced and studied in [3], [6], [7], [9] and [10].

2. Results

Remark 2.1 ([4]). Every connected dominating set in a connected graph contains every cut-vertex of G.

The next result characterizes those graphs with 1-movable doubly connected dominating sets.

Theorem 2.2. *A connected nontrivial graph G has a 1-movable doubly connected dominating set if and only if G has no cut-vertices.*

Proof. Suppose that G has a 1-movable doubly connected dominating set, say S. Then clearly, S is a connected dominating set of G. Suppose further that G has a cut-vertex, say v. Then by Remark 2.1, $v \in S$. Thus, $\langle S \setminus \{v\} \rangle$ and $\langle (S \setminus \{v\}) \cup \{u\} \rangle$ are not connected subgraphs of G for every $u \in (V(G) \setminus S) \cap N_G(v)$. Hence, S is not a 1-movable doubly connected dominating set of G. This is a contradiction to the assumption. Therefore, G has no cut-vertex.

For the converse, suppose that G has no cut-vertex. Let $S = V(G)$ and let $v \in S$. From the assumption, v is not a cut-vertex. Hence, $S \setminus \{v\}$ is a connected dominating set of G and $\langle V(G) \setminus (S \setminus \{v\}) \rangle \cong K_1$ is a connected subgraph of G. Hence, $S \setminus \{v\}$ is a doubly connected dominating set of G. Thus, $S = V(G)$ is a 1-movable doubly connected dominating set of G. □

The next result follows from the proof of Theorem 2.2.

Corollary 2.3. *If G is a connected nontrivial graph without cut-vertices, then $V(G)$ is a 1-movable doubly connected dominating set of G.*

Clearly, every 1-movable doubly connected dominating set $S \neq V(G)$ of a graph G is a doubly connected dominating set of G. Hence, $\gamma_{cc}(G) \leq \gamma^1_{mcc}(G)$ for any graph G without cut-vertices. Moreover, for any graph G without cut-vertices, $1 \leq \gamma^1_{mcc}(G) \leq n$ and these bounds are sharp. Consider the complete graph K_{10} and the cycle C_{10} as shown in Figure 2. In these graphs, $\gamma^1_{mcc}(K_{10}) = 1$ and $\gamma^1_{mcc}(C_{10}) = 10$.

Figure 2: The complete graph K_{10} and the cycle C_{10}.

The next result characterizes all graphs without cut-vertices which attained the lower bound.

Theorem 2.4. *Let G be a connected nontrivial graph without cut-vertices. Then $\gamma^1_{mcc}(G) = 1$ if and only if $G = K_2$ or $G \cong K_2 + H$ for some graph H.*

Proof. Suppose that $\gamma^1_{mcc}(G) = 1$. Let $S = \{x\}$ be a γ^1_{mcc}-set of G for some $x \in V(G)$. If $|V(G)| = 2$, then $G = K_2$. Suppose that $|V(G)| \geq 3$. Since S is a dominating set of G, $xv \in E(G)$ for all $v \in V(G) \setminus \{x\}$. Also, since S is a 1-movable doubly connected dominating set of G, there exists $u \in (V(G) \setminus S) \cap N_G(x)$ such that $(S \setminus \{x\}) \cup \{u\} = \{u\}$ is a doubly connected dominating set of G. This implies that $uw \in E(G)$ for all $w \in V(G) \setminus \{u\}$. This means that $xu \in E(G)$. Let $H = \langle V(G) \setminus \{x, u\}\rangle$. Then $G = \langle\{x, u\}\rangle + H \cong K_2 + H$. For the converse, suppose first that $G = K_2$. Then, clearly, $\gamma^1_{mcc}(G) = 1$. Suppose that $G = K_2 + H$. Let $V(K_2) = \{x, y\}$ and let $S = \{x\}$. Then S is a connected dominating set of G and $\langle V(G) \setminus S\rangle = \langle V(G) \setminus \{x\}\rangle = \langle\{y\}\rangle + H$ is connected. Hence, S is a doubly connected dominating set of G. Furthermore, since $y \in (V(G) \setminus S) \cap N_G(x)$ and $(S \setminus \{x\}) \cup \{y\} = \{y\}$ is a connected dominating set of G and $\langle V(G) \setminus \{y\}\rangle = \langle\{x\}\rangle + H$ is connected, it follows that $(S \setminus \{x\}) \cup \{y\} = \{y\}$ is a doubly connected dominating set of G. Consequently, S is a 1-movable doubly connected dominating set of G and hence, a γ^1_{mcc}-set of G. Therefore $\gamma^1_{mcc}(G) = |S| = 1$. □

Corollary 2.5. $\gamma^1_{mcc}(K_n) = 1$ *for all* $n \geq 2$.

The next result characterizes the concept of 1-movable doubly connected domination in graphs in terms of the concept of private neighbors.

