IDEAL CONVERGENT GENERALIZED DIFFERENCE SEQUENCE SPACES OF INFINITE MATRIX AND ORLICZ FUNCTION

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Abstract. In this paper we introduce some generalized difference sequence spaces by using Musielak-Orlicz function, ideal convergence and an infinite matrix defined on n-normed spaces. We study some basic topological and algebraic properties of these spaces. We also investigate some inclusion relations related to these spaces.

Keywords: Musielak-Orlicz function, ideal convergence, solid, infinite matrix, n-normed space.

1. Introduction and preliminaries

The concept of 2-normed spaces was initially developed by Gähler [10] in the mid of 1960's, while that of n-normed spaces one can see in Misiak [20]. Since then, many others have studied this concept and obtained various results, see Gunawan ([11], [12]) and Gunawan and Mashadi [13] and many others. Let $n \in \mathbb{N}$ and X be a linear space over the real field $\mathbb{R}$ of dimension d, where $d \geq n \geq 2$. A real valued function $\|\cdot, \ldots, \cdot\|$ on X^n satisfying the following four conditions:

1. $\|x_1, x_2, \ldots, x_n\| = 0$ if and only if $x_1, x_2, \ldots, x_n$ are linearly dependent in X;
2. $\|x_1, x_2, \ldots, x_n\|$ is invariant under permutation;
3. $\|\alpha x_1, x_2, \ldots, x_n\| = |\alpha| \ \|x_1, x_2, \ldots, x_n\|$ for any $\alpha \in \mathbb{R}$, and
4. $\|x + x', x_2, \ldots, x_n\| \leq \|x, x_2, \ldots, x_n\| + \|x', x_2, \ldots, x_n\|$

is called an n-norm on X, and the pair $(X, \|\cdot, \cdots, \cdot\|)$ is called an n-normed space over the field $\mathbb{R}$.

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For example, we may take $X = \mathbb{R}^n$ being equipped with the n-norm $\|x_1, x_2, \ldots, x_n\|_E$ = the volume of the n-dimensional parallelopiped spanned by the vectors $x_1, x_2, \ldots, x_n$ which may be given explicitly by the formula

$$\|x_1, x_2, \cdots, x_n\|_E = |\det(x_{ij})|,$$

where $x_i = (x_{i1}, x_{i2}, \cdots, x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, \cdots, n$, where script E denotes Euclidean space. Let $(X, \|\cdot, \cdots, \cdot\|)$ be an n-normed space of dimension $d \geq n \geq 2$ and $\{a_1, a_2, \ldots, a_n\}$ be linearly independent set in X. Then the following function $\|\cdot, \cdots, \cdot\|_\infty$ on X^{n-1} defined by

$$\|x_1, x_2, \cdots, x_{n-1}\|_\infty = \max\{\|x_1, x_2, \ldots, x_{n-1}, a_i\| : i = 1, 2, \cdots, n\}$$

defines an $(n-1)$-norm on X with respect to $\{a_1, a_2, \ldots, a_n\}$.

A sequence (x_k) in a n-normed space $(X, \|\cdot, \cdots, \cdot\|)$ is said to converge to some $L \in X$ if

$$\lim_{k\to\infty} \|x_k - L, z_1, \cdots, z_{n-1}\| = 0, \text{ for every } z_1, \ldots, z_{n-1} \in X.$$

A sequence (x_k) in a n-normed space $(X, \|\cdot, \cdots, \cdot\|)$ is said to be Cauchy if

$$\lim_{k,i\to\infty} \|x_k - x_i, z_1, \cdots, z_{n-1}\| = 0, \text{ for every } z_1, \ldots, z_{n-1} \in X.$$

If every Cauchy sequence in X converges to some $L \in X$, then X is said to be complete with respect to the n-norm. Any complete n-normed space is said to be n-Banach space.

The notions of statistical convergence and convergence in density for sequences has been in the literature, under different guises, since the early part of the last century. Over the years and under different names, statistical convergence has been discussed in the theory of Fourier analysis, ergodic theory and number theory. Statistical convergence was recently investigated by Fast [9] and Schoenberg [29] independently.

The concept of ideal convergence was first introduced by P. Kostyrko et al. [16] as a generalization of statistical convergence which was further studied in topological spaces by Das et al. [1]. More applications of ideals can be seen in ([1], [2]). We continue in this direction and introduce I-convergence of generalized sequences in more general setting.

A family $I \subset 2^X$ of subsets of a non empty set X is said to be an ideal in Xif

1. $\phi \in I$;
2. $A, B \in I$ imply $A \cup B \in I$;
3. $A \in \mathcal{I}$, $B \subset A$ imply $B \in I$, while an admissible ideal I of X further satisfies $\{x\} \in I$ for each $x \in X$ (see [14]).

