INEQUALITIES OF UNITARILY INVARIANT NORMS FOR MATRICES

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Abstract. We present some inequalities of unitarily invariant norms for matrices by using majorization, Fan dominance principle and some existing inequalities of singular values and unitarily invariant norms for matrices. Our results are refinements or generalizations of ones shown by Audenaert, Al-khlyleh, and Kittaneh.
Keywords: singular values, unitarily invariant norms, fan dominance principle.

1. Introduction

Let M_n be the space of $n \times n$ complex matrices and suppose that $s_1(A) \geq \cdots \geq s_n(A) \geq 0$ are the singular values of A, which is the eigenvalues of the positive semidefinite matrix $|A| = (A^*A)^{1/2}$, arranged in decreasing order and repeated according to multiplicity. Let $\|\cdot\|$ denote any unitarily invariant norm on M_n. For $A \in M_n$, by singular value decomposition of A, we know that the trace norm $\|A\|_1 = \sum_{j=1}^n s_j(A) = tr|A|$ and the Frobenius norm $\|A\|_F = (\sum_{j=1}^n s_j^2(A))^{1/2} = (tr|A|^2)^{1/2}$ are both unitarily invariant.

Let $A, B \in M_n$. Recently, Audenaert proved in [1] that if $v \in [0,1]$, then

$$\|AB^*\|^2 \leq \|vA^*A + (1-v)\,B^*B\|\,\|(1-v)\,A^*A + vB^*B\|, \tag{1.1}$$

which is unity of the arithmetic-geometric mean and Cauchy-Schwarz inequalities for unitarily invariant norms.

Let $A, X, B \in M_n$. Zou proved in [2] that if $v \in [0,1]$, then

$$\|AXB^*\|^2 \leq \|vA^*AX + (1-v)\,XB^*B\|\,\|(1-v)\,A^*AX + vXB^*B\|, \tag{1.2}$$

which is a generalization of inequality (1.1).

Let $A, X, B \in M_n$. Very recently, Al-Manasrah and Kittaneh proved in [3] that if $v \in [0,1]$, then

$$\begin{aligned} \|AXB^*\|_F^2 \leq & \left(\|vA^*AX + (1-v)\,XB^*B\|_F^2 - v_0^2\,\|A^*AX - XB^*B\|_F^2\right)^{1/2} \\ & \times \left(\|(1-v)\,A^*AX + vXB^*B\|_F^2 - v_0^2\,\|A^*AX - XB^*B\|_F^2\right)^{1/2}, \end{aligned} \tag{1.3}$$

where $v_0 = \min\{v,\ 1-v\}$. Inequality (1.3) is a refinement of inequality (1.2) for the Frobenius norm.

Lin [4] gave a new proof of inequality (1.1). The authors of [5, 6] showed some generalizations of inequality (1.2).

In this short note, following the idea of Lin [4], Al-Manasrah and Kittaneh [3], we first present an improvement of inequality (1.1) for the trace norm. Meanwhile, we also give a generalization of inequality (1.3).

2. Main results

In this section, we will show the main results of this paper. To do this, we need the following lemmas.

Lemma 2.1 ([7]). *Let $A \in M_n$. Then for any $k = 1, \cdots, n$, we have*

$$\prod_{j=1}^{k} |\lambda_j(A)| \le \prod_{j=1}^{k} s_j(A).$$

Lemma 2.2 ([7]). *Let $A, B \in M_n$. Then for any $k = 1, \cdots, n$, we have*

$$\prod_{j=1}^{k} s_j(AB) \le \prod_{j=1}^{k} s_j(A)\, s_j(B).$$

Lemma 2.3 ([8]). *Let $A, B \in M_n$ be positive semidefinite. If $v \in [0,1]$, then*

$$\left\|A^v B^{1-v}\right\|_1 \le \|vA + (1-v)B\|_1 - v_0\left(\sqrt{\|A\|_1} - \sqrt{\|B\|_1}\right)^2,$$

where $v_0 = \min\{v,\ 1-v\}$.

Lemma 2.4 ([9]). *Let $A, X, B \in M_n$ such that A, B are positive semidefinite. If $v \in [0,1]$, then*

$$\begin{aligned}\left\|A^v X B^{1-v}\right\|_F \le \Big(&\|vAX + (1-v)XB\|_F^p \\ &-v_0^p\left(\|AX + XB\|_F^p - 2^p\left\|A^{1/2}XB^{1/2}\right\|_F^p\right)\Big)^{1/p},\end{aligned}$$

where $v_0 = \min\{v,\ 1-v\}$.

