ON PARTIALLY TOPOLOGICAL GROUPS: EXTENSION CLOSED PROPERTIES

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Abstract. The partially (para)topological groups were defined in [9]. In this paper, we give more results in partially topological groups in the sense of H. Delfs and M. Knebusch and we prove extension closed property for connectedness, compactness, and separability of partially topological groups.

Keywords: extension closed property, Delfs-Knebusch generalized topology, partially topological groups.

1. Introduction and preliminaries

In 1985, H. Delfs and M. Knebusch [3] introduced a notion of generalized topological space. Morphisms between such spaces were named strictly continuous. In 2013, a more convenient definition of a generalized topological space (gts) was

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introduced and basic theory of gtses was developed by A. Piękosz [10, 11]. Here we will work in the category $\mathbf{GTS}_{pt}$ of partially topological gtses and strictly continuous mappings (introduced in [10]). From now on, we will use notations from [10].

Generalized topology in the sense od H. Delfs and M. Knebusch is an unknown chapter of general topology. In fact, it is a generalization of the classical concept of topology. The aim of this paper is to continue the systematic study of generalized topology in this sense. Do not mix with other meanings of generalized topology appearing in the literature.

Now we need some assumptions in this paper. We assume the existence of a universe $\mathbb{U}$ (cf. pages 22 and 23 of [8]). Sets in the sense of [8] can be called *totalities* or *collections*. A *class* is a collection $u \subseteq \mathbb{U}$. A *proper class* is a class u such that $u \notin \mathbb{U}$. Elements of the universe $\mathbb{U}$ are called $\mathbb{U}$-small sets in [8]. We denote by *ZF* the system of axioms which consists of the existence of a universe and axioms 0–8 from pages 9–10 of [7] for sets in the sense of [8]. From now on, a totality u will be called a *set* if and only if u is a $\mathbb{U}$-small set. All other set-theoretic axioms applied here, defined in [6] and independent of *ZF*, concern only $\mathbb{U}$-small sets. Axioms independent of *ZF* that are used in this work will be denoted as in [6]. When we do not involve proper classes, we can use directly ZF instead of **ZF** and still use the notation **ZF** of [6] for ZF. That is why we write $\mathbf{ZF} + \mathbf{AC} = \mathbf{ZFC}$. We recall that, in view of Exercise E2 in Section 1.1 of [6], given a finite collection of sets $X_1, \dots, X_n$, where $n \in \omega \setminus \{0\}$, the statement that every X_i with $i \in \{1, ..., n\}$ is non-void is equivalent in **ZF** to the statement that the collection $\{X_1, \dots X_n\}$ has a choice function.

The basic set-theoretic assumption of this work is **ZF**. If a theorem is unprovable in **ZF** or if we give its proof not in **ZF**, we shall clearly denote the system of axioms we use in the proof. In all other cases, theorems and their proofs are in **ZF**.

Definition 1.1. Let X be any set, τ_X be a topology on X. A family of open families $\mathrm{Cov}_X \subseteq \mathcal{P}(\tau_X)$ will be called a *partial topology* if the following conditions are satisfied:

(i) if $\mathcal{U} \subseteq \tau_X$ and $\mathcal{U}$ is finite, then $\mathcal{U} \in \mathrm{Cov}_X$;

(ii) if $\mathcal{U} \in \mathrm{Cov}_X$ and $V \in \tau_X$, then $\{U \cap V : U \in \mathcal{U}\} \in \mathrm{Cov}_X$;

(iii) if $\mathcal{U} \in \mathrm{Cov}_X$ and, for each $U \in \mathcal{U}$, we have $\mathcal{V}(U) \in \mathrm{Cov}_X$ such that $\bigcup \mathcal{V}(U) = U$, then $\bigcup_{U\in\mathcal{U}} \mathcal{V}(U) \in \mathrm{Cov}_X$;

(iv) if $\mathcal{U} \subseteq \tau_X$ and $\mathcal{V} \in \mathrm{Cov}_X$ are such that $\bigcup \mathcal{V} = \bigcup \mathcal{U}$ and, for each $V \in \mathcal{V}$ there exists $U \in \mathcal{U}$ such that $V \subseteq U$, then $\mathcal{U} \in \mathrm{Cov}_X$.