A sequence $(x_n)_{n\in\mathbb{N}}$ in X is said to be I-convergent to $x \in X$, if for each $\epsilon > 0$ the set $A(\epsilon) = \{n \in \mathbb{N} : \|x_n - x\| \geq \epsilon\}$ belongs to I (see [13]). A non empty family of sets $F \subseteq 2^X$ is said to be filter on X if and only if $\Phi \notin F$, for $A, B \in F$ we have $A \cap B \in F$ and for each $A \in F$ and $A \subseteq B$ implies $B \in F$. An ideal $I \subseteq 2^X$ is called non trivial if $I \neq 2^X$. A non-trivial ideal $I \subseteq 2^X$ is called admissible if $\{\{x\} : x \in X\} \subseteq I$. A non-trivial ideal is maximal if there cannot exist any non-trivial ideal $J \neq I$ containing I as a subset. Further details on ideals of 2^X can be found in [16].

An Orlicz function $M : [0, \infty) \to [0, \infty)$ is a continuous, non-decreasing and convex function such that $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$.

Lindenstrauss and Tzafriri [18] used the idea of Orlicz function to define the following sequence space,

$$\ell_M = \Big\{(x_k) \in w : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) < \infty, \ \text{ for some } \ \rho > 0\Big\}$$

which is called as an Orlicz sequence space. Also ℓ_M is a Banach space with the norm

$$\|(x_k)\| = \inf\Big\{\rho > 0 : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) \leq 1\Big\}.$$

Also, it was shown in [18] that every Orlicz sequence space ℓ_M contains a subspace isomorphic to $\ell_p (p \geq 1)$. An Orlicz function M satisfies Δ_2-condition if and only if for any constant $L > 1$ there exists a constant $K(L)$ such that $M(Lu) \leq K(L)M(u)$ for all values of $u \geq 0$.

A sequence $\mathcal{M} = (M_k)$ of Orlicz functions is called a Musielak-Orlicz function see ([19], [24]).

A Musielak-Orlicz function (M_k) is said to satisfy Δ_2-condition if there exist constants $a, K > 0$ and a sequence $c = (c_k)_{k=1}^{\infty} \in \ell_+^1$ (the positive cone of ℓ^1) such that the inequality

$$M_k(2u) \leq KM_k(u) + c_k$$

holds for all $k \in N$ and $u \in R_+$ whenever $M_k(u) \leq a$.

The notion of difference sequence spaces was introduced by Kızmaz [17], who studied the difference sequence spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$. The notion was further generalized by Et and Çolak [8] by introducing the spaces $l_\infty(\Delta^n)$, $c(\Delta^n)$ and $c_0(\Delta^n)$. Let w be the space of all complex or real sequences $x = (x_k)$ and let m, n be non-negative integers, then for $Z = l_\infty,\ c,\ c_0$ we have sequence spaces

$$Z(\Delta_n^m) = \{x = (x_k) \in w : (\Delta_n^m x_k) \in Z\},$$

where $\Delta_n^m x = (\Delta_n^m x_k) = (\Delta_n^{m-1} x_k - \Delta_n^{m-1} x_{k+1})$ and $\Delta_n^0 x_k = x_k$ for all $k \in \mathbb{N}$, which is equivalent to the following binomial representation

$$\Delta_n^m x_k = \sum_{v=0}^{m} (-1)^v \begin{pmatrix} m \\ v \end{pmatrix} x_{k+nv}.$$

Taking $n = 1$, we get the spaces which were studied by Et and Çolak [8]. Taking $m = n = 1$, we get the spaces which were introduced and studied by Kızmaz [17].For more details about sequence spaces (see [3], [4], [5], [6], [7], [21], [22], [23], [25], [26], [27], [28], [30], [31]) and reference therein.

Let X and Y be two sequence spaces and $A = (a_{nk})$ be an infinite matrix of real or complex numbers a_{nk}, where $n, k \in \mathbb{N}$. Then we say that A defines a matrix mapping from X into Y if for every sequence $x = (x_k)_{k=0}^{\infty} \in X$, the sequence $Ax = \{A_n(x)\}_{n=0}^{\infty}$, the A-transform of x, is in Y, where

$$A_n(x) = \sum_{k=0}^{\infty} a_{nk} x_k \quad (n \in \mathbb{N}).$$

By (X, Y), we denote the class of all matrices A such that $A : X \to Y$. Thus, $A \in (X, Y)$ if and only if the series on the right-hand side of above equention converges for each $n \in \mathbb{N}$ and every $x \in X$.

The matrix domain X_A of an infinite matrix A in a sequence space X is defined by

$$X_A = \{x = (x_k) : Ax \in X\}.$$

A sequence space E is said to be solid(or normal) if $(x_k) \in E$ implies $(\alpha_k x_k) \in E$ for all sequences of scalars (α_k) with $|\alpha_k| \leq 1$ and for all $k \in \mathbb{N}$.
Let I be an admissible ideal of $\mathbb{N}$, let $p = (p_k)$ be a bounded sequence of positive real numbers and $A = (a_{nk})$ be an infinite matrix. Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function, $u = (u_k)$ be a sequence of strictly positive real numbers and $(X, \|., ..., .\|)$ be an n-normed space. Suppose $\Lambda = (\lambda_n)$ is a non-decreasing sequence of positive real numbers such that $\lambda_{n+1} \leq \lambda_n + 1, \lambda_1 = 1, \lambda_n \to \infty$ as $n \to \infty$. Further $w(n-x)$ denotes the space of all X-valued sequences. For every $z_1, z_2, ..., z_{n-1} \in X$, for each $\epsilon > 0$ and for some $\rho > 0$ we define the following sequence spaces:

$$\begin{aligned}
&W^I\big[\Lambda, \Delta_n^m, A, \mathcal{M}, u, p, \|., ..., .\|\big] \\
&= \Bigg\{ x = (x_k) \in w(n-x) : \ \text{ for given } \ \epsilon > 0, \\
&\Big\{ n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} a_{nk} \Big[k^{-s} M_k \Big(\| \frac{u_k \Delta_n^m x_k - L}{\rho}, z_1, z_2, ..., z_{n-1} \| \Big) \Big]^{p_k} \geq \epsilon \Big\} \in I, \\
&\text{for } L \in X \ \text{ and } s \geq 0 \Bigg\},
\end{aligned}$$

$$\begin{aligned}
&W_0^I\big[\Lambda, \Delta_n^m, A, \mathcal{M}, u, p, \|., ..., .\|\big] = \Bigg\{ x = (x_k) \in w(n-x) : \ \text{ for given } \ \epsilon > 0, \\
&\Big\{ n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} a_{nk} \Big[k^{-s} M_k \Big(\| \frac{u_k \Delta_n^m x_k}{\rho}, z_1, z_2, ..., z_{n-1} \| \Big) \Big]^{p_k} \geq \epsilon \Big\} \in I, \ \text{for } s \geq 0 \Bigg\}
\end{aligned}$$

and

$$W^I_\infty\big[\Lambda,\Delta^m_n,A,\mathcal{M},u,p,\|.,...,.\|\big]=\Bigg\{x=(x_k)\in w(n-x):\exists\ K>0,$$

$$\Big\{n\in\mathbb{N}:\frac{1}{\lambda_n}\sum_{k\in I_n}a_{nk}\Big[k^{-s}M_k\Big(\|\frac{u_k\Delta^m_n x_k}{\rho},z_1,z_2,...,z_{n-1}\|\Big)\Big]^{p_k}\geq K\Big\}\in I,\ \text{for } s\geq 0\Bigg\},$$

where $I_n=[n-\lambda_n+1,n]$. Some special cases of the above defined sequence spaces are arises: If $m=0$, then we obtain the spaces as follows:

$$W^I\big[\Lambda,A,\mathcal{M},u,p,\|.,...,.\|\big]=\Bigg\{x=(x_k)\in w(n-x):\ \text{ for given }\ \epsilon>0,$$

$$\Big\{n\in\mathbb{N}:\frac{1}{\lambda_n}\sum_{k\in I_n}a_{nk}\Big[k^{-s}M_k\Big(\|\frac{u_k x_k-L}{\rho},z_1,z_2,...,z_{n-1}\|\Big)\Big]^{p_k}\geq\epsilon\Big\}\in I,$$

$$\text{for } L\in X \text{ and } s\geq 0\Bigg\},$$

$$W^I_0\big[\Lambda,A,\mathcal{M},u,p,\|.,...,.\|\big]=\Bigg\{x=(x_k)\in w(n-x):\ \text{ for given }\ \epsilon>0,$$

$$\Big\{n\in\mathbb{N}:\frac{1}{\lambda_n}\sum_{k\in I_n}a_{nk}\Big[k^{-s}M_k\Big(\|\frac{u_k x_k}{\rho},z_1,z_2,...,z_{n-1}\|\Big)\Big]^{p_k}\geq\epsilon\Big\}\in I,\ \text{for } s\geq 0\Bigg\}$$

and

$$W^I_\infty\big[\Lambda,A,\mathcal{M},u,p,\|.,...,.\|\big]=\Bigg\{x=(x_k)\in w(n-x):\exists\ K>0,$$

$$\Big\{n\in\mathbb{N}:\frac{1}{\lambda_n}\sum_{k\in I_n}a_{nk}\Big[k^{-s}M_k\Big(\|\frac{u_k x_k}{\rho},z_1,z_2,...,z_{n-1}\|\Big)\Big]^{p_k}\geq K\Big\}\in I,\ \text{for } s\geq 0\Bigg\}.$$

If $m=n=1$, then the above spaces are as follows:

$$W^I\big[\Lambda,\Delta,A,\mathcal{M},u,p,\|.,...,.\|\big]=\Bigg\{x=(x_k)\in w(n-x):\ \text{ for given }\ \epsilon>0,$$

$$\Big\{n\in\mathbb{N}:\frac{1}{\lambda_n}\sum_{k\in I_n}a_{nk}\Big[k^{-s}M_k\Big(\|\frac{u_k\Delta x_k-L}{\rho},z_1,z_2,...,z_{n-1}\|\Big)\Big]^{p_k}\geq\epsilon\Big\}\in I,$$

$$\text{for } L\in X \text{ and } s\geq 0\Bigg\},$$

$$W^I_0\big[\Lambda,\Delta,A,\mathcal{M},u,p,\|.,...,.\|\big]=\Bigg\{x=(x_k)\in w(n-x):\text{ for given }\epsilon>0,$$