Theorem 2.6. *Let G be a connected graph without cut-vertices. A proper subset S of $V(G)$ is a 1-movable doubly connected dominating set of G if and only if S is a doubly connected dominating set of G and for each $v \in S$, either*

(*i*) $epn(v; S) = ipn(v; S) = \emptyset$ *and* $\langle(V(G) \setminus S) \cup \{v\}\rangle$ *is connected or*

(*ii*) *there exists* $u \in (V(G) \setminus S) \cap N_G(v)$ *such that* $\langle[V(G) \setminus (S \cup \{u\})] \cup \{v\}\rangle$ *is connected,* $epn(v; S) \cup ipn(v; S) \subseteq N(u)$ *and* $u \in N_G[(S \setminus \{v\}) \setminus ipn(v; S)]$ *whenever* $S \setminus \{v\} \neq ipn(v; S)$.

Proof. Suppose S is a 1-movable doubly connected dominating set of G, where $S \neq V(G)$. Then S is a doubly connected dominating set of G. Let $v \in S$. Since S is a 1-movable doubly connected dominating set of G, $S \setminus \{v\}$ or $S_v = (S \setminus \{v\}) \cup \{u\}$, for some $u \in (V(G) \setminus S) \cap N_G(v)$, is a doubly connected dominating set of G. Suppose that $S \setminus \{v\}$ is a doubly connected dominating set of G. Then $\langle V(G) \setminus (S \setminus \{v\})\rangle = \langle(V(G) \setminus S) \cup \{v\})\rangle$ is connected and every vertex w in $(V(G) \setminus S) \cap N_G(v)$ is adjacent to some vertex in $S \setminus \{v\}$. This implies that $epn(v; S) = \emptyset$. Moreover, since $\langle S \setminus \{v\}\rangle$ is connected, $ipn(v; S) = \emptyset$. Suppose $S \setminus \{v\}$ is not a doubly connected dominating set of G. Then there exists a vertex $u \in (V(G) \setminus S) \cap N_G(v)$ such that S_v is a doubly connected dominating set of G. Hence, $\langle V(G) \setminus S_v\rangle = \langle[V(G) \setminus (S \cup \{u\})] \cup \{v\}\rangle$ is connected. Let $z \in epn(v; S)$. Then $z \in N(u)$ since S_v is a dominating set of G. Thus, $epn(v; S) \subseteq N(u)$. Also, if $y \in ipn(v; S)$, then $y \in N(u)$ since $\langle S_v\rangle$ is connected. Thus, $epn(v; S) \cup ipn(v; S) \subseteq N(u)$. Now, suppose that $S \setminus \{v\} \neq ipn(v; S)$. Suppose further that $u \notin N_G[(S \setminus \{v\}) \setminus ipn(v; S)]$. Then $\langle(S \setminus \{v\}) \setminus ipn(v; S)\rangle$

and $\langle\{u\}\cup ipn(v;S)\rangle$ are components of $\langle S_v\rangle$, contrary to the fact that $\langle S_v\rangle$ is connected. Therefore, $u\in N_G[(S\setminus\{v\})\setminus ipn(v;S)$ whenever $S\setminus\{v\}\neq ipn(v;S)$.

For the converse, suppose that S is a doubly connected dominating set of G satisfying one of the given conditions. Let $v\in S$. If (i) holds, then $epn(v;S)=ipn(v;S)=\emptyset$. Hence, $S\setminus\{v\}$ is a connected dominating set of G. Furthermore, since $\langle(V(G)\setminus S)\cup\{v\}\rangle=\langle V(G)\setminus(S\setminus\{v\})\rangle$ is connected, $S\setminus\{v\}$ is a doubly connected dominating set of G. Suppose that (ii) holds. Then there exists $u\in(V(G)\setminus S)\cap N_G(v)$ such that $epn(v;S)\cup ipn(v;S)\subseteq N(u)$. Set $S_v=(S\setminus\{v\})\cup\{u\}$ and let $x\in V(G)\setminus S_v$. If $x=v$ or $x\in epn(v;S)$, then $xu\in E(G)$. If $x\notin\{v\}\cup epn(v;S)$, then $xy\in E(G)$ for some $y\in S\setminus\{v\}$ since S is a dominating set of G. Thus, S_v is a dominating set of G. If $S\setminus\{v\}=ipn(v;S)$, then $S\setminus\{v\}\subseteq N_(u)$. Thus, $\langle S_v\rangle$ is connected. If $S\setminus\{v\}\neq ipn(v;S)$, then by assumption, $u\in N_G[(S\setminus\{v\})\setminus ipn(v;S)\rangle$. Hence, since $ipn(v;S)\subseteq N_G(u)$ also, $\langle S_v\rangle$ is connected. Thus, S_v is a connected dominating set of G. Furthermore, by assumption, $\langle[V(G)\setminus(S\cup\{u\})]\cup\{v\}\rangle=\langle V(G)\setminus[(S\setminus\{v\})\cup\{u\}]\rangle=\langle V(G)\setminus S_v\rangle$ is connected. Hence, S_v is a doubly connected dominating set of G. Accordingly, S is a 1-movable doubly connected dominating set of G. □

Theorem 2.7. *Let G be a connected graph without cut-vertices and let its order be $n\geq 3$. Then $\gamma^1_{mcc}(G)=2$ if and only if the following conditions hold:*