Elements of τ_X are called *open sets*, and elements of Cov_X are called *admissible families*. We say that (X, Cov_X) is a *partially topological generalized* *topological space* or simply *partially topological space*. We shall denote a partially topological space (X, Cov_X) by X when no confusion will arise. Since $\tau_X = \bigcup \mathrm{Cov}_X$, we can omit τ_X in notation.

Let X and Y be partially topological spaces and let $f : X \to Y$ be a function. Then f is called *strictly continuous* if $f^{-1}(\mathcal{U}) \in \mathrm{Cov}_X$ for any $\mathcal{U} \in \mathrm{Cov}_Y$. A bijection $f : X \to Y$ is called a *strict homeomorphism* if both f and f^{-1} are strictly continuous functions. If we have a strict homeomorphism between X and Y we say that they are *strictly homeomorphic* and we denote that by $X \cong Y$.

Remark 1.2. The above notion of partial topology is a special case of the notion of generalized topology in the sense of H. Delfs and M. Knebusch considered in [3, 10, 11, 12, 13], when the family Op_X of open sets of the generalized topology forms a topology. The category $\mathbf{GTS}_{pt}$ of partially topological spaces and their strictly continuous mappings is a topological construct (Theorem 4.4 of [13]).

Definition 1.3. Let (X, Cov_X) be a partially topological space and let Y be a subset of X. Then the partial topology

$$\mathrm{Cov}_Y = (\langle \mathrm{Cov}_X \cap_2 Y \rangle_Y)_{pt},$$

that is: the smallest partial topology containing $\mathrm{Cov}_X \cap_2 Y$, is called a *subspace partial topology* on Y, and (Y, Cov_Y) is a *subspace* of (X, Cov_X). (It is also the smallest generalized topology containing $\mathrm{Cov}_X \cap_2 Y$.)

Fact 1.4. *Let $\varphi : X \to X'$ be a mapping between partially topological spaces and let Y be a subspace of X. Then the following are equivalent:*

a) φ is strictly continuous,

b) the restriction map $\varphi|_Y : Y \to X'$ is strictly continuous.

Definition 1.5. Let (X, Cov_X) and (Y, Cov_Y) be two partially topological spaces. The *product partial topology* on $X \times Y$ is the partial topology $\mathrm{Cov}_{X \times Y} = (\langle \mathrm{Cov}_X \times_2 \mathrm{Cov}_Y \rangle_{X \times Y})_{pt}$ in the notation of Definition 4.6 of [13]; in other words: the smallest partial topology in $X \times Y$ that contains $\mathrm{Cov}_X \times_2 \mathrm{Cov}_Y$.

2. Partially topological groups

The fundamental reference for topological groups and their properties is [1]; see also [15]. In [9], the partially (para)topological groups in the sense of H. Delfs and M. Knebusch were defined. Now we discuss partially topological groups with more details and recall their basic properties.

Definition 2.1. A *partially paratopological group* G is an ordered pair $((G, *), \mathrm{Cov}_G)$ such that $(G, *)$ is a group, while Cov_G is a generalized topology on G such that $\bigcup \mathrm{Cov}_G$ is a T_1 topology on G and the multiplication map of $(G \times G, \mathrm{Cov}_{G \times G})$ into (G, Cov_G), which sends ordered pair $(x, y) \in G \times G$ to $x * y$, is strictly continuous. If also the inverse map from (G, Cov_G) into (G, Cov_G), which sends each $x \in G$ to x^{-1}, is strictly continuous, then $((G,*), \mathrm{Cov}_G)$ is called a *partially topological group*. For simplicity, when this does not lead to misunderstanding, we shall denote a partially (para)topological group $((G,*), \mathrm{Cov}_G)$ by G or by (G, Cov_G), or by $(G,*)$.

Definition 2.2. (i) A *small partially (para)topological group* is a partially (para)-topological group $((G,*), \mathrm{Cov}_G)$ such that
$\mathrm{Cov}_G = EssFin(\bigcup \mathrm{Cov}_G)$.

(ii) A *smallification* of a partially (para)topological group $G = ((G,*), \mathrm{Cov}_G)$ is the partially (para)topological group

$$G_{sm} = ((G,*), EssFin(\bigcup \mathrm{Cov}_G)).$$

(Cf. Definintion 2.3.16 of [10]).