$$\Big\{n\in\mathbb{N}:\frac{1}{\lambda_n}\sum_{k\in I_n}a_{nk}\Big[k^{-s}M_k\Big(\|\frac{u_k\Delta x_k}{\rho},z_1,z_2,...,z_{n-1}\|\Big)\Big]^{p_k}\geq\epsilon\Big\}\in I,\ \text{for } s\geq 0\Bigg\}$$

and

$$W_{\infty}^{I}\big[\Lambda, \Delta, A, \mathcal{M}, u, p, \|., ..., .\|\big] = \Bigg\{ x = (x_k) \in w(n-x) : \exists\ K > 0,$$

$$\Big\{ n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} a_{nk} \Big[k^{-s} M_k \Big(\| \frac{u_k \Delta x_k}{\rho}, z_1, z_2, ..., z_{n-1} \| \Big) \Big]^{p_k} \geq K \Big\} \in I,\ \text{for } s \geq 0 \Bigg\}.$$

If $s = 0$ and $\mathcal{M}(x) = x$ for all $x \in [0, \infty)$, then we have

$$W^{I}\big[\Lambda, \Delta_n^m, A, u, p, \|., ..., .\|\big] = \Bigg\{ x = (x_k) \in w(n-x) : \text{ for given } \epsilon > 0,$$

$$\Big\{ n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} a_{nk} \Big(\| \frac{u_k \Delta_n^m x_k - L}{\rho}, z_1, z_2, ..., z_{n-1} \| \Big)^{p_k} \geq \epsilon \Big\} \in I,$$

$$\text{for } L \in X \text{ and } s \geq 0 \Bigg\},$$

$$W_0^{I}\big[\Lambda, \Delta_n^m, A, u, p, \|., ..., .\|\big] = \Bigg\{ x = (x_k) \in w(n-x) : \text{ for given } \epsilon > 0,$$

$$\Big\{ n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} a_{nk} \Big(\| \frac{u_k \Delta_n^m x_k}{\rho}, z_1, z_2, ..., z_{n-1} \| \Big)^{p_k} \geq \epsilon \Big\} \in I,\ \text{for } s \geq 0 \Bigg\}$$

and

$$W_{\infty}^{I}\big[\Lambda, \Delta_n^m, A, u, p, \|., ..., .\|\big] = \Bigg\{ x = (x_k) \in w(n-x) : \exists\ K > 0,$$

$$\Big\{ n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} a_{nk} \Big(\| \frac{u_k \Delta_n^m x_k}{\rho}, z_1, z_2, ..., z_{n-1} \| \Big)^{p_k} \geq K \Big\} \in I,\ \text{for } s \geq 0 \Bigg\}.$$

If $p = (p_k) = 1$ for all k, then the above spaces are as follows

$$W^{I}\big[\Lambda, A, \Delta_n^m, \mathcal{M}, u, \|., ..., .\|\big] = \Bigg\{ x = (x_k) \in w(n-x) : \text{ for given } \epsilon > 0,$$

$$\Big\{ n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} a_{nk} \Big[k^{-s} M_k \Big(\| \frac{u_k \Delta_n^m x_k - L}{\rho}, z_1, z_2, ..., z_{n-1} \| \Big) \Big]$$

$$\geq \epsilon \Big\} \in I, \text{ for } L \in X \text{ and } s \geq 0 \Bigg\},$$

$$W_0^{I}\big[\Lambda, \Delta_n^m, A, \mathcal{M}, u, \|., ..., .\|\big] = \Bigg\{ x = (x_k) \in w(n-x) : \text{ for given } \epsilon > 0,$$

$$\Big\{n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} a_{nk} \Big[k^{-s} M_k \Big(\|\frac{u_k \Delta_n^m x_k}{\rho}, z_1, z_2, ..., z_{n-1}\|\Big)\Big]$$

$$\geq \epsilon\Big\} \in I, \text{ for } s \geq 0\Bigg\}$$

and

$$W_\infty^I\big[\Lambda, \Delta_n^m, A, \mathcal{M}, u, \|., ..., .\|\big] = \Bigg\{x = (x_k) \in w(n-x) : \exists\, K > 0,$$

$$\Big\{n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} a_{nk} \Big[k^{-s} M_k \Big(\|\frac{u_k \Delta_n^m x_k}{\rho}, z_1, z_2, ..., z_{n-1}\|\Big)\Big] \geq K\Big\} \in I, \text{ for } s \geq 0\Bigg\}.$$