(i) *G is not isomorphic to K_2+H for any graph H and*

(ii) *there exist two adjacent vertices x and y that dominate G such that*

(a) *$\langle V(G)\setminus\{x,y\}\rangle$ is connected*

(b) *$epn(x;\{x,y\})=\emptyset$ or $\emptyset\neq epn(x;\{x,y\})\subseteq N_G(z)$ for some $z\in N_G(x)\cap N_G(y)$ and $\langle V(G)\setminus\{z,y\}\rangle$ is connected and*

(c) *$epn(y;\{x,y\})=\emptyset$ or $\emptyset\neq epn(y;\{x,y\})\subseteq N_G(w)$ for some $w\in N_G(x)\cap N_G(y)$ and $\langle V(G)\setminus\{x,w\}\rangle$ is connected.*

Proof. Suppose that $\gamma^1_{mcc}(G)=2$. Then by Theorem 2.4, G is not isomorphic to K_2+H for any graph H. Thus, (i) holds. Let $S=\{x,y\}$ be a γ^1_{mcc}-set of G. Since S is a doubly connected dominating set of G, x and y are two adjacent vertices that dominate G and $\langle V(G)\setminus\{x,y\}\rangle$ is connected. Suppose first that $S\setminus\{x\}$ is a doubly connected dominating set of G. Then $epn(x;\{x,y\})=\emptyset$. Suppose that $S\setminus\{x\}$ is not a doubly connected dominating set of G. Then $(S\setminus\{x\})\cup\{z\}=\{y,z\}$ is a doubly connected dominating set of G for some $z\in(V(G)\setminus S)\cap N_G(x)$. This implies that $z\in N_G(y)$ and $\langle V(G)\setminus\{z,y\}\rangle$ is a connected subgraph of G. Let $v\in epn(x;\{x,y\})$. Since $vy\notin E(G)$ and since $\{z,y\}$ is a dominating set of G, it follows that $v\in N_G(z)$. Since v was arbitrarily chosen, it follows that $epn(x;\{x,y\})\subseteq N_G(z)$. Similarly, (c) holds. Hence, (ii) holds.

For the converse, suppose that (i) and (ii) hold. Then by Theorem 2.4, $\gamma^1_{mcc}(G)\geq 2$. Let $S=\{x,y\}$ such that x and y are vertices in G satisfying (ii). Then S is a doubly connected dominating set of G. Now by (b), suppose that

$epn(x; \{x, y\}) = \emptyset$. Then $S \setminus \{x\} = \{y\}$ is a connected dominating set of G. Since y is not a cut-vertex, $\langle V(G) \setminus \{y\} \rangle$ is connected. Thus, $S \setminus \{x\}$ is a doubly connected dominating set of G. Suppose that $\emptyset \neq epn(x; \{x, y\}) \subseteq N_G(z)$ for some $z \in (V(G) \setminus S) \cap [N_G(x) \cap N_G(y)]$ and $\langle V(G) \setminus \{z, y\} \rangle$ is connected. Then $(S \setminus \{x\}) \cup \{z\} = \{y, z\}$ is a doubly connected dominating set of G. Similarly, $S \setminus \{y\} = \{x\}$ or $(S \setminus \{y\}) \cup \{w\} = \{x, w\}$ is a doubly connected of G for some $w \in N_G(x) \cap N_G(y)$. Hence, S is a 1-movable doubly connected dominating set of G and thus a γ^1_{mcc}-set of G. Therefore, $\gamma^1_{mcc}(G) = |S| = 2$. □

We now characterize the 1-movable doubly connected dominating sets in the join of two connected nontrivial graphs.

Theorem 2.8 ([2]). *Let G and H be any graphs of orders $m \geq 2$ and $n \geq 2$, respectively. Then $S \subseteq V(G+H)$ is a doubly connected dominating set of $G+H$ if and only if at least one of the following holds:*

(i) $S \subseteq V(G)$ *is a connected dominating set of* G*, where* H *is connected if* $S = V(G)$*;*

(ii) $S \subseteq V(H)$ *is a connected dominating set of* H*, where* G *is connected if* $S = V(H)$*;*

(iii) $V(G) \subseteq S$*,* $V(H) \cap S \neq \emptyset$*, and* $\langle V(H) \setminus (S \cap V(H)) \rangle$ *is a connected subgraph of* H*;*

(iv) $V(H) \subseteq S$*,* $V(G) \cap S \neq \emptyset$*, and* $\langle V(G) \setminus (S \cap V(G)) \rangle$ *is a connected subgraph of* G*;*

(v) $1 \leq |S \cap V(G)| < m$ *and* $1 \leq |S \cap V(H)| < n$*.*

Theorem 2.9. *Let G and H be connected graphs of order $m \geq 2$ and $n \geq 2$, respectively. Then $S \subseteq V(G+H)$ is a 1-movable doubly connected dominating set of $G+H$ if and only if one of the following statements is true:*

(i) $S = V(G+H)$*.*

(ii) S *is a connected dominating set of* G *such that if* $|S| = 1$*, then either* S *is a 1-movable connected dominating set of* G *or there exists* $u \in V(H)$ *such that* $\{u\}$ *is a connected dominating set of* H*.*

