A BIPARTITE GRAPH ASSOCIATED TO A BI-MODULE OF A RING

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Abstract. Let R be a ring, M be a left and right R-module. We associate a bipartite graph to R-module M of ring R, denoted by $\Gamma_{R,M}$ as undirected simple graph whose two parts of vertices are $R \setminus C_R(M)$ and $M \setminus C_M(R)$ and two distinct vertices x and y are adjacent if $xy \neq yx$, where $C_R(M)$ is the set of elements of R that commute with all elements in M. Some graph theoretical properties of this graph stated in this paper.
Keywords: Bi-module, diameter, girth, bipartite, planar, vertex and edge connectivity.

1. Introduction

The study of algebraic structures, using the properties of graphs, becomes an exciting research topic in the last ten years, leading to many interesting results. There are many papers on assigning a graph to a group of ring, see [1-6]. Now we are going to define a graph which is associated to a ring R and R-module M. Assume that $C_R(M)$ is the set of elements of R which commutes with all elements of M and similarly $C_M(R)$ is the set of elements of M that commutes with all elements of R. Then we define a bipartite undirected simple graph with vertex sets $V_R \cup V_M$, where $V_R = R \setminus C_R(M)$ and $V_M = M \setminus C_M(R)$ and

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two vertices $x \in V_R$ and $y \in V_M$ are adjacent if $xy \neq yx$. We denoted this graph by $\Gamma_{R,M}$. In the rest of this section, we remind some basic definitions in graph theory which are necessary in the paper. Section 2 deals with connectivity, diameter, girth and planarity of this graph. In section 3, vertex and edge connectivity of $\Gamma_{R,M}$ will be considered.

Now we remind some basic definitions and concepts in graph theory as following. Let X be a graph. X is a simple graph if it does not have loop and multiple edges. $V(X)$ is the set of vertices of X and the degree of a vertex v of X, denoted by $deg(v)$, is the number of edges incident to v. A path P is an alternating sequence vertices and edges, $v_1e_1v_2e_2v_3 \dots v_ke_kv_{k+1}$, such that for each i, $1 \leq i \leq k$, e_i is edge between v_i and v_{i+1}. If edges between vertices in path is not important we show them by $v_1 \sim v_2 \sim v_3 \sim ... \sim v_k \sim v_{k+1}$. We denote the minimum and maximum degrees of the vertices of X ,respectively, by δ and Δ and we use the notaion $N(v)$ for the set of neighbours of vertex v. For two vertices x and y, $\mathrm{d}(x, y)$ denotes the length of the shortest path between x and y and if there is no such path $d(x, y) = \infty$. The diameter of X is defined by $\mathrm{diam}(X)=\max\{d(x, y) : x$ and y are distinct vertices of $X\}$. X is a connected graph if there is a path between every two distinct vertices of X.

The girth of X is the length of the shortest cycle in X and it is denoted by $\mathrm{gr}(X)$ and if X does not have cycles $gr(X) = \infty$. A bipartite graph is a graph whose vertices can be divided into two disjoint parts V_1 and V_2 such that every edge connects a vertex in V_1 to one in V_2 .A complete bipartite graph is a special kind of bipartite graph where every vertex of the first part is connected to every vertex of the second part. A complete bipartite graph with parts of size $\mid V_1 \mid= m$ and $\mid V_2 \mid= n$ is denoted by $K_{m,n}$.

A dominating set in a graph X is a subset S of $V(X)$ such that every vertex not in S is joined to at least one member of S. The domination number $\gamma(X)$ of X is the number of vertices in the smallest dominating set for X. An independent set of X is a subset of vertices that no each pair of distinct vertices are adjacent. A maximal independent set is an independent set such that adding any other vertex to the set forces the set to contain some edges.The notation $\alpha(X)$ is the order of maximal independent set of X . A planar graph is a graph that can be embedded in the plane so that no two edges intersect expect at the end vertices. uv-paths P and Q in a graph are internally disjoint if they have no internal vertices in common, that is, if $V(P_1) \cap V(P_2) = \{u, v\}$. The local connectivity between distinct vertices u and v is the maximum number of pairwise internally disjoint uv-paths, denoted $p(u, v)$; the local connectivity is undefined when $u = v$. A non trivial graph X is k-connected if $p(u, v) \geq k$ for any distinct vertices u and v . The connectivity $\kappa(X)$ of X is the maximum value of k for which X is k-connected. Two uv-paths P and Q in a graph are internally edge-disjoint if they have no internal edges in common, that is, if $E(P_1) \cap E(P_2) = \emptyset$. All notations and terminology are standard here and we refer to [7] and [8]. Moreover, we always assume that R is finite ring, M is finite R-module.