If $A = (C, 1)$, the Cesàro matrix, then the above spaces are as follows:

$$W^I\big[\Lambda, \Delta_n^m, \mathcal{M}, u, p, \|., ..., .\|\big] = \Bigg\{x = (x_k) \in w(n-x) : \text{ for given } \epsilon > 0,$$

$$\Big\{n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} \Big[k^{-s} M_k \Big(\|\frac{u_k \Delta_n^m x_k - L}{\rho}, z_1, z_2, ..., z_{n-1}\|\Big)\Big]^{p_k} \geq \epsilon\Big\} \in I,$$

$$\text{for } L \in X \text{ and } s \geq 0\Bigg\},$$

$$W_0^I\big[\Lambda, \Delta_n^m, \mathcal{M}, u, p, \|., ..., .\|\big] = \Bigg\{x = (x_k) \in w(n-x) : \text{ for given } \epsilon > 0,$$

$$\Big\{n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} \Big[k^{-s} M_k \Big(\|\frac{u_k \Delta_n^m x_k}{\rho}, z_1, z_2, ..., z_{n-1}\|\Big)\Big]^{p_k} \geq \epsilon\Big\} \in I, \text{ for } s \geq 0\Bigg\}$$

and

$$W_\infty^I\big[\Lambda, \Delta_n^m, \mathcal{M}, u, p, \|., ..., .\|\big] = \Bigg\{x = (x_k) \in w(n-x) : \exists\, K > 0,$$

$$\Big\{n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} \Big[k^{-s} M_k \Big(\|\frac{u_k \Delta_n^m x_k}{\rho}, z_1, z_2, ..., z_{n-1}\|\Big)\Big]^{p_k} \geq K\Big\} \in I, \text{ for } s \geq 0\Bigg\}.$$

By a lacunary sequence $\theta = (k_r); r = 0, 1, 2, ...$ where $k_0 = 0$, we shall mean an increasing sequence of non-negative integers with $k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$ and $h_r = k_r - k_{r-1}$. We finally arrived, let

$$a_{nk} = \begin{cases} \dfrac{1}{h_r}, & \text{if } k_{r-1} < k < k_r \\ 0, & \text{otherwise.} \end{cases}$$

Then the above classes of sequences are denoted by $W^I\left[\Lambda, \theta, \Delta_n^m, \mathcal{M}, p, \|., ..., .\|\right]$, $W_0^I\left[\Lambda, \theta, \Delta_n^m, \mathcal{M}, p, \|., ..., .\|\right]$ and $W_\infty^I\left[\Lambda, \theta, \Delta_n^m, \mathcal{M}, p, \|., ..., .\|\right]$.

The following inequality will be used throughout the paper. If $0 \leq p_k \leq \sup p_k = G$, $D = \max(1, 2^{G-1})$ then

$$|a_k + b_k|^{p_k} \leq D\{|a_k|^{p_k} + |b_k|^{p_k}\}, \tag{1}$$

for all k and $a_k, b_k \in \mathbb{R}$. Also $|a|^{p_k} \leq \max(1, |a|^G)$ for all $a \in \mathbb{R}$.

The main aim of this paper is to introduce some generalized difference sequence spaces defined by ideal convergence, Musielak-Orlicz function and an infinite matrix. We have also make an effort to study some inclusion relations and their topological properties.

2. Main results

Theorem 2.1. *Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function, $p = (p_k)$ be a bounded sequence of positive real numbers and $u = (u_k)$ be a sequence of strictly positive real numbers. Then $W^I\left[\Lambda, \Delta_n^m, A, \mathcal{M}, u, p, \|., ..., .\|\right]$, $W_0^I\left[\Lambda, \Delta_n^m, A, \mathcal{M}, u, p, \|., ..., .\|\right]$ and $W_\infty^I\left[\Lambda, \Delta_n^m, A, \mathcal{M}, u, p, \|., ..., .\|\right]$ are linear spaces over the real field $\mathbb{R}$.*

Proof. We shall prove the result for the space $W_0^I\left[\Lambda, \Delta_n^m, A, \mathcal{M}, u, p, \|., ..., .\|\right]$. Let $x = (x_k)$ and $y = (y_k)$ be two elements of $W_0^I\left[\Lambda, \Delta_n^m, A, \mathcal{M}, u, p, \|., ..., .\|\right]$. Then there exists $\rho_1 > 0$ and $\rho_2 > 0$ and for $z_1, z_2, ..., z_{n-1} \in X$ such that

$$A_{\frac{\epsilon}{2}} = \Big\{n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} a_{nk} \Big[k^{-s} M_k\Big(\|\frac{u_k \Delta_n^m x_k}{\rho_1}, z_1, z_2, ..., z_{n-1}\|\Big)\Big]^{p_k} \geq \frac{\epsilon}{2}\Big\} \in I$$

and

$$B_{\frac{\epsilon}{2}} = \Big\{n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} a_{nk} \Big[k^{-s} M_k\Big(\|\frac{u_k \Delta_n^m y_k}{\rho_2}, z_1, z_2, ..., z_{n-1}\|\Big)\Big]^{p_k} \geq \frac{\epsilon}{2}\Big\} \in I.$$