(iii) S *is a connected dominating set of* H *such that if* $|S| = 1$*, then either* S *is a 1-movable connected dominating set of* H *or there exists* $w \in V(G)$ *such that* $\{w\}$ *is a connected dominating set of* G*.*

(iv) $S = S_1 \cup S_2 (\neq V(G+H)$ *where* $\emptyset \neq S_1 \subseteq V(G)$ *and* $\emptyset \neq S_2 \subseteq V(H)$ *such that*

(1) *if* $S_1 = V(G)$*, then* $\langle V(H) \setminus S_2 \rangle$ *is connected and for every* $v \in S_2$*, either* $\langle V(H) \setminus (S_2 \setminus \{v\}) \rangle$ *or* $\langle V(H) \setminus [(S_2 \setminus \{v\}) \cup \{u\}] \rangle$ *is connected for some* $u \in (V(H) \setminus S_2) \cap N_H(v)$ *and*

(2) *if* $S_2 = V(H)$*, then* $\langle V(G) \setminus S_1 \rangle$ *is connected and for every* $v \in S_1$*, either* $\langle V(G) \setminus (S_1 \setminus \{v\}) \rangle$ *or* $\langle V(G) \setminus [(S_1 \setminus \{v\}) \cup \{u\}] \rangle$ *is connected for some* $u \in (V(G) \setminus S_1) \cap N_G(v)$*.*

Proof. Suppose that S is a 1-movable doubly connected dominating set of $G + H$. If $S = V(G+H)$, then (i) holds. Suppose that $S \neq V(G+H)$. Then S is a doubly connected dominating set of $G+H$. Suppose that $S \subseteq V(G)$. Then by Theorem 2.8(i), S is a connected dominating set of G. Suppose that $|S| = 1$, say $S = \{x\}$ for some $x \in V(G)$. If $\gamma(H) = 1$, then there exists $u \in V(H)$ such that $\{u\}$ is a dominating set of H. Suppose that $\gamma(H) \neq 1$. Since S is a 1-movable doubly connected dominating set of $G+H$, there exists $w \in (V(G+H) \setminus S) \cap N_G(x)$ such that $(S \setminus \{x\}) \cup \{w\} = \{w\}$ is a doubly connected dominating set of $G+H$. Since $\gamma(H) \neq 1$, it follows that $w \in V(G) \setminus S$. Hence, $(S \setminus \{x\}) \cup \{w\} = \{w\}$ is a connected dominating set of G. Thus, S is a 1-movable connected dominating set of G. This proves (ii). Similarly, (iii) holds if $S \subseteq V(H)$. Next, suppose that $S_1 = S \cap V(G) \neq \emptyset$ and $S_2 = S \cap V(H) \neq \emptyset$. Then $S = S_1 \cup S_2$. Suppose that $S_1 = V(G)$. Since $S \neq V(G+H)$, $S_2 \neq V(H)$. Since S is a doubly connected dominating set of $G+H$, $\langle V(G+H) \setminus S \rangle = \langle V(H) \setminus S_2 \rangle$ is connected. Let $v \in S_2$. With the assumption, suppose first that $S \setminus \{v\}$ is a doubly connected dominating set of $G+H$. Then $\langle V(G+H) \setminus (S \setminus \{v\}) \rangle = \langle V(H) \setminus (S_2 \setminus \{v\}) \rangle$ is connected. Suppose that $S \setminus \{v\}$ is not a doubly connected dominating set of $G+H$. Then there exists $u \in (V(G+H) \setminus S) \cap N_{G+H}(v)$ such that $(S \setminus \{v\}) \cup \{u\}$ is a doubly connected dominating set of $G+H$. Since $S_1 = V(G)$, $u \in (V(H) \setminus S_2) \cap N_H(v)$. Thus, $\langle V(G+H) \setminus [(S \setminus \{v\}) \cup \{u\}] \rangle = \langle V(H) \setminus [(S_2 \setminus \{v\}) \cup \{u\}] \rangle$ is connected. Showing that (1) holds. Similarly, (2) holds if $S_2 = V(H)$ Hence, (iv) holds.

For the converse, suppose first that (i) holds i.e., $S = V(G+H)$. Since $G+H$ has no cut-vertices, S is 1-movable doubly connected dominating set of $G+H$ by Corollary 2.3. Suppose that (ii) holds. Then by Theorem 2.8(i), S is a doubly connected dominating set of $G+H$. Let $v \in S$. If $|S| \geq 2$, then $S \setminus \{v\} \neq \emptyset$ and there exists $u \in V(H) \cap N_{G+H}(v)$ such that $(S \setminus \{v\}) \cup \{u\}$ is a doubly connected dominating set of $G+H$ by Theorem 2.8(v). Suppose that $|S| = 1$. Suppose further that S is a 1-movable connected dominating set of G. Then there exists $w \in V(G) \cap N_G(v)$ such that $(S \setminus \{v\}) \cup \{w\} = \{w\}$ is a connected dominating set of G. By Theorem 2.8(i), $(S \setminus \{v\}) \cup \{w\}$ is a doubly connected dominating set of $G+H$. Suppose that S is not a 1-movable connected dominating set of G. Then by assumption, there exists $u \in V(H)$ such that $\{u\}$ is a connected dominating set of H. Hence, $(S \setminus \{v\}) \cup \{u\} = \{u\}$ is a doubly connected dominating set of $G+H$ by Theorem 2.8(ii). Therefore, S is a 1-movable doubly connected dominating set of $G+H$. Similarly, if (iii) holds, then S is a 1-movable doubly connected dominating set of $G+H$. Suppose that (iv) holds. Then $S = S_1 \cup S_2 (\neq V(G+H)$, where $\emptyset \neq S_1 \subseteq V(G)$ and $\emptyset \neq S_2 \subseteq V(H)$. Consider the following cases:

Case 1: $S_1 \neq V(G)$ and $S_2 = V(H)$ or $S_1 = V(G)$ and $S_2 \neq V(H)$.

Suppose that $S_1 \neq V(G)$ and $S_2 = V(H)$. Then by assumption, $\langle V(G) \setminus S_1\rangle$ is connected. Hence, by Theorem 2.8(iv), S is a doubly connected dominating set of $G + H$. Let $v \in S$. Suppose that $v \in S_1$. If $|S_1| \geq 2$, then $S_1 \setminus \{v\} \neq \emptyset$. If $\langle (V(G) \setminus (S_1 \setminus \{v\})\rangle$ is connected, then $S \setminus \{v\} = (S_1 \setminus \{v\}) \cup S_2$ is a doubly connected dominating set of $G + H$ by Theorem 2.8(iv). Suppose that $\langle (V(G) \setminus (S_1 \setminus \{v\})\rangle$ is not connected. Then by assumption, there exists $u \in (V(G) \setminus S_1) \cup N_G(v)$ such that $\langle V(G) \setminus [(S_1 \setminus \{v\}) \cup \{u\}]\rangle$ is connected. Hence, $(S \setminus \{v\}) \cup \{u\} = [(S_1 \setminus \{v\}) \cup \{u\}] \cup S_2$ is a doubly connected dominating set of $G + H$ by Theorem 2.8(iv). Suppose that $|S_1| = 1$. Then $S \setminus \{v\} = S_2 = V(H)$ and $\langle V(G + H) \setminus (S \setminus \{v\})\rangle = \langle V(G)\rangle$ is connected. Hence, $S \setminus \{v\}$ is a doubly connected dominating set of $G + H$ by Theorem 2.8(ii). Next, suppose that $v \in S_2$. Since $S_2 = V(H)$, $\emptyset \neq S_2 \setminus \{v\} \neq V(H)$. Since $S_1 \neq V(H)$, $S \setminus \{v\} = S_1 \cup (S_2 \setminus \{v\})$ is a doubly connected dominating set of $G + H$ by Theorem 2.8(v). Therefore, S is a 1-movable doubly connected dominating set of $G+H$. Similarly, if $S_1 = V(G)$ and $S_2 \neq V(H)$, then S is a 1-movable doubly connected dominating set of $G + H$.

Case 2: $S_1 \neq V(G)$ and $S_2 \neq V(H)$.

Then by Theorem 2.8(v), $S = S_1 \cup S_2$ is a doubly connected dominating set of $G + H$. Let $v \in S$. Suppose that $v \in S_1$. If $|S_1| \geq 2$, then $S_1 \setminus \{v\} \neq \emptyset$. Hence, $S \setminus \{v\} = (S_1 \setminus \{v\}) \cup S_2$ is a doubly connected dominating set of $G + H$ by Theorem 2.8(v). Suppose that $|S_1| = 1$. Since G is a connected nontrivial graph, there exists $u \in V(G)$ such that $uv \in E(G)$. Hence, $|(S_1 \setminus \{v\}) \cup \{u\}| = 1$. Thus, $(S \setminus \{v\}) \cup \{u\} = [(S_1 \setminus \{v\}) \cup \{u\}] \cup S_2$ is a doubly connected dominating set of $G + H$ by Theorem 2.8(v). Similarly, if $v \in S_2$, then either $S \setminus \{v\}$ or $(S \setminus \{v\}) \cup \{u\}$ is a doubly connected dominating set of $G + H$ for some $u \in (V(G+H) \setminus S) \cap N_{G+H}(v)$. Consequently, S is a 1-movable doubly connected dominating set of $G + H$. □

Corollary 2.10. *Let G and H be connected nontrivial graphs. Then*

$$\gamma^1_{mcc}(G + H) = \begin{cases} 1, & \text{if } \gamma(G) = 1 = \gamma(H) \text{ or } \gamma^1_{mc}(G) = 1 \text{ or } \gamma^1_{mc}(H) = 1 \\ 2, & \text{otherwise} \end{cases}$$

Theorem 2.11. *Let H be a connected graph of order $n \geq 2$ and $K_1 = \langle \{x\}\rangle$. Then $S \subseteq V(K_1 + H)$ is a 1-movable doubly connected dominating set of $K_1 + H$ if and only if one of the following statements holds:*