Fact 2.3. *Let $(G,*)$ be a group and let τ be a topology on G. Let us apply the functor of smallification to (G,τ) by putting $Cov_{G_{sm}} = EssFin(\tau)$. Then $G = ((G,*),\tau)$ is a (para)topological group if and only if $G_{sm} = ((G,*), Cov_{G_{sm}})$ is a partially (para)topological group.*

The group G_{sm} can also be denoted by G_{st}, applying the conventions from Definition 1.2 of [13]. Similarly, we have

Example 2.4. Since $(\mathbb{R}^n, +)$ is a group and the localized smallified euclidean space $\mathbb{R}^n_{lst}$ (the localization of $\mathbb{R}^n_{st}$; see Definition 2.1.15 in [11]) is still partially topological, we get partially topological groups $(\mathbb{R}^n_{lst}, +)$ for $n \in \mathbb{N}$.

Example 2.5. The usual topological as well as smallified Sorgenfrey lines (see Definition 4.3 of [14]) with addition as group action form partially paratopological groups $(\mathbb{R}^S_{ut}, +)$, $(\mathbb{R}^S_{st}, +)$.

Definition 2.6. Any subgroup H of a partially (para)topological group G is a partially (para)topological group again, and is called a *partially (para)topological subgroup of* G.

Definition 2.7. Let $\varphi : G \to G'$ be a function. Then φ is called a *morphism of partially (para)topological groups* if φ is both strictly continuous and a group homomorphism. Moreover, φ is an *isomorphism* if it is a strict homeomorphism and group isomorphism.

If we have an isomorphism between two partially (para)topological groups G and G' then we say that they are isomorphic and we denote that by $G \cong G'$.

Remark 2.8. It is obvious that composition of two morphisms of partially (para)topological groups is again a morphism. Also, the identity map is an isomorphism. So partially paratopological groups and their morphisms form a category **PPTGr**, while partially topological groups and their morphisms form a category **PTGr**.

3. Extension closed properties of partially topological groups

In this section, we prove extension closed property for connectedness, compactness, and separability of partially topological groups.

From [9] we know that Cartesian product of compact(small) sets is compact.

Definition 3.1. A partially topological space (X, Cov_X) is *topologically compact (admissibly compact)* if each open (admissible) cover of X admits a finite subcover, and is *small* if it is hereditary admissibly compact.

Fact 3.2. *Let (X, Cov_X) and (Y, Cov_Y) be two partially topological spaces and $f : X \to Y$ a surjective strictly continuous function. If (X, Cov_X) is*

i) topologically compact,

ii) admissibly compact,

iii) small

then so is (Y, Cov_Y).

The famous Tychonoff's theorem may be stated in the following way

Theorem 3.3. *Assume* **ZF**. *Let X be the product of partially topological spaces $X_k, k \in K$, where K is a finite set. If every X_k is topologically compact (small, respectively), then so is X.*

Remark 3.4. It is still unknown whether admissible compactness is finitely productive even in **ZFC**.

Corollary 3.5. *Assume* **ZF**. *Let G be a partially topological group and A and B subsets of G. If A and B are topologically compact (small, respectively), then so is AB.*

Theorem 3.6. *For any two compact(small) subsets E and F of a partial topological group G, their product EF in G is a compact(small) subspace of G.*

Proof. Since multiplication is strictly continuous, the subspace EF of G is a strictly continuous image of the Cartesian product $E \times F$ of the spaces E and F. Since $E \times F$ is compact(small), the space EF is compact. □

Theorem 3.7. *Assume* **ZF**. *Let G be a normal compact partial topological group, F a compact subset of G, and P a closed subset of G. Then the sets FP and PF are closed.*

Proof. Since G is compact then $P \times F$ is compact and by previous result PF and FP are $\mathcal{G}$-compact. Since G is normal, FP and PF are closed. □

From[9] we know that Cartesian product of connected sets is connected.

Definition 3.8. Let X be a partially topological space and let $U, V \subseteq X$. Then we say that the pair U, V is *separated* if $Cl(U) \cap V = Cl(V) \cap U = \emptyset$. We say that X is *connected* if it can not be written as a union of two separated sets.

Definition 3.9. A function $f : X \to Y$ between partially topological spaces is called *open*(*closed*) if for every open (closed) set $U \subseteq X$, $f(U)$ is open (closed) in Y.