2. Basic properties of $\Gamma_{R,M}$

In this section, we start with the definition of graph $\Gamma_{R,M}$

Definition. Let R be a ring and M be a left and right R-module such that M and R are distinct and have different object. The non-commutative graph of R with respect to R-module M, denoted by $\Gamma_{R,M}$, is defined as a bipartite graph with vertex sets $R \setminus C_R(M)$, $M \setminus C_M(R)$ as its parts such that $C_R(M) = \{x \in R \mid xy = yx \text{ for all } y \in M\}$ and $C_M(R) = \{x \in M | xy = yx \text{ for all } y \in R\}$ in such a way that two vertices $x \in R \setminus C_R(M)$ and $y \in M \setminus C_M(R)$ are adjacent if $xy \neq yx$.

It is clear that if we have one of the cases $R = \{0\}$, $M = \{0\}$, $R = C_R(M)$ or $M = C_M(R)$ then $V_R = \emptyset$ or $V_M = \emptyset$ and so the graph is null. Moreover, if R is a commutative ring, then graph is null. Thus we always assume that $R \neq \{0\}$ and $M \neq \{0\}$. We may also assume that R and M are finite and we do not mention it in the lemmas and theorems. If R is infinite, then we will state specifically. Let us state the following simple lemmas.

Lemma 2.1. *Graph $\Gamma_{R,M}$ does not have an isolated vertex.*

Proof. Assume that $V(\Gamma_{R,M}) = V_R \cup V_M$ where $V_R = R \setminus C_R(M)$ and $V_M = M \setminus C_M(R)$. If $x \in V_R$ is an arbitrary vertex in the first part, then we have $x \in R$ and $x \notin C_R(M)$ thus there exists an element $y \in M$ such that $xy \neq yx$. Hence $y \in M \setminus C_M(R)$ which it implies that y is vertex in V_M. Therefore x and y are adjacent. Similary, for every vertex of V_M there exists a vertex in V_R. □

Lemma 2.2. *Let x be a vertex in graph $\Gamma_{R,M}$. Then $deg(x) = |M| - |C_M(x)|$ if $x \in R \setminus C_R(M)$ and $deg(x) = |R| - |C_R(x)|$ if $x \in M \setminus C_M(R)$.*

Proof. Let $x \in R \setminus C_R(M)$ then $\deg(x) = |\{y \in M \setminus C_M(R) \text{ such that } xy \neq yx\}| = |M| - |\{y \in M \text{ such that } xy = yx\}| = |M| - |C_M(x)|$. If $x \in M \setminus C_M(R)$ thus it is similar to above. □

Lemma 2.3. *The graph $\Gamma_{R,M}$ is connected.*

Proof. Let x and y be two arbitrary vertices. We have two cases. The first case is that x and y are in the same part, for instance assume that $x, y \in V_R$. Then by Lemma 2.2, there are vertices $m, n \in V_M$ such that $x \sim m$, $y \sim n$. If $y \sim m$ or $x \sim n$ then we have a path between x and y and the proof is done. Suppose that $y \nsim m$ and $x \nsim n$ then put $z = x + y$ and we have $x \sim m \sim z \sim n \sim y$. The second case is that $x, \in V_R$ and $y \in V_M$. If x is adjacent to y then the proof follows. Otherwise, there are vertices $r \in V_R$ such that $r \sim y$, by Lemma 2.2. So we have path between x and r by the previous case and $r \sim y$. Therefore we have a path between x and y and the proof is completed. □

Lemma 2.4. *Let $a \in R$, $b \in M$ and $a + C_R(M)$, $b + C_M(R)$ are two cosets of R and R-module M,respectively, and a is adjacent to b if and only if every $x \in a + C_R(M)$ and every $y \in b + C_M(R)$ are adjacent.*

Proof. Assume that $x \in a+C_R(M)$ and $y \in b+C_M(R)$ then there are elements $z \in C_R(M)$ and $m \in C_M(R)$ such that $x = a+z$ and $y = b+m$. If x is not adjacent to y, then we have $xy = yx$ or $(a+z)(b+m) = (b+m)(a+z)$ or $ab = ba$ which is a contradiction. Hence x is adjacent to y as required. □

By the above lemma, one can see that induce subgraph of $\Gamma_{R,M}$ to the cosets $a + C_R(M)$ and $b + C_M(R)$ is a complete bipartite graph, where $a \in V_R$ and $b \in V_M$.

Lemma 2.5. *diam* $(\Gamma_{R,M}) \leq 3$.