(i) *$S = V(K_1)$ and there exists $w \in V(H)$ such that $\{w\}$ is a connected dominating set of H.*

(ii) *$S = V(K_1) \cup S_1$ where $\emptyset \neq S_1 \subseteq V(H)$ such that if $S_1 \neq V(H)$, then*

(1) *$\langle V(H) \setminus S_1\rangle$ is connected;*

(2) *either S_1 is a connected dominating set of H or $S_1 \cup \{a\}$ is a connected dominating set of H for some $a \in V(H) \setminus S_1$; and*

(3) *for every* $v \in S_1$, *either* $\langle V(H)\setminus(S_1\setminus\{v\})\rangle$ *or* $\langle V(H)\setminus[(S_1\setminus\{v\})\cup\{u\}]\rangle$ *is connected for some* $u \in (V(H)\setminus S_1)\cap N_H(v)$.

(*iii*) *S is a connected dominating set of H such that for every $v \in S$, $S\setminus\{v\}$ or $(S\setminus\{v\})\cup\{u\}$ is a connected dominating set of H for some $u \in (V(H)\setminus S)\cap N_H(v)$ or $\langle V(H)\setminus(S\setminus\{v\})\rangle$ is connected.*

Proof. Suppose that $S \subseteq V(K_1+H)$ is a 1-movable doubly connected dominating set of K_1+H. Consider the following cases:

Case 1: $x \in S$.

Suppose that $S=\{x\}$. Since S is a 1-movable doubly connected dominating set of K_1+H, there exists $w \in V(H)$ such that $(S\setminus\{x\})\cup\{w\}=\{w\}$ is a doubly connected dominating set of K_1+H. It follows that $\{w\}$ is a connected dominating set of K_1+H (and hence of H). This proves (*i*). Suppose that $S=V(K_1)\cup S_1$, where $\emptyset \neq S_1 \subseteq V(H)$. Suppose that $S_1 \neq V(H)$. Since S is a doubly connected dominating set of K_1+H, it follows that $\langle V(K_1+H)\setminus S\rangle = \langle V(H)\setminus S_1\rangle$ is connected. Suppose that $S\setminus\{x\}$ is a doubly connected dominating set in K_1+H. Then $S\setminus\{x\}=S_1$ is a connected dominating set in H. Next, suppose that $S\setminus\{x\}$ is not a doubly connected dominating set of K_1+H. Since S is a 1-movable doubly connected dominating set of K_1+H, there exists $a \in V(H)\setminus S_1$ such that $(S\setminus\{x\})\cup\{a\}=S_1\cup\{a\}$ is a doubly connected dominating set of K_1+H. This implies that $S_1\cup\{a\}$ is a connected dominating set of H. Let $v \in S_1$. Suppose that $S\setminus\{v\}$ is a doubly connected dominating set of K_1+H. Then $\langle V(K_1+H)\setminus(S\setminus\{v\})\rangle = \langle V(H)\setminus(S_1\setminus\{v\})\rangle$ is connected. Suppose that $S\setminus\{v\}$ is not a doubly connected dominating set of K_1+H. Then by assumption, $(S\setminus\{v\})\cup\{u\}$ is a doubly connected dominating set of K_1+H for some $u \in V(H)\setminus S_1)\cap N_H(v)$. Hence, $\langle V(K_1+H)\setminus[(S\setminus\{v\})\cup\{u\}]\rangle = \langle V(H)\setminus[(S_1\setminus\{v\})\cup\{v\}]\rangle$ is connected. This proves (*ii*).

Case 2: $x \notin S$.

Then $S \subseteq V(H)$. By assumption, S is a connected dominating set of H. Let $v \in S$. If $S\setminus\{v\}$ is a doubly connected dominating set of K_1+H, then $S\setminus\{v\}$ is a connected dominating set of H. Suppose that $S\setminus\{v\}$ is not a doubly connected dominating set of K_1+H. Then by assumption, there exists $u \in (V(K_1+H)\setminus S)\cap N_{K_1+H}(v)$ such that $(S\setminus\{v\})\cup\{u\}$ is a doubly connected dominating set of K_1+H. If $u \in V(H)\setminus S$, then $(S\setminus\{v\})\cup\{u\}$ is a connected dominating set of H. If $u=x$, then $\langle V(H)\setminus(S\setminus\{v\})\rangle$ is connected. Thus, (*iii*) holds.

For the converse, suppose first that (*i*) holds. Then $S=\{x\}=V(K_1)$ is a connected dominating set of K_1+H and $\langle V(K_1+H)\setminus S\rangle = H$ is connected. Thus, S is a doubly connected dominating set of K_1+H. By assumption, there exists $w \in V(H)$ such that $\{w\}$ is a connected dominating set of H. This implies that $(S\setminus\{x\})\cup\{w\}=\{w\}$ is a connected dominating set of K_1+H and $\langle V(K_1+H)\setminus\{w\}\rangle = \langle (V(H)\setminus\{w\})\cup\{x\}\rangle$ is connected. Hence, $(S\setminus\{x\})\cup\{w\}=\{w\}$ is a doubly connected dominating set of K_1+H. Therefore, S is a 1-movable doubly connected dominating set of K_1+H. Suppose that (*ii*) holds. Suppose first that $S = V(K_1 + H)$. Since $K_1 + H$ has no cut-vertices, by Corollary 2.3, S is a 1-movable doubly connected dominating set of $K_1 + H$. Suppose that $S_1 \neq V(H)$. Then by (1), S is a doubly connected dominating set of $K_1 + H$. Let $v \in S$. Consider the following cases:

Case 1: $v = x$.

