ON POSITIVE WEAK SOLUTIONS FOR A CLASS OF NONLINEAR SYSTEMS

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Abstract. We study the positive weak solutions for the system

$$\left.\begin{array}{rl} -\Delta_{P,p}u = \lambda a(x)f(v) & \text{in } \Omega, \\ -\Delta_{P,p}v = \lambda b(x)g(u) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega. \end{array}\right\}$$

where $\lambda > 0$ is a parameter, $\Delta_{P,p}$ with $p > 1$ and $P = P(x)$ is a weight function, denotes the weighted p-Laplacian defined by $\Delta_{P,p}u \equiv div[P(x)|\nabla u|^{p-2}\nabla u]$, $a(x)$, $b(x)$ are weight functions and $\Omega \subset \Re^N$ is a bounded domain with smooth boundary $\partial\Omega$. We discuss the existence of positive weak solutions for large λ when

$$\lim_{x\to+\infty} \frac{f^{\frac{1}{p-1}}(M(g(x))^{\frac{1}{p-1}})}{x} = 0, \qquad \text{for every } \; M > 0.$$

In particular, we do not assume any sign-changing conditions on $a(x)$ or $b(x)$. Our approach depends on the method of sub–supersolutions.
Keywords: weak solution, p-Laplacian, nonlinear system, sub-supersolutions.

1. Introduction

In this paper, we study the existence of positive weak solutions for λ large for the following nonlinear system

$$\left.\begin{array}{rl} -\Delta_{P,p}u = \lambda a(x)f(v) & \text{in } \Omega, \\ -\Delta_{P,p}v = \lambda b(x)g(u) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{array}\right\} \tag{1.1}$$

where $\Delta_{P,p}$ with $p > 1$ and $P = P(x)$ is a weight function, denotes the weighted p-Laplacian defined by $\Delta_{P,p}u \equiv div[P(x)|\nabla u|^{p-2}\nabla u]$, λ is a positive parameter, $a(x)$ and $b(x)$ are weight functions and that there exist positive constants a_0, b_0 such that $a(x) \geq a_0$, $b(x) \geq b_0$, f and g are given functions and $\Omega \subset \Re^N$ is a bounded domain with smooth boundary $\partial\Omega$. Our approach is based on the method of sub-supersolutions (see e.g. [2]).

Recently many results concerning the existence of positive weak solutions for the nonlinear systems involving Laplacian, p-Laplacian or weighted p-Laplacian operators were obtained by various authors with the help of the sub-supersolutions method (see [1, 3, 6],[8]-[16]).

Dalmasso [4] have been studied system (1.1) when $p = 2$, $P(x) = a(x) = b(x) = 1$, f and g are increasing functions and $f, g \geq 0$. Results of [4] extended in [7] to the case when no sigh conditions on $f(0)$ or $g(0)$ were required and without assuming monotonicity conditions on f or g.

This paper is organized as follows:

In section 2, we introduce some technical results and notations, which are established in [5]. In section 3, we give some assumptions on the functions f, g to insure the validity of the existence of the positive weak solutions for system (1.1) in a suitable weighted Sobolev space. Also, we prove the existence of positive weak solutions for system (1.1) by using the method of sub–supersolutions. In section 4, we give some related results and examples.

2. Technical results

Now, we introduce some technical results to the weighted homogeneous eigenvalue problem (see [5])

$$\left.\begin{aligned} -\Delta_{P,p}u \equiv div[P(x)|\nabla u|^{p-2}\nabla u] = \lambda a(x)|u|^{p-2}u \quad & \text{in}\,\Omega, \\ u = 0 \quad & \text{on}\,\partial\Omega. \end{aligned}\right\} \tag{2.1}$$

The function $P(x)$ is a weight function (measurable and positive a.e. in Ω), satisfying the conditions

$$\begin{aligned} & P(x),\, (P(x))^{-\frac{1}{p-1}} \in L^1_{Loc}(\Omega),\ \text{with}\ p > 1, (P(x))^{-s} \in L^1(\Omega), \\ & \text{with}\ s \in (\frac{N}{p}, \infty) \cap [\frac{1}{p-1}, \infty), \end{aligned} \tag{2.2}$$

and $a(x)$ is a measurable function satisfies

$$a(x) \in L^{\frac{k}{k-r}}(\Omega), \tag{2.3}$$

with some k satisfies $p < k < p_s^*$ where $p_s^* = \frac{Np_s}{N-p_s}$ with $p_s = \frac{ps}{s+1} < p < p_s^*$ and $meas\ \{x \in \Omega : a(x) > 0\} > 0$. Examples of functions satisfying (2.2) are mentioned in [5].