Let $\alpha, \beta \in \mathbb{R}$. Since $\|., ..., .\|$ is a n-norm, Δ_n^m is linear and the contributing of $\mathcal{M} = (M_k)$, the following inequality holds:

$$\begin{aligned}
&\frac{1}{\lambda_n} \sum_{k \in I_n} a_{nk} \Big[k_{-s} M_k\Big(\|\frac{u_k \Delta_n^m (\alpha x_k + \beta y_k)}{|\alpha|\rho_1 + |\beta|\rho_2}, z_1, z_2, ..., z_{n-1}\|\Big)\Big]^{p_k} \\
&\leq D \frac{1}{\lambda_n} \sum_{k \in I_n} a_{nk} \Big[\frac{|\alpha|}{|\alpha|\rho_1 + |\beta|\rho_2} k^{-s} M_k\Big(\|\frac{u_k \Delta_n^m x_k}{\rho_1}, z_1, z_2, ..., z_{n-1}\|\Big)\Big]^{p_k} \\
&+ D \frac{1}{\lambda_n} \sum_{k \in I_n} a_{nk} \Big[\frac{|\beta|}{|\alpha|\rho_1 + |\beta|\rho_2} k^{-s} M_k\Big(\|\frac{u_k \Delta_n^m y_k}{\rho_2}, z_1, z_2, ..., z_{n-1}\|\Big)\Big]^{p_k} \\
&\leq DK \frac{1}{\lambda_n} \sum_{k \in I_n} a_{nk} \Big[k^{-s} M_k\Big(\|\frac{u_k \Delta_n^m x_k}{\rho_1}, z_1, z_2, ..., z_{n-1}\|\Big)\Big]^{p_k} \\
&+ DK \frac{1}{\lambda_n} \sum_{k \in I_n} a_{nk} \Big[k^{-s} M_k\Big(\|\frac{u_k \Delta_n^m y_k}{\rho_2}, z_1, z_2, ..., z_{n-1}\|\Big)\Big]^{p_k},
\end{aligned}$$

where $K = \max\{1, \frac{|\alpha|}{|\alpha|\rho_1+|\beta|\rho_2}, \frac{|\beta|}{|\alpha|\rho_1+|\beta|\rho_2}\}$.

From the above relation, we get

$$\Big\{n \in \mathbb{N} : \frac{1}{\lambda_n}\sum_{k\in I_n} a_{nk}\Big[k^{-s}M_k\Big(\|\frac{u_k\Delta_n^m(\alpha x_k+\beta y_k)}{|\alpha|\rho_1+|\beta|\rho_2}, z_1, z_2, ..., z_{n-1}\|\Big)\Big]^{p_k} \geq \epsilon\Big\}$$
$$\subseteq \Big\{n \in \mathbb{N} : DK\frac{1}{\lambda_n}\sum_{k\in I_n} a_{nk}\Big[k^{-s}M_k\Big(\|\frac{u_k\Delta_n^m x_k}{\rho_1}, z_1, z_2, ..., z_{n-1}\|\Big)\Big]^{p_k} \geq \frac{\epsilon}{2}\Big\}$$
$$\cup \Big\{n \in \mathbb{N} : DK\frac{1}{\lambda_n}\sum_{k\in I_n} a_{nk}\Big[k^{-s}M_k\Big(\|\frac{u_k\Delta_n^m y_k}{\rho_2}, z_1, z_2, ..., z_{n-1}\|\Big)\Big]^{p_k} \geq \frac{\epsilon}{2}\Big\}.$$

Since both the sets on the R.H.S of above relation are belongs to I, so the set on the L.H.S of the inclusion relation belongs to I. Similarly we can prove other cases. This completes the proof of the theorem.

Theorem 2.2. *Let $\mathcal{M}' = (M_k')$ and $\mathcal{M}'' = (M_k'')$ be two Musielak-orlicz functions. Then we have $W_0^I\big[\Lambda, \Delta_n^m, A, \mathcal{M}', u, p, \|., ..., .\|\big] \cap W_0^I\big[\Lambda, \Delta_n^m, A, \mathcal{M}'', u, p, \|., ..., .\|\big] \subseteq W_0^I\big[\Lambda, \Delta_n^m, A, \mathcal{M}' + \mathcal{M}'', u, p, \|., ..., .\|\big]$.*

Proof. Let $x = (x_k) \in W_0^I\big[\Lambda, \Delta_n^m, A, \mathcal{M}', u, p, \|., ..., .\|\big] \cap W_0^I\big[\Lambda, \Delta_n^m, A, \mathcal{M}'', u, p, \|., ..., .\|\big]$. Then we get the result by the following inequality:

$$\frac{1}{\lambda_n}\sum_{k\in I_n} a_{nk}\Big[k^{-s}(M_k'+M_k'')\Big(\|\frac{u_k\Delta_n^m x_k}{\rho}, z_1, z_2, ..., z_{n-1}\|\Big)\Big]^{p_k}$$
$$\leq D\frac{1}{\lambda_n}\sum_{k\in I_n} a_{nk}\Big[k^{-s}M_k'\Big(\|\frac{u_k\Delta_n^m x_k}{\rho}, z_1, z_2, ..., z_{n-1}\|\Big)\Big]^{p_k}$$
$$+ D\frac{1}{\lambda_n}\sum_{k\in I_n} a_{nk}\Big[k^{-s}M_k''\Big(\|\frac{u_k\Delta_n^m x_k}{\rho}, z_1, z_2, ..., z_{n-1}\|\Big)\Big]^{p_k}.$$