Suppose that S_1 is a connected dominating set of H. Then $S \setminus \{v\} = S \setminus \{x\} = S_1$ is a connected dominating set of $K_1 + H$. Since $\langle V(K_1+H) \setminus S_1 \rangle = \langle \{x\} \rangle + \langle V(H) \setminus S_1 \rangle$ is connected, $S \setminus \{v\}$ is a doubly connected dominating set of $K_1 + H$. Suppose that S_1 is not a connected dominating set of H. Then by assumption, $S_1 \cup \{a\}$ is a connected dominating set of H for some $a \in V(H) \setminus S_1$. It follows that $(S \setminus \{x\}) \cup \{a\} = S_1 \cup \{a\}$ is a connected dominating set of $K_1 + H$. Moreover, since $\langle V(K_1+H) \setminus [(S_1 \setminus \{v\}) \cup \{a\}] \rangle = \langle V(H) \setminus [(S_1 \setminus \{v\}) \cup \{a\}] \cup \{x\} \rangle$ is connected, $(S \setminus \{x\}) \cup \{a\}$ is a doubly connected dominating set of $K_1 + H$.

Case 2: $v \neq x$.

Then $v \in S_1$. Clearly, $S \setminus \{v\} = \{x\} \cup (S_1 \setminus \{v\})$ is a connected dominating set of $K_1 + H$. If $\langle V(H) \setminus (S_1 \setminus \{v\}) \rangle$ is connected, then $\langle V(K_1+H) \setminus (S \setminus \{v\}) \rangle = \langle V(H) \setminus (S_1 \setminus \{v\}) \rangle$ is connected. Thus, $S \setminus \{v\}$ is a doubly connected dominating set of $K_1 + H$. Suppose that $\langle V(H) \setminus (S_1 \setminus \{v\}) \rangle$ is not connected. Then by assumption, $\langle V(H) \setminus [(S_1 \setminus \{v\}) \cup \{u\} \rangle$ is connected for some $u \in (V(H) \setminus S_1) \cap N_H(v)$. Hence, $\langle V(K_1 + H) \setminus [(S \setminus \{v\}) \cup \{u\}] \rangle = \langle V(H) \setminus [(S_1 \setminus \{v\}) \cup \{u\} \rangle$ is connected. Moreover, $(S \setminus \{v\}) \cup \{u\} = \{x\} \cup [(S_1 \setminus \{v\}) \cup \{u\}]$ is a connected dominating set of $K_1 + H$. Thus, $(S \setminus \{v\}) \cup \{u\}$ is a doubly connected dominating set of $K_1 + H$. Therefore, S is a 1-movable doubly connected dominating set of $K_1 + H$.

Finally, suppose that (iii) holds. Since S is a connected dominating set of H, it is a connected domonating set of $K_1 + H$. Moreover, since $\langle V(K_1+H) \setminus S \rangle = \langle \{x\} \rangle + \langle V(H) \setminus S \rangle$ is connected, S is a doubly connected dominating set of $K_1 + H$. Now, let $v \in S$. Suppose that $S \setminus \{v\}$ is a connected dominating set of H. Since $\langle V(K_1+H) \setminus (S \setminus \{v\}) \rangle = K_1 + \langle V(H) \setminus (S \setminus \{v\}) \rangle$ is connected, $S \setminus \{v\}$ is a doubly connected dominating set of $K_1 + H$. Suppose that $(S \setminus \{v\}) \cup \{u\}$ is a connected dominating set of H for some $u \in (V(H) \setminus S) \cap N_H(v)$. Then it is a connected dominating set of $K_1 + H$. Since $\langle V(K_1 + H) \setminus [(S \setminus \{v\}) \cup \{u\}] \rangle = K_1 + \langle V(H) \setminus [(S \setminus \{v\}) \cup \{u\}] \rangle$ is connected, $(S \setminus \{v\}) \cup \{u\}$ is a doubly connected dominating set of $K_1 + H$. Suppose that $\langle V(H) \setminus (S \setminus \{v\}) \rangle$ is connected in H. Let $S^* = (S \setminus \{v\}) \cup \{x\}$. Then S^* is a connected dominating set of $K_1 + H$. Since $\langle V(K_1 + H) \setminus S^* \rangle = \langle V(H) \setminus (S \setminus \{v\}) \rangle$ is connected in $K_1 + H$, it follows that S^* is a doubly connected dominating set of $K_1 + H$. Thus, S is a 1-movable doubly connected dominating set of $K_1 + H$. □

Corollary 2.12. *Let H be a connected nontrivial graph. Then the following holds:*

(i) $\gamma^1_{mcc}(K_1 + H) = 1$ *if and only if* $\gamma(H) = 1$.