Theorem 2.1. *Let $A, B \in M_n$. If $v \in [0,1]$, then*

$$\begin{aligned}\|AB^*\|_1^2 \le\ & \left(\|vA^*A + (1-v)B^*B\| - v_0\left(\sqrt{\|A^*A\|_1} - \sqrt{\|B^*B\|_1}\right)^2\right) \\ \times\ & \left(\|(1-v)A^*A + vB^*B\| - v_0\left(\sqrt{\|A^*A\|_1} - \sqrt{\|B^*B\|_1}\right)^2\right), \end{aligned} \tag{2.1}$$

where $v_0 = \min\{v,\ 1-v\}$.

Proof. By Lemmas 2.1 and 2.2, we know that for any $k = 1, \cdots, n$, we have

$$\begin{aligned}\prod_{j=1}^{k} s_j^2 \left(AXB^*\right) &= \prod_{j=1}^{k} \lambda_j \left(BX^*A^*AXB^*\right) \\ &= \prod_{j=1}^{k} \lambda_j \left(A^*AXB^*BX^*\right) \\ &= \prod_{j=1}^{k} \lambda_j \left((A^*A)^v X (B^*B)^{1-v} (B^*B)^v X^* (A^*A)^{1-v}\right) \\ &\leq \prod_{j=1}^{k} s_j \left((A^*A)^v X (B^*B)^{1-v} (B^*B)^v X^* (A^*A)^{1-v}\right) \\ &\leq \prod_{j=1}^{k} s_j \left((A^*A)^v X (B^*B)^{1-v}\right) s_j \left((B^*B)^v X^* (A^*A)^{1-v}\right).\end{aligned}$$

That is

$$\prod_{j=1}^{k} s_j(AXB^*) \leq \prod_{j=1}^{k} s_j^{1/2}((A^*A)^v X (B^*B)^{1-v}) s_j^{1/2}((B^*B)^v X^* (A^*A)^{1-v}). \tag{2.2}$$

Let

$$Y_1 = diag\left(s_1^{1/2}\left((A^*A)^v X (B^*B)^{1-v}\right), \cdots, s_n^{1/2}\left((A^*A)^v X (B^*B)^{1-v}\right)\right),$$

$$Y_2 = diag\left(s_1^{1/2}\left((B^*B)^v X^* (A^*A)^{1-v}\right), \cdots, s_n^{1/2}\left((B^*B)^v X^* (A^*A)^{1-v}\right)\right).$$

Then, it follows from (2.2) that

$$\prod_{j=1}^{k} s_j \left(AXB^*\right) \leq \prod_{j=1}^{k} s_j \left(Y_1\right) s_j \left(Y_2\right) = \prod_{j=1}^{k} s_j \left(Y_1 Y_2\right).$$

Since weak log-majorization implies weak majorization, we obtain

$$\sum_{j=1}^{k} s_j \left(AXB^*\right) \leq \sum_{j=1}^{k} s_j \left(Y_1 Y_2\right). \tag{2.3}$$

By Fan's dominance principle [7], we know that inequality (2.3) is equivalent to

$$\left\|AXB^*\right\| \leq \left\|Y_1 Y_2\right\|. \tag{2.4}$$

Putting $X = I$ and $v = 0$ or $v = 1$ in inequality (1.1), we get

$$\left\|Y_1 Y_2^*\right\|^2 \leq \left\|Y_1^* Y_1\right\| \left\|Y_2^* Y_2\right\|.$$

It follows from (2.4) and this last inequality that

$$\|AXB^*\|^2 \le \left\|(A^*A)^v X (B^*B)^{1-v}\right\| \left\|(B^*B)^v X^* (A^*A)^{1-v}\right\|. \tag{2.5}$$

Since trace is unitarily invariant, we have

$$\|AB^*\|_1^2 \le \left\|(A^*A)^v (B^*B)^{1-v}\right\|_1 \left\|(B^*B)^v (A^*A)^{1-v}\right\|_1.$$

Lemma 2.3 and the above inequality complete the proof. □

Remark 2.1. Obviously, inequality (2.1) is a refinement of inequality (1.1).