Hence,

$$\Big\{n \in \mathbb{N} : \frac{1}{\lambda_n}\sum_{k\in I_n} a_{nk}\Big[k^{-s}(M_k'+M_k'')\Big(\|\frac{u_k\Delta_n^m x_k}{\rho}, z_1, z_2, ..., z_{n-1}\|\Big)\Big]^{p_k} \geq \epsilon\Big\}$$
$$\subseteq \Big\{n \in \mathbb{N} : D\frac{1}{\lambda_n}\sum_{k\in I_n} a_{nk}\Big[k^{-s}M_k'\Big(\|\frac{u_k\Delta_n^m x_k}{\rho}, z_1, z_2, ..., z_{n-1}\|\Big)\Big]^{p_k} \geq \frac{\epsilon}{2}\Big\}$$
$$\cup \Big\{n \in \mathbb{N} : D\frac{1}{\lambda_n}\sum_{k\in I_n} a_{nk}\Big[k^{-s}M_k''\Big(\|\frac{u_k\Delta_n^m x_k}{\rho}, z_1, z_2, ..., z_{n-1}\|\Big)\Big]^{p_k} \geq \frac{\epsilon}{2}\Big\}.$$

Since both the sets on the R.H.S of above relation are belongs to I, so the set on the L.H.S of the inclusion relation belongs to I. This completes the proof of the theorem.

Theorem 2.3. *The inclusions $Z\big[\Lambda, \Delta_n^{m-1}, A, \mathcal{M}, u, p, \|., ..., .\|\big] \subseteq Z\big[\Lambda, \Delta_n^m, A, \mathcal{M}, u, p, \|., ..., .\|\big]$ are strict for $m \geq 1$. In general $Z\big[\Lambda, \Delta_n^{m-1}, \mathcal{M}, u, p, \|., ..., .\|\big] \subseteq Z\big[\Lambda, \Delta_n^m, A, \mathcal{M}, u, p, \|., ..., .\|\big]$, for $m = 0, 1, 2, ...$ where $Z = W^I, W_0^I, W_\infty^I$.*

Proof. We give the proof for $W_0^I\left[\Lambda, \Delta_n^{m-1}, A, \mathcal{M}, u, p, \|., ..., .\|\right]$ only. The others can be proved by similar argument. Let $x = (x_k)$ be any element in the space $W_0^I\left[\Lambda, \Delta_n^{m-1}, A, \mathcal{M}, u, p, \|., ..., .\|\right]$. Let $\epsilon > 0$ be given. Then there exists $\rho > 0$ such that the set

$$\Big\{n \in \mathbb{N} : \frac{1}{\lambda_n}\sum_{k\in I_n} a_{nk}\Big[k^{-s}M_k\Big(\|\frac{u_k\Delta_n^{m-1}x_k}{\rho}, z_1, z_2, ..., z_{n-1}\|\Big)\Big]^{p_k} \geq \epsilon\Big\} \in I.$$

Since $\mathcal{M} = (M_k)$ is non-decreasing and convex for every k, it follows that

$$\begin{aligned}
&\frac{1}{\lambda_n}\sum_{k\in I_n} a_{nk}\Big[k^{-s}M_k\Big(\|\frac{u_k\Delta_n^{m}x_k}{2\rho}, z_1, z_2, ..., z_{n-1}\|\Big)\Big]^{p_k} \\
&= \frac{1}{\lambda_n}\sum_{k\in I_n} a_{nk}\Big[k^{-s}M_k\Big(\|\frac{u_k\Delta_n^{m-1}x_{k+1} - u_k\Delta_n^{m-1}x_k}{2\rho}, z_1, z_2, ..., z_{n-1}\|\Big)\Big]^{p_k} \\
&\leq D\frac{1}{\lambda_n}\sum_{k\in I_n} a_{nk}\Big[\frac{1}{2}k^{-s}M_k\Big(\|\frac{u_k\Delta_n^{m-1}x_{k+1}}{\rho}, z_1, z_2, ..., z_{n-1}\|\Big)\Big]^{p_k} \\
&+ D\frac{1}{\lambda_n}\sum_{k\in I_n} a_{nk}\Big[\frac{1}{2}k^{-s}M_k\Big(\|\frac{u_k\Delta_n^{m-1}x_k}{\rho}, z_1, z_2, ..., z_{n-1}\|\Big)\Big]^{p_k} \\
&\leq DH\frac{1}{\lambda_n}\sum_{k\in I_n} a_{nk}\Big[k^{-s}M_k\Big(\|\frac{u_k\Delta_n^{m-1}x_{k+1}}{\rho}, z_1, z_2, ..., z_{n-1}\|\Big)\Big]^{p_k} \\
&+ DH\frac{1}{\lambda_n}\sum_{k\in I_n} a_{nk}\Big[k^{-s}M_k\Big(\|\frac{u_k\Delta_n^{m-1}x_k}{\rho}, z_1, z_2, ..., z_{n-1}\|\Big)\Big]^{p_k},
\end{aligned}$$