Lemma 1 ([5]). *There exists the first eigenvalue $\lambda_{1p} > 0$ and at least one corresponding eigenfunction $\phi_{1p} \geq 0$ a.e. in Ω of the eigenvalue problem* (2.1).

Theorem 2 ([5])**.** *Let $P(x)$ satisfies* (2.2) *and $a(x)$ satisfies* (2.3), *then* (2.1) *admits a positive eigenvalue λ_{1p}. Moreover, it is characterized by*

$$\lambda_{1p} \int_\Omega a(x)|\phi_{1p}|^p \leq \int_\Omega P(x)|\nabla\phi_{1p}|^p. \tag{2.4}$$

Moreover, let us consider the weighted Sobolev space $W^{1,p}(P,\Omega)$ which is the set of all real valued functions u defined in Ω with the norm

$$\|u\|_{W^{1,p}(P,\Omega)} = \left(\int_\Omega |u|^p + \int_\Omega P(x)|\nabla u|^p\right)^{\frac{1}{p}} < \infty, \tag{2.5}$$

and the space $W_0^{1,p}(P,\Omega)$ which is the closure of $C_0^\infty(\Omega)$ in $W^{1,p}(P,\Omega)$ with respect to the norm

$$\|u\|_{W_0^{1,p}(P,\Omega)} = \left(\int_\Omega P(x)|\nabla u|^p\right)^{\frac{1}{p}} < \infty, \tag{2.6}$$

which is equivalent to the norm given by (2.5). The two spaces $W^{1,p}(P,\Omega)$ and $W_0^{1,p}(P,\Omega)$ are well defined reflexive Banach Spaces.

3. Existence results

In this section, we prove the existence of positive weak solutions (u,v) for system (1.1) via the method of sub-supersolutions. We shall establish our results by constructing a subsolution $(\psi_1,\psi_2) \in (W_0^{1,p}(P,\Omega))^2$ and a supersolution $(z_1,z_2) \in (W_0^{1,p}(P,\Omega))^2$ of (1.1) such that $\psi_i \leq z_i$ for $i = 1,2$. That is, ψ_i, $i = 1,2$, satisfies

$$\begin{aligned}
\int_\Omega P(x)|\nabla\psi_1|^{p-2}\nabla\psi_1\nabla\zeta dx &\leq \lambda\int_\Omega a(x)f(\psi_2)\zeta dx\\
\int_\Omega P(x)|\nabla\psi_2|^{p-2}\nabla\psi_2\nabla\zeta dx &\leq \lambda\int_\Omega b(x)g(\psi_1)\zeta dx
\end{aligned}$$

and z_i, $i = 1,2$, satisfies

$$\begin{aligned}
\int_\Omega P(x)|\nabla z_1|^{p-2}\nabla z_1\nabla\zeta dx &\geq \lambda\int_\Omega a(x)f(z_2)\zeta dx\\
\int_\Omega P(x)|\nabla z_2|^{p-2}\nabla z_2\nabla\zeta dx &\geq \lambda\int_\Omega b(x)g(z_1)\zeta dx,
\end{aligned}$$

for all test functions $\zeta \in W_0^{1,p}(P,\Omega)$ with $\zeta \geq 0$. Then the following result holds:

Lemma 3 ([2])**.** *Suppose there exist sub and supersolutions (ψ_1,ψ_2) and (z_1,z_2) respectively of system (1.1) such that $(\psi_1,\psi_2) \leq (z_1,z_2)$. Then system* (1.1) *has a solution (u,v) such that $(u,v) \in [(\psi_1,\psi_2),(z_1,z_2)]$.*

We give the following hypotheses:

($\mathbf{H}_1$) $f, g : [0,\infty) \longrightarrow [0,\infty)$ are C^1 nondecreasing functions such that $f(s), g(s) > 0$ for $s > 0$.