Proof. Suppose that x and y are two arbitrary vertices. Then we have the following cases:

Case1: $x, y \in R\backslash C_R(M)$. Then there are vertices n and m in part $M\backslash C_M(R)$ such that $x \sim n$ and $y \sim m$. If $x \sim m$ or $y \sim n$., then $x \sim m \sim y$ or $x \sim n \sim y$ and so $d(x, y) = 2$ otherwise, we will have $x \sim n + m \sim y$ which implies that $d(x, y) = 2$ thus in case 1, $d(x, y) = 2$.

Case 2: $x \in R \setminus C_R(M)$ and $y \in M \setminus C_M(R)$. If $x \sim y$ then $d(x, y) = 1$. So assume that $x \nsim y$. By Lemma 2.2, there are vertices $n \in R \setminus C_R(M)$ such that $n \sim y$. Hence $d(x, y) \leq d(x, n) + d(n, y) \leq 3$. Thus $d(x, y) \leq 3$.

The case that $x, y \in M \setminus C_M(R)$, then the proof is very similar to case 1. □

Lemma 2.6. *If diam*$(\Gamma_{R,M}) = 2$ *then* $\Gamma_{R,M}$ *is complete bipartite graph.*

Proof. If there are two vertices that not adjacent so diam$(\Gamma_{R,M}) > 2$ which is a contradiction. □

In the following lemma, we compute the girth of $\Gamma_{R,M}$.

Lemma 2.7. *gr*$(\Gamma_{R,M}) = 4$ *or* 6.

Proof. We have three cases to follow:

Case 1: If $\mid V_R \mid= 1$ or $\mid V_M \mid= 1$ then gr$(\Gamma_{R,M}) = \infty$. So $\mid V_R \mid\geq 2$ and $\mid V_M \mid\geq 2$.

Case 2: If r_1, r_2 are in one part. Assume that $r_1, r_2 \in V_R$. By lemma 2.2 there exist $m_1, m_2 \in V_M$ such that if $r_1 \sim m_1, r_2 \sim m_2$ then gr$(\Gamma_{R,M}) = 4$.

Case 3: If $r_1 \in V_R$ and $m_1 \in V_M$. By lemma 2.2 there exist $r_2 \in V_R$ and $m_2 \in V_M$ such that $r_1 \sim m_1, r_2 \sim m_2$ so we will have $r_1 \sim m_1 + m_2 \sim r_2 \sim m_2 \sim r_1 + r_2 \sim m_1 \sim r_1$. The proof is completed. □

Lemma 2.8. $R = \cup_{a\in R} a + C_R(M)$.

Proof. The proof follows the fact that $C_R(M)$ is an additive subgroup of R. □

Lemma 2.9. *If* $x \in V(\Gamma_{R,M})$, *if* $x \in V_R$ *then* $d(x) \geq \lceil \frac{|M\backslash C_M(R)|}{2} \rceil$. *or if* $x \in V_M$ *then* $d(x) \geq \lceil \frac{|R\backslash C_R(M)|}{2} \rceil$.

Proof. Case 1: Assume that $x \in V_R$. Let x be adjacent to y in $M \setminus C_M(R)$ and the vertex m_i not be adjacent to x in $M \setminus C_M(R)$, for each $1 \leq i \leq k$. Obviously, the vertices y, $y+m_1$, $y+m_2$, $\cdots$, $y+m_k$ are in $M \setminus C_M(R)$ and are adjacent to x. Since they are distinguished, then x is adjacent to $k+1$ vertices.

Case 2: If $x \in V_M$ then the proof is similar above. □

Lemma 2.10. $\alpha(\Gamma_{R,M}) = max\{\mid R \setminus C_R(M) \mid, \mid M \setminus C_M(R) \mid\}$.

Proof. Without loss of generality, suppose that $\mid M \setminus C_M(R) \mid \leq \mid R \setminus C_R(M) \mid = k$. Let H be the largest independent set and $V(H) = V_1 \cup V_2$, where $V_1 \subseteq V(R \setminus C_R(M))$ and $V_2 \subseteq V(M \setminus C_M(R))$. Clearly, $V_1 \neq \emptyset$ or $V_2 \neq \emptyset$. By Lemma 2.9, each vertex of V_1 or V_2 is adjacent to at least $\lceil \frac{|M \setminus C_M(R)|}{2} \rceil$ or $\lceil \frac{|R \setminus C_R(M)|}{2} \rceil$ vertices, respectively. Therefore,

$$\mid V_1 \mid + \mid V_2 \mid \leq \lceil \frac{\mid M \setminus C_M(R) \mid}{2} \rceil + \lceil \frac{\mid R \setminus C_R(M) \mid}{2} \rceil.$$