Fact 3.10. *Let $f : X \to Y$ be an open and injective function between partially topological spaces and let $A \subseteq X$. If $f(A)$ is connected, so is A.*

Theorem 3.11. *For any two connected subsets E and F of a partial topological group G, their product EF in G is a connected subspace of G.*

Proof. Since multiplication is strictly continuous, the subspace EF of G is a strictly continuous image of the Cartesian product $E \times F$ of the spaces E and F. Since $E \times F$ is connected from [9], the space EF is connected. □

Definition 3.12. A closed strictly continuous mapping with compact pre-images of points is called perfect.

Theorem 3.13. *The quotient mapping π of G onto the quotient space G/H is perfect where H is a compact subgroup of a normal partial topological group G.*

Proof. Take any closed subset P of G. Then, by Theorem 3.7, PH is closed in G. However, PH is the union of cosets, that is $PH = \pi^{-1}\pi(P)$. It follows by definition of a quotient mapping, that the set $\pi(P)$ is closed in the quotient space G/H. Thus π is a closed mapping. In addition, if $y \in G/H$ and $\pi(x) = y$ for some $x \in G$, then $\pi^{-1}(y) = xH$ is a compact subset of G. Hence the fibers of π are compact and π is perfect. □

Corollary 3.14. *Let H be a compact subgroup of a normal partial topological group G such that the quotient space G/H is compact. Then G is also compact.*

Theorem 3.15. *Let $f : G \to H$ be a strictly continuous mapping of partial topological spaces. If G is compact and H is normal, then f is closed.*

Proof. Let K be a closed set in G. Since G is compact then K is compact. So by strictly continuity of f, $f(K)$ is compact in H. By assumption, H is normal, so $f(K)$ is closed. □

Definition 3.16. Let $X$and Y be partially topological spaces. A strictly continuous onto mapping $f : X \to Y$ is called *identification map* if f is open or closed.

Theorem 3.17. *Let $f : G \to H$ be a strictly continuous onto homomorphism of partial topological groups. If G is compact and H is normal, then f is open.*

Proof. By Theorem 3.15, the mapping f is closed, and hence it is quotient. Let K be the kernel of f. If U is open in G, then $f^{-1}(f(U)) = KU$ is open in G. Since f is quotient, it follows that the image $f(U)$ is open in H. Therefore, f is an open mapping. □

Lemma 3.18. *Suppose that $f : X \to Y$ is an open strictly continuous mapping of a space X onto a space Y, $x \in X$, $B \subseteq Y$, and $f(x) \in Cl(B)$ where $Cl(B)$ is closure of B. Then $x \in f^{-1}(Cl(B))$.*

Proof. Take $y = f(x)$, and let O be an open neighborhood of x. Then $f(O)$ is an open neighborhood of y. Therefore, $f(O) \cap B \neq \emptyset$ and, hence $O \cap f^{-1}(B) \neq \emptyset$. It follows that $x \in Cl(f^{-1}(B))$. Equality is evident. □

Theorem 3.19. *Let H be a closed subgroup of a partial topological group G. If the spaces H and G/H are separable, then the space G is also separable.*

Proof. Let π be the natural homomorphism of G onto the quotient space G/H. Since G/H is separable, we can fix a dense countable subset B of G/H. Since H is separable and every coset xH is strictly homeomorphic to H, we can fix a dense countable subset M_y of $\pi^{-1}(y)$, for each $y \in B$. Put $M = \bigcup\{M_y : y \in B\}$. Then M is a countable subset of G and M is dense in $\pi^{-1}(B)$. Since π is an open mapping of G onto G/H, it follows from Lemma 3.18 that $Cl(\pi^{-1}(B)) = G$. Hence, M is dense in G and G is separable. □

Theorem 3.20. *Let H be a closed invariant subgroup of a partial topological group G. If H and G/H are connected, then so is G.*

Proof. Suppose that H and G/H are connected and $f : G \to \{0, 1\}$ be an arbitrary strictly continuous map. We have to show that f is constant. The restriction of f to H must be constant and since each coset gH is connected, f must be constant on gH as well taking value $f(g)$. Thus we have a well-defined map $\tilde{f} : G/H \to \{0, 1\}$ such that $\tilde{f} \circ \pi = f$. By the fundamental property of quotient spaces, it follows that $\tilde{f}$ is strictly continuous and so must be constant since G/H is connected. Hence f is also constant and we conclude that G is connected. □

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