($\mathbf{H}_2$) For all $M > 0$,

$$\lim_{x\to+\infty} \frac{f^{\frac{1}{p-1}}(M(g(x))^{\frac{1}{p-1}})}{x} = 0.$$

Theorem 4. *Let* $(H.1)$ *and* $(H.2)$ *hold. Then system* (1.1) *has a positive weak solution* $(u,v) \in (W_0^{1,p}(P,\Omega))^2$ *for* λ *large.*

Proof. Let λ_{1p} be the first eigenvalue of the eigenvalue problem (2.1) and ϕ_{1p} the corresponding positive eigenfunction with $\|\phi_{1p}\|_\infty = 1$. Let k_0, m, $\delta > 0$ be such that $f(x), g(x) \geq -k_0$ for all $x \geq 0$, $P(x)|\nabla\phi_{1p}|^p - \lambda_{1p}a(x)\phi_{1p}^p \geq m$ on $\overline{\Omega}_\delta = \{x \in \Omega : d(x,\partial\Omega) \leq \delta\}$. We shall verify that $(\psi_1,\psi_2) = (\frac{p-1}{p}(\frac{\lambda a_0 k_0}{m})^{\frac{1}{p-1}}\phi_{1p}^{\frac{p}{p-1}}, \frac{p-1}{p}(\frac{\lambda b_0 k_0}{m})^{\frac{1}{p-1}}\phi_{1p}^{\frac{p}{p-1}})$ is a subsolution of (1.1) for λ large. Let $\zeta \in W_0^{1,p}(P,\Omega)$ with $\zeta \geq 0$.

A calculation shows that

$$\begin{aligned}
&\int_\Omega P(x)|\nabla\psi_1|^{p-2}\nabla\psi_1\cdot\nabla\zeta dx \\
&= \frac{\lambda a_0 k_0}{m}\int_\Omega P(x)\phi_{1p}|\nabla\phi_{1p}|^{p-2}\nabla\phi_{1p}\cdot\nabla\zeta dx \\
&= \frac{\lambda a_0 k_0}{m}\left\{\int_\Omega P(x)|\nabla\phi_{1p}|^{p-2}\nabla\phi_{1p}\nabla(\phi_{1p}\zeta)dx - \int_\Omega P(x)|\nabla\phi_{1p}|^p\zeta dx\right\} \\
&= \frac{\lambda a_0 k_0}{m}\int_\Omega(\lambda_{1p}a(x)\phi_{1p}^p - P(x)|\nabla\phi_{1p}|^p)\zeta dx.
\end{aligned}$$

Similarly, we have

$$\int_\Omega P(x)|\nabla\psi_2|^{p-2}\nabla\psi_2\cdot\nabla\zeta dx = \frac{\lambda b_0 k_0}{m}\int_\Omega(\lambda_{1p}b(x)\phi_{1p}^p - P(x)|\nabla\phi_{1p}|^p)\zeta dx.$$

Now, on $\overline{\Omega}_\delta$, we have $P(x)|\nabla\phi_{1p}|^p - \lambda_{1p}a(x)\phi_{1p}^p \geq m$. Hence,

$$\frac{\lambda a_0 k_0}{m}(\lambda_{1p}a(x)\phi_{1p}^p - P(x)|\nabla\phi_{1p}|^p) \leq -\lambda a_0 k_0 \leq \lambda a(x) f(\psi_2).$$

A similar argument shows that

$$\frac{\lambda b_0 k_0}{m}(\lambda_{1p}b(x)\phi_{1p}^p - P(x)|\nabla\phi_{1p}|^p) \leq -\lambda b_0 k_0 \leq \lambda b(x) g(\psi_1).$$

Next, on $\Omega - \overline{\Omega}_\delta$, we have $\phi_{1p} \geq \mu$ for some $\mu > 0$. Also $f(\psi_2)$ and $g(\psi_1)$ are depending on λ and nondecreasing functions and therefore for λ large we have, using (2.4),

$$\begin{aligned}
f(\psi_2) &\geq \frac{k_0}{m}\lambda_{1p} \geq \frac{k_0}{m}(\lambda_{1p}a(x)\phi_{1p}^p - P(x)|\nabla\phi_{1p}|^p), \\
g(\psi_1) &\geq \frac{k_0}{m}\lambda_{1p} \geq \frac{k_0}{m}(\lambda_{1p}b(x)\phi_{1p}^p - P(x)|\nabla\phi_{1p}|^p).
\end{aligned}$$

Hence

$$\int_\Omega P(x)|\nabla\psi_1|^{p-2}\nabla\psi_1\cdot\nabla\zeta dx \le \lambda\int_\Omega a(x)f(\psi_2)\zeta dx.$$

Similarly, we have

$$\int_\Omega P(x)|\nabla\psi_2|^{p-2}\nabla\psi_2\cdot\nabla\zeta dx \le \lambda\int_\Omega b(x)g(\psi_1)\zeta dx,$$

i.e. (ψ_1,ψ_2) is a subsolution of (1.1) for λ large.