Thus, $V(H)$ is lower than k. □

Lemma 2.11. *If graph $\Gamma_{R,M}$ has a pendent then $\Gamma_{R,M}$ is star graph.*

Proof. Suppose that $x \in R \setminus C_R(M)$ and $\deg(x) = 1$ then by Lemma 2.3, $|M| - |C_M(x)| = 1$ or $|C_M(x)| = |M| - 1$. Since $C_M(x) \subset M$ so $|C_M(x)|||M|$ or $|M| - 1||M|$ then there exists a positive integer k such that $|M| = k(|M| - 1)$. Thus $k = \frac{|M|}{|M|-1} \in Z$ and we should have $|M| - 1 = 1$ consequently $|M| = 2$ or $|M| - 1 = |M|$ that is a contradicthion. We know that $C_M(R) \subseteq M$ then we have two cases: $C_M(R) = M$ that it is a contradiction and $C_M(R) = \{0\}$ then $M \setminus C_M(R) = \{a\}$ so $|M \setminus C_M(R)| = 1$ therefore $\Gamma_{R,M}$ is star graph. □

Theorem 2.12. *If $|C_R(M)| \geq 3$ and $|C_M(R)| \geq 3$ then $\Gamma_{R,M}$ is not planar graph.*

Proof. Since $|C_R(M)| \geq 3$ and $|C_M(R)| \geq 3$ then the order of cosets of $C_R(M)$ and $C_M(R)$ are at least 3. Let $x \in C_R(M)$ so by Lemma 2.2, there exists $a \in M \setminus C_M(R)$ such that $x \sim a$ and by Lemma 2.5, x is adjacent to all members of coset $a + C_M(R)$ and because the order of cosets of R is at least 3 so we have a subgraph $K_{3,3}$.Therefore $\Gamma_{R,M}$ is not planar. □

Corollary 2.13. *If $|C_R(M)| \geq 2$ and $|C_M(R)| \geq 3$ then $\Gamma_{R,M}$ is not 1-plannar graph.*

Proof. It is similar to proof of Theorem 2.12. □

3. Vertex and edge connectivity

In this section, we discuss about vertex and edge k-connectivity of $\Gamma_{R,M}$. Let us state the following lemma which plays an important role in the proof of next theorem.

Lemma 3.1. *Every connected graph G satisfies the inequalities $\kappa(G) \leq \kappa^{'}(G) \leq \delta$.*

Definition. For two vertices x and y which are in the same part, put $Z = N(x) \cap N(y)$, $X = N(x) \setminus Z$ and $Y = N(y) \setminus Z$.

Lemma 3.2. *Let x and y be the vertices in the same part. Then $|Z| > |X|$ and $|Z| > |Y|$.*

Proof. If $X = \emptyset$ or $Y = \emptyset$, then the assertion holds by Lemma 2.9. So, suppose that $1 \leq |X| \leq |Y|$ and $x_1 \in X$. If $Y = \{y_1, y_2, \cdots, y_k\}$, then $\{y_1 + x_1, y_2 + x_1, \cdots, y_k + x_1\} \subseteq Z$. □

Theorem 3.3. *Graph $\Gamma_{R,M}$ is δ-edge-connected.*

Proof. Let x, y be two vertices of $\Gamma_{R,M}$. We show that there exists $\min\{d(x), d(y)\}$ edge disjoint paths between x and y. Consider the two following cases:

Case 1: x and y belong to the same part. Assume that $x, y \in R \setminus C_R(M)$ and $|X| \leq |Y|$. Put $X = \{x_1, x_2, \cdots, x_k\}$, $Y = \{y_1, y_2, \cdots, y_l\}$ and $Z = \{z_1, z_2, \cdots, z_t\}$. For vertices x_i and y_i, there is vertex u_i such that $x_i \sim u_i \sim y_i$, $1 \leq i \leq k$. Now, consider the paths $x \sim y_i \sim y$ $(1 \leq i \leq l)$ and $x \sim x_i \sim u_i \sim y_i \sim y$ $(1 \leq i \leq k)$. Therefore, there exists $d(x)$ edge disjoint paths between x and y.