(ii) *If H has no cut-vertices, then* $\gamma^1_{mcc}(K_1 + H) \leq \gamma^1_{mc}(H)$.

Corollary 2.13. *Let n be a positive integer. Then*

$$\gamma^1_{mcc}(F_n) = \begin{cases} 1, & if\, 1 \leq n \leq 3 \\ n, & if\, n \geq 4 \end{cases}$$

and

$$\gamma^1_{mcc}(W_n) = \begin{cases} 1, & if\, n = 3 \\ 2, & if\, n = 4 \\ 3, & if\, n = 5 \\ n, & if\, n \geq 6 \end{cases}$$

Theorem 2.14. *Let $m \geq 2$ and $n \geq 2$ be integers. Then $S \subseteq V(K_{m,n})$ is a 1-movable doubly connected dominating set of $K_{m,n} = \bar{K}_m + \bar{K}_n$ if and only if $S = S_1 \cup S_2$, where $\emptyset \neq S_1 \subseteq V(\bar{K}_m)$ with $|S_1| \geq 2$ and $\emptyset \neq S_2 \subseteq V(\bar{K}_n)$ with $|S_2| \geq 2$ such that $S_1 = V(\bar{K}_m)$ if and only if $S_2 = V(\bar{K}_n)$.*

Proof. Suppose that $S \subseteq V(K_{m,n})$ is a 1-movable doubly connected dominating set of $K_{m,n}$. Let $S_1 = S \cap V(\bar{K}_m)$ and $S_2 = S \cap V(\bar{K}_n)$. Then $S = S_1 \cup S_2$. Since S is a (doubly) connected dominating set of $K_{m,n}$, $S_1 \neq \emptyset$ and $S_2 \neq \emptyset$. Suppose that $|S_1| = 1$, say $S_1 = \{x\}$ for some $x \in V(\bar{K}_m)$. Then $\langle S \setminus \{x\} \rangle = \langle S_2 \rangle$ is an empty subgraph of $\bar{K}_n$. Hence, $S \setminus \{x\}$ is not a (doubly)connected dominating set of $K_{m,n}$. Also, $\langle (S \setminus \{x\}) \cup \{u\} \rangle = \langle S_2 \cup \{u\} \rangle$ is an empty subgraph of $\bar{K}_n$ for all $u \in V(\bar{K}_n) \setminus S_2$. Hence, $(S \setminus \{x\}) \cup \{u\}$ is not a (doubly)connected dominating set of $K_{m,n}$ for all $u \in V(\bar{K}_n) \setminus S_2$. This implies that S is not a 1-movable doubly connected dominating set of $K_{m,n}$, contrary to our assumption. Hence, $|S_1| \geq 2$. Similarly, $|S_2| \geq 2$. Suppose that $S_1 = V(\bar{K}_m)$. Since S is a doubly connected dominating set of $K_{m,n}$, $\langle V(K_{m,n}) \setminus S \rangle = \langle V(\bar{K}_n) \setminus S_2 \rangle$ is connected if $S_2 \neq V(\bar{K}_n)$. Since this is not possible, $S_2 = V(\bar{K}_n)$. Similarly, $S_1 = V(\bar{K}_m)$ if $S_2 = V(\bar{K}_n)$.

For the converse, suppose that $S = S_1 \cup S_2$, where S_1 and S_2 are the sets satisfying the given conditions. If $S_1 = V(\bar{K}_m)$ and $S_2 = V(\bar{K}_n)$, then $S = V(K_{m,n})$. Since $m \geq 2$ and $n \geq 2$, $K_{m,n}$ has no cut-vertices. Hence by Corollary 2.3, S is a 1-movable doubly connected dominating set of $K_{m,n}$. Suppose that $S_1 \neq V(\bar{K}_m)$ or $S_2 \neq V(\bar{K}_n)$. Then by assumption, $2 \leq |S_1| < m$ and $2 \leq |S_2| < n$. Hence, by Theorem 2.8(v), $S = S_1 \cup S_2$ is a doubly connected dominating set of $K_{m,n}$. Let $v \in S$. Suppose that $v \in S_1$. Since $2 \leq |S_1| < m$, $S_1 \setminus \{v\} \neq \emptyset$. Since $1 \leq |S_1 \setminus \{v\}| < m$, it follows from Theorem 2.8(v) that $S \setminus \{v\}$ is a doubly connected dominating set of $K_{m,n}$. Similarly, $S \setminus \{v\}$ is a doubly connected dominating set of $K_{m,n}$ if $v \in S_2$. Consequently, S is a 1-movable doubly connected dominating set of $K_{m,n}$. □

Corollary 2.15. *Let $m \geq 2$ and $n \geq 2$ be integers. Then*

$$\gamma^1_{mcc}(K_{m,n}) = \begin{cases} n+2, & if \quad m = 2 \,and\, n \geq 2 \\ m+2, & if \quad n = 2 \,and\, m \geq 2 \\ 4, & otherwise \end{cases}$$

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