Next, let e_p be the solution of (see [17])

$$-\Delta_{P,p}e_p = 1 \quad \text{in } \Omega, \qquad e_p = 0 \qquad \text{on } \partial\Omega.$$

Let

$$(z_1,z_2)=\left(\frac{C}{\mu_p}\lambda^{\frac{1}{p-1}}e_p,(l_b\lambda)^{\frac{1}{p-1}}\ [g(C\lambda^{\frac{1}{p-1}})]^{\frac{1}{p-1}}e_p\right)$$

where $\mu_p=\|e_p\|_\infty$, $l_b=\|b(x)\|_\infty$and $C>0$ is a large number to be chosen later, We shall verify that (z_1,z_2) is a supersolution of (1.1) for λ large. To this end, let $\zeta\in W_0^{1,p}(P,\Omega)$ with $\zeta\ge 0$. Then we have

$$\begin{aligned}\int_\Omega P(x)|\nabla z_1|^{p-2}\nabla z_1\cdot\nabla\zeta dx &= \lambda\left(\frac{C}{\mu_p}\right)^{p-1}\int_\Omega P(x)|\nabla e_p|^{p-2}\nabla e_p\cdot\nabla\zeta dx\\ &= \frac{1}{\mu_p^{p-1}}\left(C\lambda^{\frac{1}{p-1}}\right)^{p-1}\int_\Omega\zeta dx.\end{aligned}$$

By $(\mathbf{H}_2)$, we can choose C large enough so that

$$(C\lambda^{\frac{1}{p-1}})^{p-1}\ge(\mu_p^{p-1}l_a\lambda)f([(l_b\lambda)^{\frac{1}{p-1}}\mu_p][g(C\lambda^{\frac{1}{p-1}})]^{\frac{1}{p-1}}),$$

where $l_a=\|a(x)\|_\infty$, and therefore,

$$\begin{aligned}\int_\Omega P(x)|\nabla z_1|^{p-2}\nabla z_1\cdot\nabla\zeta dx &\ge \lambda l_a\int_\Omega f([(l_b\lambda)^{\frac{1}{p-1}}\mu_p][g(C\lambda^{\frac{1}{p-1}})]^{\frac{1}{p-1}})\zeta dx\\ &\ge \lambda\int_\Omega a(x)f(z_2)\zeta dx.\end{aligned}$$

Next, we have

$$\begin{aligned}\int_\Omega P(x)|\nabla z_2|^{p-2}\nabla z_2\cdot\nabla\zeta dx &= \lambda l_b g(C\lambda^{\frac{1}{p-1}})\int_\Omega P(x)|\nabla e_p|^{p-2}\nabla e_p\cdot\nabla\zeta dx\\ &= \lambda l_b g(C\lambda^{\frac{1}{p-1}})\int_\Omega\zeta dx\\ &\ge \lambda l_b\int_\Omega g(C\mu_p^{-1}\lambda^{\frac{1}{p-1}}e_p)\zeta dx\\ &\ge \lambda\int_\Omega b(x)g(z_1)\zeta dx\end{aligned}$$

i.e. (z_1,z_2) is a supersolution of (1.1) with $z_i\ge\psi_i$ for C large, $i=1,2$. Thus, there exists a positive weak solution (u,v) of (1.1) with $\psi_1\le u\le z_1,\ \psi_2\le v\le z_2$. This completes the proof.

4. Example and related result

4.1 Example

Many illustrative examples for the results obtained in this paper can be easily constructed. We just give the following one below.

Let

$$f(x) = ax^r, \; g(x) = bx^s,$$

where, $a, b, r, s > 0$ and $rs < (p-1)^2$. Then it is easy to see that f and g satisfy $(\mathbf{H}_1)$, $(\mathbf{H}_2)$.