Remark 2.2. Putting $X = I$ in (2.2), we have

$$\prod_{j=1}^{k} s_j^2 (AB^*) \le \prod_{j=1}^{k} s_j \left((A^*A)^v (B^*B)^{1-v}\right) s_j \left((B^*B)^v (A^*A)^{1-v}\right). \tag{2.6}$$

Ando proved in [10] that if $v \in [0,1]$, then

$$s_j \left(A^v B^{1-v}\right) \le s_j (vA + (1-v) B), j = 1, \cdots, n.$$

Combining inequality (2.6) with Ando's result, we get

$$\prod_{j=1}^{k} s_j^2 (AB^*) \le \prod_{j=1}^{k} s_j (vA^*A + (1-v) B^*B) s_j ((1-v) A^*A + vB^*B),$$

which implies inequality (1.1).

Next, we shall give a generalization of inequality (1.3).

Theorem 2.2. *Let $A, X, B \in M_n$. If $v \in [0,1]$, then*

$$\begin{aligned} \|AXB^*\|_F^2 &\le \left(\|vA^*AX + (1-v) XB^*B\|_F^p - v_0^p f(A,X,B,p)\right)^{1/p} \\ &\times \left(\|(1-v) A^*AX + vXB^*B\|_F^p - v_0^p f(A,X,B,p)\right)^{1/p}, \end{aligned} \tag{2.7}$$

where

$$v_0 = \min\{v,\ 1-v\},$$

$$f(A,X,B,p) = \|A^*AX + XB^*B\|_F^p - 2^p \left\|(A^*A)^{1/2} X (B^*B)^{1/2}\right\|_F^p.$$

Proof. Note that for any $Y \in M_n$, we have $\|Y\| = \|Y^*\|$. Since Frobenius norm is unitarily invariant, it follows from inequality (2.5) and Lemma 2.5, we obtain

$$\begin{aligned} \|AXB^*\|_F^2 &\le \left\|(A^*A)^v X (B^*B)^{1-v}\right\|_F \left\|(B^*B)^v X^* (A^*A)^{1-v}\right\|_F \\ &= \left\|(A^*A)^v X (B^*B)^{1-v}\right\|_F \left\|(A^*A)^{1-v} X (B^*B)^v\right\|_F \\ &\le \left(\|vA^*AX + (1-v) XB^*B\|_F^p - v_0^p f(A,X,B,p)\right)^{1/p} \\ &\times \left(\|(1-v) A^*AX + vXB^*B\|_F^p - v_0^p f(A,X,B,p)\right)^{1/p}, \end{aligned}$$

where

$$v_0 = \min\{v,\ 1-v\},$$

$$f(A,X,B,p) = \|A^*AX + XB^*B\|_F^p - 2^p \left\|(A^*A)^{1/2} X (B^*B)^{1/2}\right\|_F^p .$$

This completes the proof. □

Remark 2.3. Putting $p = 2$ in inequality (2.7), we obtain

$$\begin{aligned} \|AXB^*\|_F^2 &\leq \left(\|vA^*AX + (1-v)XB^*B\|_F^2 - v_0^2 f(A,X,B,2)\right)^{1/2} \\ &\times \left(\|(1-v)A^*AX + vXB^*B\|_F^2 - v_0^2 f(A,X,B,2)\right)^{1/2}. \end{aligned} \tag{2.8}$$

Note that

$$\|A^*AX + XB^*B\|_F^2 = \|A^*AX - XB^*B\|_F^2 + 4\left\|(A^*A)^{1/2} X (B^*B)^{1/2}\right\|_F^2,$$

then, we can rewrite inequality (2.8) as follows

$$\begin{aligned} \|AXB^*\|_F^2 &\leq \left(\|vA^*AX + (1-v)XB^*B\|_F^2 - v_0^2\|A^*AX - XB^*B\|_F^2\right)^{1/2} \\ &\times \left(\|(1-v)A^*AX + vXB^*B\|_F^2 - v_0^2\|A^*AX - XB^*B\|_F^2\right)^{1/2}. \end{aligned}$$

This is inequality (1.3) and so we know that inequality (2.7) is a generalization of inequality (1.3).

Acknowledgements

The author wishes to express her heartfelt thanks to the referees for their detailed and helpful suggestions for revising the manuscript.

Competing interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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Accepted: 21.01.2016