where $H = \max\Big\{1, (\frac{1}{2})^G\Big\}$. Thus we have

$$\begin{aligned}
&\Big\{n \in \mathbb{N} : \frac{1}{\lambda_n}\sum_{k\in I_n} a_{nk}\Big[k^{-s}M_k\Big(\|\frac{u_k\Delta_n^{m}x_k}{2\rho}, z_1, z_2, ..., z_{n-1}\|\Big)\Big]^{p_k} \geq \epsilon\Big\} \\
&\subseteq \Big\{n \in \mathbb{N} : \frac{1}{\lambda_n}\sum_{k\in I_n} a_{nk}\Big[k^{-s}M_k\Big(\|\frac{u_k\Delta_n^{m-1}x_{k+1}}{\rho}, z_1, z_2, ..., z_{n-1}\|\Big)\Big]^{p_k} \geq \frac{\epsilon}{2}\Big\} \\
&\cup \Big\{n \in \mathbb{N} : \frac{1}{\lambda_n}\sum_{k\in I_n} a_{nk}\Big[k^{-s}M_k\Big(\|\frac{u_k\Delta_n^{m-1}x_k}{\rho}, z_1, z_2, ..., z_{n-1}\|\Big)\Big]^{p_k} \geq \frac{\epsilon}{2}\Big\}.
\end{aligned}$$

Since both the sets in right hand side of the above relation belongs to I, therefore we get the set

$$\Big\{n \in \mathbb{N} : \frac{1}{\lambda_n}\sum_{k\in I_n} a_{nk}\Big[k^{-s}M_k\Big(\|\frac{u_k\Delta_n^{m}x_k}{\rho}, z_1, z_2, ..., z_{n-1}\|\Big)\Big]^{p_k} \geq \epsilon\Big\} \in I.$$

This inclusion is strict follows from the following example.

Example. Let $M_k(x) = x$, for all $k \in \mathbb{N}$, $u_k = p_k = 1$ for all $k \in \mathbb{N}$, $s = 0$, $\lambda_n = 1$ and $A = (C, 1)$, the Cesaro matrix. Now consider a sequence $x =$

$(x_k) = (k^t)$. Then for $n = 1, x = (x_k)$ belongs to $W_0^I\big[\Lambda, \Delta_n^m, \mathcal{M}, u, p, \|., ..., .\|\big]$ but does not belongs to $W_0^I\big[\Lambda, \Delta_n^{m-1}, \mathcal{M}, u, p, \|., ..., .\|\big]$, because $\Delta_n^m x_k = 0$ and $\Delta_n^{m-1} x_k = (-1)^{m-1}(m-1)!$.

Theorem 2.4. *For any two sequences $p = (p_k)$ and $q = (q_k)$ of positive real numbers and for any two n-norms $\|., ..., .\|_1$ and $\|., ..., .\|_2$ on X, we have the following $Z\big[\Lambda, \Delta_n^m, A, \mathcal{M}, u, p, \|., ..., .\|_1\big] \cap Z\big[\Lambda, \Delta_n^m, A, \mathcal{M}, u, q, \|., ..., .\|_2\big] \neq \phi$ where $Z = W^I, W_0^I$ and W_∞^I.*

Proof. Since the zero element belongs to both the classes of sequences, so the intersection is non-empty.

Theorem 2.5. *The sequence spaces $W_0^I\big[\Lambda, \Delta_n^m, A, \mathcal{M}, u, p, \|., ..., .\|\big]$ and $W_\infty^I\big[\Lambda, \Delta_n^m, A, \mathcal{M}, u, p, \|., ..., .\|\big]$ are normal as well as monotone.*

Proof. We shall prove the theorem for $W_0^I\big[\Lambda, \Delta_n^m, A, \mathcal{M}, u, p, \|., ..., .\|\big]$. Let $x = (x_k) \in W_0^I\big[\Lambda, \Delta_n^m, A, \mathcal{M}, u, p, \|., ..., .\|\big]$ and $\alpha = (\alpha_k)$ be a sequence of scalars such that $|\alpha_k| \leq 1$ for all $k \in \mathbb{N}$. Then for given $\epsilon > 0$, we have

$$\Big\{n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} a_{nk} \Big[k^{-s} M_k\Big(\|\frac{u_k \Delta_n^m(\alpha_k x_k)}{\rho}, z_1, z_2, ..., z_{n-1}\|\Big)\Big]^{p_k} \geq \epsilon\Big\}$$
$$\subseteq \Big\{n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} a_{nk} \Big[k^{-s} M_k\Big(\|\frac{u_k \Delta_n^m(x_k)}{\rho}, z_1, z_2, ..., z_{n-1}\|\Big)\Big]^{p_k} \geq \epsilon\Big\} \in I.$$

Hence, $\alpha_k x_k \in W_0^I\big[\Lambda, \Delta_n^m, A, \mathcal{M}, u,\ p, \|., ..., .\|\big]$. Thus, the space $W_0^I\big[\Lambda, \Delta_n^m, A, \mathcal{M}, u,\ p, \|., ..., .\|\big]$ is normal. Therefore, $W_0^I\big[\Lambda, \Delta_n^m, A, \mathcal{M}, u, p, \|., ..., .\|\big]$ is monotone also (see [15]). Similarly, we can prove the theorem for other case. This completes the proof of the theorem.

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