4.2 Related result

Existence results obtained in this article can be established in a similar way for the following nonlinear system

$$\left.\begin{array}{rl} -\Delta_{P,p} u = \lambda a(x) v^{\beta} & \text{in } \Omega, \\ -\Delta_{P,p} v = \lambda b(x) u^{\alpha} & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{array}\right\}$$

under the assumptions that

$(\boldsymbol{A}_1)$ $a(x)$ and $b(x)$ are weight functions such that $a(x) \geq a_0 > 0$, $b(x) \geq b_0 > 0$.

$(\boldsymbol{A}_2)$ $0 < \alpha < p-1$ and $0 < \beta < p-1$.

Remark 5. Existence results of positive weak solutions for system (1.1) still hold if we replace the condition $\lim_{x\to+\infty} \frac{f^{\frac{1}{p-1}}(M(g(x))^{\frac{1}{p-1}})}{x} = 0$, for every $M > 0$, given in (H_2), by the condition $\lim_{x\to+\infty} \frac{f[M(g(x))^{\frac{1}{p-1}}]}{x^{p-1}} = 0$, for every $M > 0$.

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References

[1] G. Afrouzi, S. Ala, *An existence result of positive solutions for a class of Laplacian system,* Int. Journal of Math. Analysis, 4 (42) (2010), 2075-2078.

[2] A. Canada, P. Drabek, J. Games, *Existence of Positive solutions for some problems with nonlinear diffusion*, Trans. Amer. Math. Soc., 349 (1997), 4231-4249.

[3] M. Chhetri, D. Hai, R. Shivaji, *On positive solutions for classes of p-Laplacian semipositone system*, Discrete and Dynamical Systems, 9(4) (2003), 1063-1071.

[4] R. Dalmasso, *Existence and uniqueness of positive solutions of semilinear elliptic systems*, Nonlinear Anal., 39 (2000), 559-568.

[5] P. Drabek, A. Kufner, F. Nicolosi, *Quasilinear elliptic equation with degenerations and singularities*, Walter de Gruyter, Bertin, New York, 1997.

[6] D. Hai, R. Shivaji, *An existence result on positive solutions for a class of p-Laplacian systems*, Nonlinear Anal., 56 (2004), 1007-1010.

[7] D. Hai, R. Shivaji, *An existence result on positive solutions for a class of semilinear elliptic systems*, Proc. Roy. Soc. Edinburgh Sect. A, 134 (2004), 137-141.

[8] S. Khafagy, *Existence and nonexistence of positive weak solutions for a class of (p, q)-Laplacian with different weights*, Int. J. Contemp. Math. Sciences, 6 (48) (2011), 2391-2400.

[9] S. Khafagy, *Existence and nonexistence of positive weak solutions for a class of weighted (p, q)-Laplacian nonlinear system*, Global Journal of Pure and Applied Mathematics, 9(4) (2013), 379-387.

[10] S. Khafagy, *Maximum principle and existence of weak solutions for nonlinear system involving weighted (p, q)-Laplacian,* Southeast Asian Bulletin of Mathematics, 40 (2016), 353-364.

[11] S. Khafagy, *Maximum principle and existence of weak solutions for nonlinear system involving singular p-Laplacian operators,* Journal of Partial differential equations, 29 (2) (2016), 89-101.

[12] S. Khafagy, *Non-existence of positive weak solutions for some weighted p-Laplacian system*, Journal of Advanced Research in Dynamical and Control Systems, 7 (2015), 71-77.

[13] S. Khafagy, *On positive weak solutions for nonlinear elliptic system involving singular p-Laplacian operator,* Journal of Mathematical Analysis, 7 (5) (2016), 10-17.

[14] S. Khafagy, *On the stability of positive weak solution for weighted p-Laplacian nonlinear system*, New Zealand Journal of Mathematics, 45 (2015), 39-43.

[15] E. Lee, R. Shivaji, J. Ye, *Positive solutions for elliptic equations involving nonlinearities with falling zeroes*, Applied Mathematics Letters, 22 (2009), 846-851.

[16] S. Rasouli, Z. Halimi, Z. Mashhadban, *A note on the existence of positive solution for a class of Laplacian nonlinear system with sign-changing weight*, The Journal of Mathematics and Computer Science, 3 (3) (2011), 339-354.

[17] H. Serag, S. Khafagy, *On maximum principle and existence of positive weak solutions for $n \times n$ nonlinear systems involving degenerated p-Laplacian operator*, Turk. J. M., 34 (2010), 59-71.

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