Case 2: x and y are in the different parts. Let $x \in R \setminus C_R(M)$ and $y \in M \setminus C_M(R)$ and consider the following cases:

Case 2.1: $x \nsim y$. Put H the induced bipartite subgraph on $A \cup B$, where $A = N(x) \cup \{y\}$ and $B = N(y) \cup \{x\}$. For each set $S \subseteq A$, we claim that $N(S) \geq S$. If $|S| < |A|/2$, then we have $N(S) \geq S$, by Lemma 2.9. If $|S| \geq |A|/2$ and $a \in B \setminus N(S)$, then we have $N(a) \subset A \setminus S$, that it is a contradiction by Lemma 2.9. From Hall's Theorem we can conclude that graph H has a perfect matching. If vertex x is not in perfect matching, then assume that $x_i \sim y_i$, for every integer i, $1 \leq i \leq k$. In this case, there are k disjoint paths $x \sim x_i \sim y_i \sim y$. If vertex x is in perfect matching, then consider $x \sim x_1$ and $x_i \sim y_i$ and $y \sim y_{k+1}$, for each $2 \leq i \leq k$. Consider the following cases:

Case 2.1.1: y_{k+1} is adjacent to x_1. Then consider the paths $x \sim x_i \sim y_i \sim y$, for $2 \leq i \leq k$ and $x \sim x_1 \sim y_{k+1} \sim y$.

Case 2.1.2: y_{k+1} is not adjacent to x_1. By Lemma 2.9, there is integer t such that $2 \leq t \leq k$, $x_1 \sim y_t$ and $y_{k+1} \sim x_t$. Now, there is k disjoint paths between x to y as $x \sim x_1 \sim y_t \sim y$, $x \sim x_t \sim y_{k+1} \sim y$, and $x \sim x_i \sim y_i \sim y$, for integers $2 \leq i \leq k$ except $i = t$.

Case 3: $x \sim y$. Each vertex of the set A or B is adjacent to half of vertices of other part, by Lemma 2.9. It is not difficult to see that similar to Case

2.1, graph H has a perfect matching by Hall's Theorem. Let the edge xy be in the matching. Without of lose of generality, assume that $x_i \sim y_i$, for every $1 \leq i \leq k$.. Thus, $x \sim y$ and $x \sim x_i \sim y_i \sim y$ are the disjoint xy-paths. If the edge xy is not in the matching, then similar to Case 1, we have k edge disjoint paths. □

The following theorem gives the maximum value of k for which $\Gamma_{R,M}$ is k-connected.

Theorem 3.4. *$\kappa(\Gamma_{R,M}) = \delta$.*

Proof. Let x, y be two vertices of $\Gamma_{R,M}$. If x and y are in different parts, then there are δ disjoint paths between x and y, by Theorem 3.3. Now, suppose that x and y are in the same part. Without lose of generality, assume that $x, y \in R \setminus C_R(M)$ and $d(x) = \delta$. Put $X = \{x_1, x_2, \cdots, x_k\}$, $Y = \{y_1, y_2, \cdots, y_l\}$ and $Z = \{z_1, z_2, \cdots, z_t\}$. By Lemma 3.2, $t > l$ and $t > k$. Put $A_i = N(x_i) \cap N(y_i)$, for $1 \leq i \leq k$. Since $d(x) = \delta$, we have $|A_i| > k$. Thus, there exist k distinguished vertices $h_1, h_2, \cdots, h_k$ such that $x_i \sim h_i \sim y_i$, $1 \leq i \leq k$. Now, consider xy-paths $x \sim z_i \sim y$, $1 \leq i \leq t$ and $x \sim x_i \sim h_i \sim y_i \sim y$, $1 \leq i \leq k$. Thus, there is δ disjoint xy-paths. If $d(x) \neq \delta$, then it is enough to put δ neighbours of x and do like above. □

References

[1] A. Abdollahi, S. Akbari and H. R. Maimani, *Non-commuting graph of a group*, J. Algebra, 298 (2006), 468-492.

[2] S. Akbari and A. Mohammadian, *Zero-divisor graphs of non-commutative rings*, J. Algebra, 296 (2006), 462-479.

[3] D. F. Anderson and P. S. Livingston, *The zero-divisor graph of a commutative ring*, J. Algebra, 217 (2) (1999) 434-447.

[4] P. J. Cameron, *The power graph of a finite group*, J. Group Theory, 13 (2010), 779-783.

[5] I. Chakrabarty, S. Ghosh, T. K. Mukherjee and M. K. Sen,*Intersection graphs of ideals of rings*, Discrete Math., 309 (2009), 5381-5392.

[6] H. Maimani, M. Salimi, A. Sattari and S. Yassemi, *Comaximal graph of commutative rings*, J. Algebra, 319 (2008), 1801-1808.

[7] R. Y. Sharp, *Steps in Commutative Algebra*, Cambridge University Press, Cambridge, 1990.

[8] West D.B., *Introduction to Graph Theory*, 2000.

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